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Differential algebra on lattice Green functions and Calabi–Yau operators

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Abstract

We revisit miscellaneous linear differential operators mostly associated with lattice Green functions in arbitrary dimensions, but also Calabi–Yau operators and order-7 operators corresponding to exceptional differential Galois groups. We show that these irreducible operators are not only globally nilpotent, but are such that they are homomorphic to their (formal) adjoints. Considering these operators, or, sometimes, equivalent operators, we show that they are also such that, either their symmetric square or their exterior square, have a rational solution. This is a general result: an irreducible linear differential operator homomorphic to its (formal) adjoint is necessarily such that either its symmetric square, or its exterior square has a rational solution, and this situation corresponds to the occurrence of a special differential Galois group. We thus define the notion of being 'Special Geometry' for a linear differential operator if it is irreducible, globally nilpotent, and such that it is homomorphic to its (formal) adjoint. Since many derived from geometry n-fold integrals ('Periods') occurring in physics, are seen to be diagonals of rational functions, we address several examples of (minimal order) operators annihilating diagonals of rational functions, and remark that they also seem to be, systematically, associated with irreducible factors homomorphic to their adjoint.

Keywords: lattice Green functions, Calabi–Yau ODEs, Ising model operators, differential Galois groups, self-adjoint operators

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1. Introduction

When one considers all the irreducible factors of the globally nilpotent linear differential operators encountered in the study of n-folds integrals of the Ising class [1] (or the ones displayed by other authors in an enumerative combinatorics framework [2, 3], or in a Calabi–Yau framework [4–6]), one could expect their differential Galois groups to be generically $SL(N, \mathbb{C})$ or extensions of $SL(N, \mathbb{C})$. However, it turns out that their differential Galois groups are of 'selected' types: they are either the orthogonal group $O(N, \mathbb{C})$, the symplectic group $Sp(N, \mathbb{C})$ or subgroups of these such as $SO(N)$, G_2 (see [7]), etc. The aim of this paper is to demonstrate and study this phenomenon, the ubiquity of either an orthogonal or symplectic geometry on the solutions of these operators.

Along this line it is worth recalling that *globally nilpotent linear differential operators* associated with generic ${}_nF_{n-1}$ *hypergeometric functions with rational parameters*[6], have $SL(N, \mathbb{C})$ (or extensions of $SL(N, \mathbb{C})$) as their differential Galois groups. For instance, the differential Galois group for the ${}_3F_2$ hypergeometric function

$$ {}_3F_2\Big(\Big[\tfrac{191}{479}, \tfrac{359}{311}, \tfrac{503}{89}\Big], \Big[\tfrac{521}{151}, \tfrac{401}{67}\Big], x\Big), \tag{1} $$

is $SL(3, \mathbb{C})$[7]. In contrast, in simple examples, the emergence of 'selected' differential Galois groups can be seen very explicitly [10], and understood (from a physicist's viewpoint) as the emergence of some 'invariant'. To illustrate this on the $SO(3, \mathbb{C})$ group[8], let us consider the *non-Fuchsian* operator (in $\theta = x \cdot \mathrm{d}/\mathrm{d}x$):

$$ 2\theta \cdot (3\theta - 2)(3\theta - 4) - 9x \cdot (2\theta + 1), \tag{2} $$

with the three ${}_1F_2$ hypergeometric solutions

$$ {}_1F_2\left(\left[\tfrac{1}{2}\right], \left[-\tfrac{1}{3}, \tfrac{1}{3}\right], x\right), \quad x^{2/3} \cdot {}_1F_2\left(\left[\tfrac{7}{6}\right], \left[\tfrac{1}{3}, \tfrac{5}{3}\right], x\right), \quad x^{4/3} \cdot {}_1F_2\left(\left[\tfrac{11}{6}\right], \left[\tfrac{5}{3}, \tfrac{7}{3}\right], x\right). $$

If f denotes a solution of this operator (in the above closed form or as a formal solution at the origin or at ∞), one has the following *quadratic relation* $Q(f, f', f'') = \text{const.}$, where:

$$ Q(X_0, X_1, X_2) = 9 \cdot (36x + 5) \cdot x^2 \cdot X_2^2 - 324 \cdot x^2 \cdot X_2 \cdot X_1 - 648x^2 \cdot X_2 \cdot X_0 $$
$$ +(81x - 5) \cdot X_1^2 + 9 \cdot (36x - 5) \cdot X_0 \cdot X_1 + 9 \cdot (36x - 5) \cdot X_0^2. $$

The constant depends on the linear combination of solutions used. For instance, with the first ${}_1F_2$ hypergeometric solution one has $Q(f, f', f'') = 225/4$, while with the two other ${}_1F_2$ solutions it reads $Q(f, f', f'') = 0$. In other words, Q is a *first integral*.

The emergence of such 'special' differential Galois groups in so many domains of theoretical physics is clearly something *we need to understand better*.

We have provided a large number of linear ODEs on various problems of lattice statistical mechanics, in particular for the magnetic susceptibility of the two-dimensional Ising model [8, 11–20]. These linear ODEs factorize into many factors of order ranging from 1, to 12 (for $\chi^{(5)}$) and even 23 (for $\chi^{(6)}$). As far as the factors of smallest orders (2, 3 and 4) are concerned,

[6] Their corresponding linear differential operators are necessarily globally nilpotent [8].
[7] One shows that there are no rational solutions of symmetric powers in degree 2, 3, 4, 6, 8, 9, 12, using an algorithm in van Hoeij *et al* [9].
[8] This operator is actually homomorphic to its adjoint (see below) with non-trivial order-2 intertwiners.

one can verify that *all these linear differential operators are homomorphic to their adjoint*. Furthermore, one remarks experimentally, that their exterior square or symmetric square, either have a *rational solution*, or are of an *order smaller than the order one would expect generically*. Quite often these differential operators are simply *conjugated to their adjoint*, i.e. the intertwiner between the operator and its adjoint, is just an order-zero operator, namely a function. In this case they can easily be recast into *self-adjoint* operators. A large set of linear differential operators *conjugated to their adjoint*, can be found in the very large list of Calabi–Yau order-4 operators obtained by Almkvist *et al* [5], or displayed by Batyrev and van Straten [4], or some simple order-3 operators displayed in a paper by Golyshev [21, 22] (see also Sanabria Malagon [23]).

Throughout this paper we will see examples of *irreducible* operators where these two differential algebra properties occur simultaneously. On the one hand, these operators are *homomorphic to their adjoint*, and on the other hand, their symmetric or exterior square have a *rational*[9]*solution*. These simultaneous properties correspond to special differential Galois groups. In fact, these properties are equivalent[10].

In this paper, we will have a learn-by-example approach of all these concepts. In this respect, we will display, for pedagogical reasons, a set of enumerative combinatorics examples corresponding to miscellaneous *lattice Green functions* [2, 3, 25–27] as well as Calabi–Yau examples, together with order-7 operators [28, 29] associated with exceptional differential Galois groups. We will show that these lattice Green operators, Calabi–Yau operators and order-7 operators associated with exceptional groups, are a perfect illustration of differential operators with *selected differential algebra structures*: they are homomorphic to their adjoint; also, either their symmetric or exterior powers (most of the time squares) have a *rational solution*, or the previous symmetric, or exterior, powers of some *equivalent operators*[11] have a rational solution. This situation corresponds to the emergence of *selected differential Galois groups* (orthogonal or symplectic), a situation we could call 'Special Geometry'. Among the derived from geometry n-fold integrals ('Periods') occurring in physics, we have seen that they are quite often *diagonals of rational functions* [18, 19]. We will also address, in this paper, examples of (minimal order) operators annihilating diagonals of rational functions; we will remark that they also seem to have *irreducible factors homomorphic to their adjoint*.

2. Adjoint of differential operators and invertible homomorphisms of an operator with its adjoint

In the next section, examples of linear differential operators corresponding to lattice Green functions on various lattices are displayed according to their order N and their complexity. We focus on the differential algebra structures of these linear differential operators, in particular with respect to an important 'duality' with amounts to performing the *adjoint*, or, more precisely (see 2.1 in [30]), the 'formal adjoint' of the operator (D_x in the whole paper denotes the derivative d/dx):

[9] They may have hyperexponential solutions [24] (command expsols in DEtools), i.e. Nth root of rational solutions, when one considers homomorphisms *up to algebraic extensions*.

[10] In a Tannakian formulation, one could say that the homomorphisms of an operator L_1 with another operator L_2 are isomorphic to the product $\mathrm{Hom}(L_1, L_2) \simeq L_1 \otimes L_2^\star$, giving, in the case of the homomorphisms of an operator L with its adjoint $L^\star$, $\mathrm{Hom}(L, L^\star) \simeq L \otimes (L^\star)^\star \simeq L \otimes L$, which is isomorphic to the direct sum $L \otimes L \simeq Ext^2(L) \oplus \mathrm{Sym}^2(L)$.

[11] For an operator L, an equivalent operator $\tilde{L}$ can be built from $L \oplus O_n = \tilde{L} \cdot O_n$, where the operator O_n can (for simplicity) be taken as $O_1 = D_x$, or $O_2 = D_x^2$, etc.

$$L = \sum_{n=0}^{N} a_n(x) \cdot D_x^n \longrightarrow adjoint(L) = (-1)^N \cdot \sum_{n=0}^{N} (-1)^n \cdot D_x^n \cdot a_n(x), \tag{3}$$

that is:

$$\sum_{n=0}^{N} a_n(x) \cdot \frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} = 0 \longrightarrow (-1)^N \cdot \sum_{n=0}^{N} (-1)^n \cdot \frac{\mathrm{d}^n (a_n(x) \cdot f(x))}{\mathrm{d}x^n} = 0. \tag{4}$$

2.1. Homomorphisms of an operator with its adjoint

Recall that two operators L and $\tilde{L}$, of the same order, are called homomorphic (see [30, 31]) when there exist two operators T and S, whose *order is smaller* than the one of L and $\tilde{L}$, such that[12]:

$$\tilde{L} \cdot T = S \cdot L. \tag{5}$$

The intertwiner T maps the solutions of L into the solutions of $\tilde{L}$. When T and L have *no common right-factor* (or equivalently when S and $\tilde{L}$ has no common left factor), for example when L is *irreducible*, one can show that this map is bijective. When (5) holds, and T and L have *no common right-factor*, one says that L and $\tilde{L}$ are *equivalent*. Thus, one also has *intertwiners* $\tilde{T}$, $\tilde{S}$ such that

$$L \cdot \tilde{T} = \tilde{S} \cdot \tilde{L}, \tag{6}$$

We say that L is *self-adjoint* when $L = adjoint(L)$. We say that L is *conjugated with its adjoint* when there exists a rational, or Nth root of rational, function f such that $L \cdot f = f \cdot adjoint(L)$, i.e. $L \cdot f$ is self-adjoint. More generally, a differential operator L is *homomorphic to its adjoint* (in the above sense) when there exists an (intertwiner) operator T (of *order less than that of* L) such that[13]

$$L \cdot T = adjoint(T) \cdot adjoint(L). \tag{7}$$

Again, this means that the operator $L \cdot T$ is self-adjoint.

The typical situation which we encounter in physics is that the differential operators are of a rather large order and factorize into many factors of various orders (see the minimal order operators [11, 13] annihilating the $\chi^{(n)}$'s). For these large order differential operators, we will systematically factorize the operator. *The interesting concept amounts to seeing if each irreducible factor in the factorization, is homomorphic to its adjoint*.

We end this section with two comments. For irreducible L, one deduces, from (5) and (6), the equality

$$L \cdot \tilde{T} \cdot T = \tilde{S} \cdot S \cdot L, \tag{8}$$

so the remainder of the right division of $\tilde{T} \cdot T$ by L is a constant. When $\tilde{L}$ is the adjoint of L, we will see, in the sequel, that this relation on the intertwiners T and $\tilde{T}$ makes a remarkable 'decomposition' of L emerge. The second comment is on the homomorphisms of an operator with its adjoint in the *reducible* case. For two reducible differential operators, L and $\tilde{L}$, of the *same order*, the relation (5) may hold. For a reducible operator [33] having the unique

[12] The *intertwiner* T is given by the command `Homomorphisms`$(L, \tilde{L})$ of the DEtools package in Maple [32].

[13] It is easy to show, in the case of an homomorphism of an operator L with its adjoint, that the intertwiner on the right-hand side of (7) is necessarily equal to the adjoint of the intertwiner on the left-hand side. Actually, from the equivalence $L \cdot T = S \cdot adjoint(L)$, taking adjoint on both sides gives $adjoint(T) \cdot adjoint(L) = L \cdot adjoint(S)$. For irreducible L, the intertwiner is unique, so $S = adjoint(T)$.

factorization $L = L_n \cdot L_p$, with $n \neq p$, one can show that the homomorphism with the adjoint *just reduces* to a homomorphism of the *right factor* L_p with its adjoint. The corresponding rational solution for the symmetric or exterior square is precisely the rational solution induced by the right factor, since $\mathrm{Sym}^2(L_p)$ (resp. $Ext^2(L_p)$) is a right-factor of $\mathrm{Sym}^2(L_n \cdot L_p)$ (resp. $Ext^2(L_n \cdot L_p)$).

In the sequel, when studying homomorphism of an operator with its adjoint, we will restrict to *irreducible* operators.

3. Special ODEs from lattice statistical mechanics and enumerative combinatorics: lattice Green functions

We are going to display miscellaneous examples of linear differential operators corresponding to *lattice Green functions* on various lattices. We will denote these lattice Green operators G_n^{latt}, where n is the order of the operator, and where *latt* refers to the lattice that one considers.

3.1. Special lattice Green ODEs: simple cubic lattice

The most well-known example of lattice Green function has been obtained [34] for the *simple cubic lattice*. The lattice Green function corresponds to the order-3 operator (see equation (19) in [3])

$$G_3^{\mathrm{sc}} = \theta^3 - 2x \cdot (10\theta^2 + 10\theta + 3)(2\theta + 1) + 18x^2 \cdot (2\theta + 3)\,(2\theta + 2)\,(2\theta + 1), \qquad (9)$$

This order-3 operator (9), when divided by x on the left, is *exactly self-adjoint*. The symmetric square of G_3^{sc} is of *order* 5 (instead of the generic order 6).

The solution of (9), which corresponds to a series expansion with *integer coefficients*, is the Hadamard product of $(1-4x)^{-1/2}$ with a Heun function, and is also the square of another Heun function which can also be written in terms of ${}_2F_1$ hypergeometric functions with *two possible algebraic pullbacks*:

$$\begin{aligned} HeunG\left(9,\ \tfrac{3}{4},\ \tfrac{1}{4},\ \tfrac{3}{4},\ 1,\ \tfrac{1}{2};\ 36x\right)^2 &= HeunG\left(\tfrac{1}{9},\ \tfrac{1}{12},\ \tfrac{1}{4},\ \tfrac{3}{4},\ 1,\ \tfrac{1}{2};\ 4x\right)^2 \\ &= (1-4x)^{-1/2} \star HeunG(1/9, 1/3, 1, 1, 1, 1;\ x) \\ &= C_\pm^{1/2} \cdot {}_2F_1\left(\left[\tfrac{1}{6},\ \tfrac{1}{3}\right],\ [1];\ P_\pm\right)^2 \\ &= 1 + 6x + 90x^2 + 1860x^3 + 44\,730x^4 + 1172\,556x^5 + \cdots \end{aligned} \qquad (10)$$

where the algebraic pull-backs $P_\pm$ and algebraic prefactors $C_\pm$ read:

$$\begin{aligned} P_\pm &= 54 \cdot x \cdot (1 - 27x + 108x^2 \pm (1 - 9x)((1-36x)(1-4x))^{1/2}), \\ C_\pm &= -18x + \tfrac{5}{2} \pm \tfrac{3}{2} \cdot ((1-36x)\cdot(1-4x))^{1/2}. \end{aligned} \qquad (11)$$

The fact that these selected Heun functions (10) correspond to *modular forms* [20] can be seen on the relation between the two algebraic pullbacks, $y = P_+$ and $z = P_-$, namely the genus-zero *modular curve*[14]:

$$\begin{aligned} 4 \cdot y^3 z^3 - 12y^2 z^2 \cdot (z+y) + 3\,yz \cdot (4y^2\,4z^2 - 127yz) \\ -4 \cdot (y+z)(y^2 + z^2 + 83\,yz) + 432 \cdot yz = 0. \end{aligned} \qquad (12)$$

Remark 3.1. If one compares two Heun functions with the same singular points and the same critical exponents, which just differ by their *accessory parameter*, namely

[14] Which is *exactly a rational modular curve* already found for the order-3 operator F_3 in [20].

$HeunG(9, 3/4, 1/4, 3/4, 1, 1/2, 36x)$ and[15] $HeunG(9, -3/4, 1/4, 3/4, 1, 1/2, 36x)$, one sees that the first one corresponds to a *modular form* and to series with *integer coefficients*, while the second one *is not even a globally bounded series* [18, 19]. These two Heun functions $HeunG(9, \pm 3/4, 1/4, 3/4, 1, 1/2, 36x)$ are solutions of order-2 linear differential operators

$$H_2^{(\pm)} = \theta^2 - x \cdot (40\,\theta^2 + 20\,\theta\, \pm 3) + 9 \cdot x^2 \cdot (4\,\theta + 3)(4\,\theta + 1), \qquad (13)$$

which are, both, conjugated to their adjoint:

$$f(x) \cdot \, adjoint\big(H_2^{(\pm)}\big) = H_2^{(\pm)} \cdot f(x) \quad \text{with:} \quad f(x) = x \cdot \, ((1-36\,x) \cdot (1-4\,x))^{1/2}.$$

3.2. Special lattice Green ODEs: face-centered cubic lattice

A third order linear differential operator corresponds to the *lattice Green function* of the *face-centered cubic* lattice (see equation (19) in Guttmann's paper [3]):

$$G_3^{\rm fcc} = \theta^3 - 2\,x \cdot \theta \cdot (\theta + 1)(2\,\theta + 1) - 16\,x^2 \cdot (\theta + 1)(5\,\theta^2 + 10\,\theta + 6)$$
$$-96\,x^3 \cdot (\theta + 1)(\theta + 2)(2\,\theta + 3). \qquad (14)$$

This operator, once divided by x, is *exactly self-adjoint*, namely: $1/x \cdot G_3^{\rm fcc} = adjoint(1/x \cdot G_3^{\rm fcc})$. The *symmetric square* of $G_3^{\rm fcc}$ is of order 5 (instead of the order 6 one could expect for generic order-3 operators).

Let us introduce, instead of $G_3^{\rm fcc}$, the equivalent operator $\tilde{G}_3^{\rm fcc}$ such that

$$S_1^{\rm fcc} \cdot G_3^{\rm fcc} = \tilde{G}_3^{\rm fcc} \cdot D_x. \qquad (15)$$

where the order-1 intertwiner $S_1^{\rm fcc}$ and the Wronskian $\rho(x)$ read

$$S_1^{\rm fcc} = D_x - \frac{{\rm d}\ln(\rho(x))}{{\rm d}x}, \qquad \rho(x) = \frac{6\,x + 1}{x \cdot \, (4\,x + 1)^2 \cdot \, (12\,x - 1)}. \qquad (16)$$

We find that the *symmetric square* of the equivalent operator $\tilde{G}_3^{\rm fcc}$ has a *rational solution* $r(x)$:

$$r(x) = \frac{1}{x^2 \cdot \, (4\,x + 1)^2\,(12\,x - 1)}. \qquad (17)$$

More precisely, the *symmetric square* of the equivalent operator $\tilde{G}_3^{\rm fcc}$ is the *direct sum* of an order-1 operator and an order-5 operator:

$$Sym^2(\tilde{G}_3^{\rm fcc}) = M_1 \oplus M_5 \qquad \text{where:} \qquad M_1 = D_x - \frac{{\rm d}\ln(r(x))}{{\rm d}x}. \qquad (18)$$

The Wronskian of $G_3^{\rm fcc}$ is the square root of a rational function. The differential Galois group is not the generic $SL(3, \mathbb{C})$ one could expect for a generic order-3 operator, but is equal to the orthogonal group $O(3, \mathbb{C})$: the rational solution (17) of ${\rm Sym}^2(\tilde{G}_3^{\rm fcc})$, comes from an invariant of degree 2 for the differential Galois group.

In fact the operator (14) is the *symmetric square* of an order-2 operator[16]:

$$\theta^2 - 2\,x \cdot \theta \cdot (4\,\theta + 1) - 24\,x^2 \cdot \, (\theta + 1)\,(2\,\theta + 1). \qquad (19)$$

From that last remark, one immediately deduces that the differential Galois group must be the differential Galois group of an order-2 operator, generically $SL(2, \mathbb{C})$. Indeed, $O(3, \mathbb{C})$ is a symmetric square of $SL(2, \mathbb{C})$ (see [36]). It is shown in [36] that a third order operator has a symmetric square of order 5 (instead of order 6) if, and only if, it is the symmetric square of a second order operator.

[15] In [34] Joyce adopted the Heun function notation used by Snow [35], which corresponds to a *change of sign in the accessory parameter* q in the Heun function $HeunG(a, q, \alpha, \beta, \gamma, \delta, x)$. Therefore $HeunG(9, 3/4, 1/4, 3/4, 1, 1/2, *)$ is denoted $F(9, -3/4, 1/4, 3/4, 1, 1/2, *)$ in [34]. Unfortunately this old notation, different from the one used, for instance, in Maple, may contribute to some confusion in the literature.

[16] Conjugated to its adjoint by the function $(1 - 12\,x)^{1/2} \cdot x$.

3.3. Order-3 operators conjugated to their adjoint

In fact, the above results, for the sc, fcc lattices, can be seen as the consequence of the following general result on *order*-3 linear differential operators (without any loss of generality we restrict to monic operators)

$$L_3 = D_x^3 + a_2(x) \cdot D_x^2 + a_1(x) \cdot D_x + a_0(x). \tag{20}$$

Any (monic) order-3 operator which is conjugated to its adjoint, namely $L_3 \cdot f(x) = f(x) \cdot adjoint(L_3)$, is the symmetric square of an order-2 operator

$$L_2 = D_x^2 + b_1(x) \cdot D_x + b_0(x), \qquad \text{with:} \quad b_1(x) = -\frac{1}{2}\frac{1}{f(x)}\frac{\mathrm{d}f(x)}{\mathrm{d}x},$$

$$\text{where:} \qquad b_0(x) = \frac{a_1(x)}{4} + \frac{1}{8}\frac{1}{f(x)} \cdot \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} - \frac{1}{4} \cdot \left(\frac{1}{f(x)} \cdot \frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2. \tag{21}$$

Note that one necessarily has $a_2(x) = 3\, b_1(x)$. The Wronskian of L_3 is necessarily equal to $f(x)^{3/2}$, and the order-2 operator (21) is conjugated to its adjoint by a function:

$$f(x)^{1/2} \cdot adjoint(L_2) = L_2 \cdot f(x)^{1/2}. \tag{22}$$

The symmetric square of such an order-3 operator L_3, conjugated to its adjoint, is of order 5 (in contrast to order 6 for symmetric squares of generic order-3 operators).

3.4. Special lattice Green ODEs: 4D face-centered cubic lattice

A slightly more involved example, corresponding to the *four-dimensional face-centered cubic lattice Green function*, can be found in paragraph 2.5 of Guttmann's paper [3] (it is also ODE no 366 in the list of Almkvist *et al* [5]). This order-4 linear differential operator

$$\begin{aligned} G_4^{\rm 4Dfcc} = \; & \theta^4 + x \cdot (39 \cdot \theta^4 - 30 \cdot \theta^3 - 19 \cdot \theta^2 - 4\,\theta) \\ & + 2\,x^2 \cdot (16 \cdot \theta^4 - 1070 \cdot \theta^3 - 1057 \cdot \theta^2 - 676\,\theta - 192) \\ & -36\,x^3 \cdot (171 \cdot \theta^3 + 566 \cdot \theta^2 + 600\,\theta + 316) \cdot (3\,\theta + 2) \\ & - 2^5\,3^3\,x^4 \cdot (384 \cdot \theta^4 + 1542 \cdot \theta^3 + 2635 \cdot \theta^2 + 2173\,\theta + 702) \\ & -2^6\,3^3\,x^5 \cdot (1393 \cdot \theta^3 + 5571 \cdot \theta^2 + 8378\,\theta + 4584) \cdot (\theta + 1) \\ & - 2^{10}\,3^5\,x^6 \cdot (31 \cdot \theta^2 + 105\,\theta + 98)(\theta + 1) \cdot (\theta + 2) \\ & -2^{12}\,3^7\,x^7 \cdot (\theta + 1)(\theta + 2)^2 \cdot (\theta + 3) \\ = \; & x^4 \cdot \; (1 + 3\,x)\,(1 + 4\,x)\,(1 + 8\,x)(1 + 12\,x)\,(1 + 18\,x)^2\,(1 - 24\,x) \cdot D_x^4 + \cdots \end{aligned} \tag{23}$$

can be seen to be conjugated to its adjoint by a function $f^{\rm 4Dfcc}$:

$$G_4^{\rm 4Dfcc} \cdot f^{\rm 4Dfcc} = f^{\rm 4Dfcc} \cdot adjoint\big(G_4^{\rm 4Dfcc}\big),$$
$$\text{with:} \quad f^{\rm 4Dfcc} = x \cdot (1 + 18\,x)^3.$$

The *exterior square* of operator (23) is an irreducible *order*-5 operator (not order-6 as could be expected): one easily checks that the 'order-5 Calabi–Yau condition' (see [6] and (72) below) is actually satisfied for operator (23). If one considers an operator $\tilde{G}_4^{\rm 4Dfcc}$, non-trivially homomorphic [30, 31] to $G_4^{\rm 4Dfcc}$, its *exterior square* is, now, an operator of (the generic) order 6, and it has a *rational solution*. For instance, if we consider the operator $\tilde{G}_4^{\rm 4Dfcc}$ equivalent to $G_4^{\rm 4Dfcc}$

$$S_1^{\rm 4Dfcc} \cdot G_4^{\rm 4Dfcc} = \tilde{G}_4^{\rm 4Dfcc} \cdot D_x. \tag{24}$$

where

$$S_1^{\rm 4Dfcc} = -\frac{r(x)}{(1+18\,x)^3\cdot x}\cdot\Big(D_x - \frac{{\rm d}\ln(\rho(x))}{{\rm d}x}\Big), \qquad \text{with}$$
$$r(x) = \frac{18\,x+1}{x^3\cdot(3\,x+1)\,(4\,x+1)\,(8\,x+1)\,(12\,x+1)\,(24\,x-1)}, \qquad \text{and}$$
$$\rho(x) = (1119\,744\,x^5 + 508\,032\,x^4 + 82\,512\,x^3 + 6318\,x^2 + 237\,x + 4)\cdot x, \tag{25}$$

we find that the *exterior square* of $\tilde{G}_4^{\rm 4Dfcc}$ has the *rational solution* $r(x)$.

This situation will be encountered many times. For an operator whose exterior (resp. symmetric) power has an order which is the generic order minus one, one can always switch to an equivalent operator where the corresponding exterior (resp. symmetric) power annihilates a rational solution.

The Wronskian of $G_4^{\rm 4Dfcc}$ is a rational function. As the exterior square of $\tilde{G}_4^{\rm 4Dfcc}$ has a *rational solution*, the differential Galois group is included in the *symplectic group* $Sp(4, \mathbb{C})$. Moreover, its symmetric square being irreducible, theorems A.5 and A.7 of Beukers *et al* [37] show that the differential Galois group is exactly $Sp(4, \mathbb{C})$.

3.5. Special lattice Green ODEs: 5D staircase polygons

Another example of Guttmann and Prellberg [3, 25], corresponding to the generating function of the five-dimensional staircase polygons, is the order-4 operator

$$\begin{aligned} G_4^{\rm 5D} &= \theta^4 \; - x\cdot(35\,\theta^4 + 70\,\theta^3 + 63\,\theta^2 + 28\,\theta + 5) \\ &\qquad + x^2\cdot(259\,\theta^2 + 518\,\theta + 285)(\theta+1)^2 - 225\,x^3\cdot(\theta+1)^2\cdot(\theta+2)^2 \\ &= x^4\cdot(1 + 35\,x + 259\,x^2 - 225\,x^3)\cdot D_x^4 + \cdots \end{aligned} \tag{26}$$

which can be seen to be conjugated to its adjoint:

$$G_4^{\rm 5D}\cdot x = x\cdot adjoint\big(G_4^{\rm 5D}\big).$$

The exterior square operator of the order-4 operator (26) is an irreducible *order*-5 operator (not order-6 as could be expected): the ‘order-5 Calabi–Yau condition’ (see (72) below) is satisfied for (26). Let us introduce, instead of $G_4^{\rm 5D}$, the equivalent operator $\tilde{G}_4^{\rm 5D}$ corresponding to the intertwining relation

$$S_1^{\rm 5D}\cdot G_4^{\rm 5D} = \tilde{G}_4^{\rm 5D}\cdot D_x, \tag{27}$$

where the order-1 intertwiner $S_1^{\rm 5D}$ reads

$$S_1^{\rm 5D} = -\frac{r(x)}{x}\cdot\Big(D_x - \frac{{\rm d}\ln((60\,x+1)\,(3\,x-1)\,x)}{{\rm d}x}\Big), \tag{28}$$

and where $r(x)$ is the rational function:

$$r(x) = \frac{1}{(225\,x^3 - 259\,x^2 - 35\,x - 1)\cdot x^3}. \tag{29}$$

We find, again, that the exterior square of the equivalent operator $\tilde{G}_4^{5D}$ has the *rational solution* $r(x)$. The Wronskian of $G_4^{\rm 5D}$ is a rational function. The differential Galois group is, again (see A.5 and A.7 in appendix A of [37]), the *symplectic group* $Sp(4, \mathbb{C})$.

3.6. Order-6 operator by Broadhurst: the lattice Green function of the five-dimensional fcc lattice

A more involved example of order-6, can be found in Koutschan's paper [27] and in an unpublished paper of Broadhurst (see equation (74) in [38]) and corresponds to a *five-dimensional fcc lattice*

$$G_6^{5\mathrm{Dfcc}} = 3^4 \cdot \theta^5 \cdot (\theta - 1) + \sum_{j=1}^{12} x^j \cdot Q_j(\theta) = h_6 \cdot D_x^6 + \cdots, \qquad (30)$$

where the polynomials Q_j are degree-6 polynomials with integer coefficients, and where the head polynomial h_6 reads:

$$h_6 = x^6 \cdot \lambda(x) \cdot p_6,$$

with: $\quad \lambda(x) = (1-4x)\,(1-8x)\,(1+16x)\,(1-16x)\,(1-48x)\,(3-16x),$

and:

$$p_6 = 916\,586\,496\,x^6 - 571\,981\,824\,x^5 + 67\,242\,496\,x^4 - 8372\,096\,x^3 + 315\,096\,x^2 - 6840\,x + 27. \qquad (31)$$

This order-6 linear differential operator has, at the origin $x = 0$, *two independent analytic solutions* (it is *not* MUM[17] [19]). One can build, from these two solutions, a one-parameter family of analytic solutions:

$$1 + 8 \cdot x \cdot c + \tfrac{8}{3} \cdot (41 \cdot c - 2) \cdot x^2 + \tfrac{32}{27} \cdot (1933 \cdot c - 286) \cdot x^3 + \cdots$$

which, for $c = 1$, (and only this value) becomes a series with *integer*[18] coefficients:

$$1 + 8 \cdot x + 104 \cdot x^2 + 1952 \cdot x^3 + 46\,696 \cdot x^4 + 1301\,248 \cdot x^5 + \cdots.$$

The question of the integrality of such *D*-finite series, emerging from physics, is addressed in previous papers [18, 19].

Remark 3.2. The other unique independent no-log series starting with x reads:

$$z_0(x) = x + \tfrac{41}{3} \cdot x^2 + \tfrac{7732}{27} \cdot x^3 + \tfrac{183\,136}{27} \cdot x^4 + \tfrac{386\,626\,144}{2025} \cdot x^5 + \cdots.$$

It is *not a globally bounded* series [18, 19], i.e. it is *not* a series that can be recast into a series with integer coefficients after a rescaling of the variable.

This order-6 linear differential operator is *globally nilpotent* [8, 39].

We found that the order-6 operator $G_6^{5\mathrm{Dfcc}}$ is *non-trivially homomorphic to its adjoint*, with a simple *order*-1 intertwiner

$$G_6^{5\mathrm{Dfcc}} \cdot T_1^{5\mathrm{Dfcc}} = adjoint\big(T_1^{5\mathrm{Dfcc}}\big) \cdot adjoint\big(G_6^{5\mathrm{Dfcc}}\big), \qquad (32)$$

with:

$$T_1^{5\mathrm{Dfcc}} = x^2 \cdot p_2 \cdot p_6 \cdot \left(D_x - \frac{1}{2} \cdot \frac{\mathrm{d}\ln(R(x))}{\mathrm{d}x}\right), \qquad \text{where}$$

$$R(x) = \frac{p_2^5}{x^4 \cdot p_6^4} \qquad \text{with} \quad p_2 = 1152\,x^2 - 56\,x - 3. \qquad (33)$$

[17] MUM means maximally unipotent monodromy [3, 20, 40].

[18] The integrality of these coefficients has been checked with 2000 coefficients, and the coefficients $c_{c\cdot 10\,000} \cdot x^{c\cdot 10\,000}$ coefficients, for $c = 1, 2, 3, 4$, have also been seen to be integers.

Introducing

$$\rho(x) = \frac{{p_2}^6}{{p_6}^3 \cdot x^2}, \tag{34}$$

the previous order-1 intertwiner $T_1^{5\text{Dfcc}}$, can be seen as the product of the rational function $\rho(x)$, and of a *self-adjoint* order-1 operator Y_1^s:

$$T_1^{5\text{Dfcc}} = \rho(x) \cdot Y_1^s, \qquad Y_1^s = \frac{1}{R(x)} \cdot \left(D_x - \frac{1}{2} \cdot \frac{\mathrm{d}\ln(R(x))}{\mathrm{d}x}\right). \tag{35}$$

The other intertwining relation is a bit more involved since the intertwiner is an *order*-5 linear differential operator $S_5^{5\text{Dfcc}}$

$$adjoint\big(S_5^{5\text{Dfcc}}\big) \cdot G_6^{5\text{Dfcc}} = adjoint\big(G_6^{5\text{Dfcc}}\big) \cdot S_5^{5\text{Dfcc}}, \tag{36}$$

where

$$S_5^{5\text{Dfcc}} = \frac{x^2 \cdot \lambda(x) \cdot p_2^5}{p_6^3} \cdot \left(D_x^5 - \frac{1}{2} \cdot \frac{\mathrm{d}\ln(\mu(x))}{\mathrm{d}x} \cdot D_x^4 + \cdots\right)$$

with $\lambda(x)$ as above in (31), and:

$$\mu(x) = -\frac{p_2^5}{\lambda(x)^5 \cdot x^{20}}.$$

Quite remarkably, introducing the *same* function $\rho(x)$ as for $T_1^{5\text{Dfcc}}$ (see (34)), the previous order-5 intertwiner $S_5^{5\text{Dfcc}}$, can be seen as the product $S_5^{5\text{Dfcc}} = \rho(x) \cdot Y_5^s$, of the rational function $\rho(x)$ (see (34)) and of a *self-adjoint* order-5 operator

$$Y_5^s = \frac{x^4 \cdot \lambda(x)}{p_2} \cdot \left(D_x^5 - \frac{1}{2} \cdot \frac{\mathrm{d}\ln(\mu(x))}{\mathrm{d}x} \cdot D_x^4 + \cdots\right). \tag{37}$$

The self-adjoint order-5 irreducible operator Y_5^s has a solution which is analytic at $x = 0$ and has the following expansion

$$1 + 8x + 102x^2 + \tfrac{487\,192}{243}x^3 + \tfrac{86\,597\,215}{1944}x^4 + \tfrac{22\,841\,991\,292}{16\,875}x^5 + \cdots.$$

This solution-series is *not globally bounded* [18, 19]. The study of the formal series solutions at $x = 0$ shows a MUM structure.

The self-adjoint order-5 irreducible operator Y_5^s is such that its *symmetric square is of order 14 instead of the order 15 expected generically* (its exterior square is of order 10, as it should, with no rational solution).

The Wronskian of this order-6 linear differential operator $G_6^{5\text{Dfcc}}$ is the square root of a rational function:

$$W\big(G_6^{5\text{Dfcc}}\big) = \left(\frac{p_6^2}{x^{28} \cdot \lambda(x)^7}\right)^{1/2}.$$

The previous homomorphisms of the order-6 operator $G_6^{5\text{Dfcc}}$ with its adjoint, namely (32) and (36), can be simply rewritten in terms of the self-adjoint operators Y_1^s and Y_5^s:

$$G_6^{5\text{Dfcc}} \cdot \rho(x) \cdot Y_1^s = Y_1^s \cdot \rho(x) \cdot adjoint\big(G_6^{5\text{Dfcc}}\big), \tag{38}$$

$$Y_5^s \cdot \rho(x) \cdot G_6^{5\text{Dfcc}} = adjoint\big(G_6^{5\text{Dfcc}}\big) \cdot \rho(x) \cdot Y_5^s. \tag{39}$$

From these two intertwining relations it is straightforward[19] to see that an operator of the form

$$\Omega_6 = Y_1^s \cdot \rho(x) \cdot Y_5^s + \frac{\alpha}{\rho(x)}, \tag{40}$$

[19] Using the identity $adjoint(\Omega + f(x)) = adjoint(\Omega) + f(x)$ valid for *any even order operator* Ω, and for *any* function $f(x)$.

satisfies the *same* intertwining relations (38) and (39), as $G_6^{5\mathrm{Dfcc}}$. A simple calculation shows that the order-6 operator $G_6^{5\mathrm{Dfcc}}$ is *actually of the form* (40) with $\alpha = -192$:

$$G_6^{5\mathrm{Dfcc}} = Y_1^s \cdot \rho(x) \cdot Y_5^s - \frac{192}{\rho(x)}. \qquad (41)$$

Recalling section 2, and, more precisely, the fact that the right division of $\tilde{T} \cdot T$ by L is a constant (see (8)), one can rewrite (41) as:

$$\rho(x) \cdot Y_1^s \cdot \rho(x) \cdot Y_5^s = 192 + \rho(x) \cdot G_6^{5\mathrm{Dfcc}}. \qquad (42)$$

In other words, the two intertwiners $\rho(x) \cdot Y_1^s$ and $\rho(x) \cdot Y_5^s$ are *inverse of each other modulo the operator* $\rho(x) \cdot G_6^{5\mathrm{Dfcc}}$. The operator Y_5^s is *not globally nilpotent* [8].

The *exterior square* $Ext^2(G_6^{5\mathrm{Dfcc}})$ is an order-15 linear differential operator which *does not have a rational solution* (or a hyperexponential solution, see chapter 4 of [30] and [41]), thus *excluding a symplectic structure* with an $Sp(6, \mathbb{C})$ differential Galois group.

In contrast, its *symmetric square* $\mathrm{Sym}^2(G_6^{5\mathrm{Dfcc}})$, which does not have a rational solution, is of *order* 20 instead of the generic order 21. In fact, the *associated differential system does have a rational solution* (see next section (3.6.1) below). The emergence (for the system) of a rational solution for the symmetric square means that the differential Galois group is included[20] in the orthogonal group $O(6, \mathbb{C})$.

From that viewpoint, the order-6 operator $G_6^{5\mathrm{Dfcc}}$ seems to contradict an 'experimental' principle[21] that orthogonal groups occur from odd order operators, and symplectic groups occur from even order operators. In fact, the exceptional character of this even order operator comes from this decomposition (41) in terms of *odd order* intertwiners (see (38) and (39)).

The log structure of the solutions is exactly the same as the one of a symmetric square of an order-3 operator, $\mathrm{Sym}^2(L_3)$, which might suggest that the differential Galois group could be the differential Galois group of a MUM order-3 operator (generically $SL(3, \mathbb{C})$).

3.6.1. System representation of $G_6^{5\mathrm{Dfcc}}$. The calculations are performed using the differential system associated with the operator $G_6^{5\mathrm{Dfcc}}$. One gets[22] the following *rational solution* for the *symmetric square* of the companion system associated with the operator $G_6^{5\mathrm{Dfcc}}$:

$$\begin{aligned}
&[c_1, \; c_2, \; c_3, \; \dots, \; c_{21}] \\
&\quad = \Big[0, \; 0, \; 0, \; 0, \; \frac{2 \cdot Q_5}{\delta}, -\frac{20\, x^3 \cdot Q_6}{\delta^2}, 0, 0, -\frac{2 \cdot Q_5}{\delta}, \; \frac{12\, x^3\, Q_6}{\delta^2}, \\
&\quad -\frac{2\, x^6\, Q_{11}}{\delta^3}, \; \frac{Q_5}{\delta}, -\frac{4\, x^7 \cdot Q_6}{\delta^2}, \; \frac{2\, x^6\, Q_{14}}{\delta^3}, \; \frac{6\, x^9\, Q_{15}}{\delta^4}, \; \frac{x^6\, Q_{16}}{\delta^3}, \; \frac{-2\, x^9\, Q_{17}}{\delta^4}, \\
&\quad \frac{8\, x^{12}\, Q_{18}}{\delta^5}, \; \frac{x^{12}\, Q_{19}}{\delta^5}, \; \frac{-8\, x^{15}\, Q_{20}}{\delta^6}, \; \frac{4\, x^{18}\, Q_{21}}{\delta^6}\Big],
\end{aligned} \qquad (43)$$

where

$$c_1 = c_2 = c_3 = c_4 = c_7 = c_8 = 0, \qquad c_5 = c_9 = c_{12}, \qquad (44)$$

and where, recalling p_2 in (33) and λ in (31)

$$\delta = -x^4 \cdot \lambda(x), \qquad Q_5 = p_2,$$

[20] In fact, an argument of Katz [7] enables, in principle, to see whether the differential Galois group is included in $O(6, \mathbb{C})$ or actually equal to $O(6, \mathbb{C})$. This argument is difficult to work out here.

[21] See Katz's book [7] and most of the explicit examples known in the literature.

[22] In order to do these calculations on the linear differential systems, download the Maple Tools files TensorConstructions.m and IntegrableConnections.m in the web page [42]. Using DEtools, you will need to use, on the order-6 operator $G_6^{5\mathrm{Dfcc}}$, the command companion-system,then the command symmetric-power-system(,2)) and finally the command RationalSolutions([],[x]).

$$Q_6 = 14\,495\,514\,624\,x^8 - 8191\,475\,712\,x^7 + 1552\,941\,056\,x^6 - 94\,273\,536\,x^5$$
$$-3440\,640\,x^4 + 498\,624\,x^3 - 3632\,x^2 - 609\,x + 9,$$

the other Q_n's being much larger polynomials.

If one wants to stick with an operator description, similarly to (24) or (27), one can switch, by operator equivalence, to an operator such that its symmetric square is of the generic order 21 and has a *rational* solution.

The denominator of the monic order-20 operator $Sym^2(G_6^{5\mathrm{Dfcc}})$ is of the form $x^{16} \cdot \lambda(x)^5 \cdot p_{278}$, where p_{278} is a polynomial of degree 278 in x.

Let us introduce, for $n \geqslant 2$, an equivalent operator $G_6^{(n)}$, corresponding to an intertwining by D_x^n

$$S_2^{(n)} \cdot G_6^{5\mathrm{Dfcc}} = G_6^{(n)} \cdot D_x^n. \tag{45}$$

For $n = 2$, the *symmetric square* of $G_6^{(n)}$ has the *rational solution*

$$\frac{p_2}{x^4 \cdot \lambda(x)} = \frac{1152\,x^2 - 56\,x - 3}{x^4 \cdot (16\,x + 1)\,(8\,x - 1)\,(4\,x - 1)\,(16\,x - 1)\,(48\,x - 1)\,(16\,x - 3)}, \tag{46}$$

which is nothing but $c_5/2$ in (43). For the symmetric square of the other $\tilde{G}_6^{(n)}$'s one finds, respectively for $n = 3$, $n = 4$ and $n = 5$, the rational solutions c_{16}, c_{19} and c_{21} in (43). More generally the rational solution reads:

$$\frac{P_{12\,n-22}(x) \cdot x^{8\,n-12}}{x^{2\,n} \cdot \delta^{2\,n-3}}, \tag{47}$$

where $P_m(x)$ is a polynomial of degree m in x.

Getting (or even only checking) the rational solution (46) for the symmetric square of the equivalent operator (45), paradoxically, corresponds to massive calculations compared to obtaining the rational solution on the symmetric square of the companion system (see (43)). As far as practical calculations are concerned, computing with the linear differential systems turns out to be *drastically more efficient*, and allows to handle symmetric and exterior powers constructions on larger examples.

3.7. Koutschan's order-8 operator: the lattice Green function of the six-dimensional fcc lattice

A slightly more spectacular[23] example of order-8, $G_8^{6\mathrm{Dfcc}}$, has been found by Koutschan [27] for a *six-dimensional face-centered cubic lattice*. The irreducibility of this order-8 operator is hard to check[24]. One finds again, at the origin $x = 0$, that there are two independent analytical solutions (no logarithms). Since the order-8 operator $G_8^{6\mathrm{Dfcc}}$ has *two analytical solutions*, it *cannot be* MUM [19] at $x = 0$.

A linear combination of these solutions is globally bounded [18, 19]. It is such that, after rescaling, it can be recast into a series with *integer* coefficients [18, 19]:

$$1 + 60\,x^2 + 960\,x^3 + 30\,780\,x^4 + 996\,480\,x^5 + 36\,560\,400x^6 + \cdots. \tag{48}$$

We normalize $G_8^{6\mathrm{Dfcc}}$ to a monic form: $G_8^{6\mathrm{Dfcc}} = D_x^8 + \cdots$ This order-8 operator is (non-trivially) *homomorphic to its adjoint*; one intertwiner is of order 6, the other one is of order 2:

$$adjoint\big(S_6^{6\mathrm{Dfcc}}\big) \cdot G_8^{6\mathrm{Dfcc}} = adjoint\big(G_8^{6\mathrm{Dfcc}}\big) \cdot S_6^{6\mathrm{Dfcc}}, \tag{49}$$

[23] It is a rather large [43] order-8 linear differential operator of 52 megabytes.
[24] One can, however, check that this operator has no rational solutions.

$$G_8^{6\mathrm{Dfcc}} \cdot T_2^{6\mathrm{Dfcc}} = adjoint\big(T_2^{6\mathrm{Dfcc}}\big) \cdot adjoint\big(G_8^{6\mathrm{Dfcc}}\big). \qquad (50)$$

Once again, after the *same* rescaling, the two intertwiners turn out to be *self-adjoint operators*. Let us introduce

$$a(x) = \frac{x^6 \cdot p_5^4}{p_{25}} \cdot \lambda(x), \qquad (51)$$

where the polynomial p_5 reads

$$p_5 = 56\,x^5 + 625\,x^4 - 1251\,x^3 - 24\,840\,x^2 - 65\,556\,x - 38\,880, \qquad (52)$$

where $\lambda(x)$ reads:

$$\lambda(x) = (x-1)\,(x-3)\,(x+24)\,(2\,x+15)\,(7\,x+60)\,(2\,x+3)\,(4\,x+15) \\ \times (x+9)\,(x+5)\,(x+4)\,(x+15)^4, \qquad (53)$$

and where the polynomial p_{25} is a rather large polynomial of degree 25.

The intertwiners $T_2^{6\mathrm{Dfcc}}$ and $S_6^{6\mathrm{Dfcc}}$ are, respectively, of the form $T_2^{6\mathrm{Dfcc}} = a(x) \cdot Y_2^s$ and $S_6^{6\mathrm{Dfcc}} = a(x) \cdot Y_6^s$, where Y_2^s and Y_6^s are two irreducible *self-adjoint* order-2 and order-6 operators

$$Y_2^s = \frac{1}{W_2(x)} \cdot \Big(D_x^2 - \frac{\mathrm{d}\ln(W_2(x))}{\mathrm{d}x} \cdot D_x + \cdots\Big), \qquad (54)$$

and:

$$Y_6^s = \frac{1}{W_6(x)^{1/3}} \cdot \Big(D_x^6 - \frac{\mathrm{d}\ln(W_6(x))}{\mathrm{d}x} \cdot D_x^5 + \cdots\Big). \qquad (55)$$

Their corresponding Wronskians $W_2(x)$ and $W_6(x)$ read respectively:

$$W_2(x) = \frac{x^{11} \cdot \lambda(x)^2 \cdot p_5^3}{(x+15)^3 \cdot p_{25}}, \qquad W_6(x)^{1/3} = \frac{(x+15)^3 \cdot p_5}{\lambda(x) \cdot x^5}. \qquad (56)$$

These self-adjoint operators are *not globally nilpotent* [8].

The intertwining relations (49) give, in terms of the self-adjoint operators (54) and (55):

$$Y_6^s \cdot a(x) \cdot G_8^{6\mathrm{Dfcc}} = adjoint\big(G_8^{6\mathrm{Dfcc}}\big) \cdot a(x) \cdot Y_6^s, \\ G_8^{6\mathrm{Dfcc}} \cdot a(x) \cdot Y_2^s = Y_2^s \cdot a(x) \cdot adjoint\big(G_8^{6\mathrm{Dfcc}}\big), \qquad (57)$$

which yield

$$\mathcal{K}_8 \cdot \mathcal{M}_8 = \mathcal{M}_8 \cdot \mathcal{K}_8 \qquad \text{and} \qquad (58)$$

$$adjoint(\mathcal{M}_8) \cdot adjoint(\mathcal{K}_8) = adjoint(\mathcal{K}_8) \cdot adjoint(\mathcal{M}_8), \qquad \text{where:}$$
$$\mathcal{K}_8 = a(x) \cdot G_8^{6\mathrm{Dfcc}} \qquad \text{and:} \qquad \mathcal{M}_8 = a(x) \cdot Y_2^s \cdot a(x) \cdot Y_6^s. \qquad (59)$$

A commutation relation of linear differential operators, like (58), is a drastic constraint on the operators. As $\mathcal{K}_8$ is *irreducible*, the commutation (58) forces $\mathcal{K}_8$ to be of the form $\alpha \cdot \mathcal{M}_8 + \beta$, where α and β are constants. We may thus guess, from the intertwining relations (57), a decomposition of the order-8 operator $G_8^{6\mathrm{Dfcc}}$, similar to the one we had for $G_6^{5\mathrm{Dfcc}}$, of the form

$$G_8^{6\mathrm{Dfcc}} = Y_2^s \cdot a(x) \cdot Y_6^s + \frac{\alpha}{a(x)}, \qquad (60)$$

where Y_2^s and Y_6^s are two self-adjoint operators of *even* order (instead of odd order for $G_6^{5\mathrm{Dfcc}}$). This is, indeed, the case. The operator $G_8^{6\mathrm{Dfcc}}$ has the noticeable decomposition:

$$G_8^{6\mathrm{Dfcc}} = Y_2^s \cdot a(x) \cdot Y_6^s + \frac{87\,480}{a(x)}. \qquad (61)$$

Again, and similarly to what has been done for $G_6^{5\mathrm{Dfcc}}$ (see (42)), one can rewrite (61) as:

$$a(x) \cdot Y_2^s \cdot a(x) \cdot Y_6^s = -87\,480 + a(x) \cdot G_8^{6\mathrm{Dfcc}}, \qquad (62)$$

which means that the two intertwiners $a(x) \cdot Y_2^s$ and $a(x) \cdot Y_6^s$ are inverse of each other modulo the operator $a(x) \cdot G_8^{6\mathrm{Dfcc}}$. From (62) one sees that a solution of $G_8^{6\mathrm{Dfcc}}$ is an eigenfunction of $a(x) \cdot Y_2^s \cdot a(x) \cdot Y_6^s$ with the eigenvalue $-87\,480$.

The examination of the formal series solutions, at $x = 0$, of the self-adjoint order-6 Y_6^s operator shows a MUM structure. The Y_6^s operator has one analytic solution at $x = 0$, which has the following expansion

$$Sol\big(Y_6^s\big) = 1 + \tfrac{197}{11\,520}\,x^2 + \tfrac{8559\,443}{1889\,568\,000}\,x^3 + \tfrac{381\,585\,241\,573}{154\,793\,410\,560\,000}\,x^4 + \cdots. \qquad (63)$$

This solution-series (63), again, is *not*[25] *globally bounded* [18, 19]. One deduces immediately, from decomposition (61), the eigenvalue result: the order-8 operator $a(x) \cdot G_8^{6\mathrm{Dfcc}}$ has the *nonglobally bounded* eigenfunction (63), corresponding to the *integer eigenvalue* 87 480.

The self-adjoint order-5 irreducible operator Y_6^s is such that its *exterior square is of order 14 instead of the order 15 expected generically* (its symmetric square is of order 21 as it should, with no rational solution).

The symmetric square of $G_8^{6\mathrm{Dfcc}}$ is of the (generic) order 36. However the *exterior square* of $G_8^{6\mathrm{Dfcc}}$ is of order 27 *instead of the (generic) order* 28.

Remark 3.3. The adjoint of $G_8^{6\mathrm{Dfcc}}$ has the following decomposition, straightforwardly deduced from (61):

$$adjoint\big(G_8^{6\mathrm{Dfcc}}\big) = Y_6^s \cdot a(x) \cdot Y_2^s + \frac{87\,480}{a(x)}. \qquad (64)$$

So we can expect the Wronskian of Y_2^s to be a (rational) solution of its exterior square. We have verified that *this is indeed the case*.

Similarly to the previous order-6 operator $G_6^{5\mathrm{Dfcc}}$, one could try to switch to equivalent operators (see (45)), calculate the exterior square of these equivalent operators, and try to find the corresponding rational solution (see (46)). These, at first sight, straightforward calculations are, in fact, too 'massive'. The way to get the rational solution is, in fact, to switch to *differential systems* (see (43)).

3.7.1. System representation of $G_8^{6\mathrm{Dfcc}}$. In fact, even after switching to a differential system using the same tools [42] that we used for obtaining (43), we found that the resulting calculation of rational solutions exceeded our computational capacity. These calculations mostly amount to finding a transformation that reduces the system to a system with simple poles. We need a second 'trick' to be able to achieve these calculations and get the rational solution of the differential system. An easy way is to rewrite the system[26] in terms of the *homogeneous derivative* $\theta = x \cdot D_x$. Switching to this companion system[27] in θ, one automatically has simple poles for the system.

[25] This is also the case for the self-adjoint order-2 Y_2^s operator. Its solution analytic at $x = 0$ is *not* globally bounded [18, 19].

[26] In the Maple Tensor Construction tools found at [42], the command `Theta_companion_system`(L) returns two matrices $\frac{1}{p(x)}A_\theta$ and P_θ such that, for $Y = (y, y', \ldots, y^{(n-1)})^T$, we have $Y = P_\theta Y_\theta$ and $Y'_\theta = \frac{1}{p(x)}\,A_\theta\, Y_\theta$, where A_θ has no finite poles and $p(x)$ is squarefree, it has only simple roots. This gives the correspondence between the original companion system and the θ-companion system.

[27] If one is reluctant to switch to companion systems in θ, another way to achieve these calculations is to perform a reduction on the matrix of the corresponding system (Moser-reduce in Maple, or Isolde in [42]) *before* calculating the symmetric powers of the system (an operation that preserves the order of the poles).

With all these tricks and tools, we finally found that the linear *differential system* for the *exterior square* of the order-8 $G_8^{6\mathrm{Dfcc}}$ operator is of *order 28 and has a rational solution*. Note that the intertwiners (49) have been found from this rational solution: looking directly for the intertwiners (49) requires calculations which are too massive .

The rational solution reads

$$[c_1,\ c_2,\ c_3,\ \ldots,\ c_{28}]$$
$$= \Big[0,\ 0,\ 0,\ 0,\ \frac{P_5}{\delta},\ \frac{x^4 P_6}{\delta^2},\ \frac{x^8 P_7}{\delta^3},\ 0,\ 0,\ \frac{P_{10}}{\delta},\ \frac{x^4 P_{11}}{\delta^2},\ \frac{x^8 P_{12}}{\delta^3},\ \frac{x^{12} P_{13}}{\delta^4},$$
$$\times \frac{P_{14}}{\delta},\ \frac{x^4 P_{15}}{\delta^2},\ \frac{x^8 P_{16}}{\delta^3},\ \frac{x^{12} P_{17}}{\delta^4},\ \frac{x^{16} P_{18}}{\delta^5},\ \frac{x^3 P_{19}}{\delta^2},\ \frac{x^7 P_{20}}{\delta^3},\ \frac{x^{11} P_{21}}{\delta^4},$$
$$\times \frac{x^{15} P_{22}}{\delta^5},\ \frac{x^6 P_{23}}{\delta^3},\ \frac{x^{10} P_{24}}{\delta^4},\ \frac{x^{14} P_{25}}{\delta^5},\ \frac{x^9 P_{26}}{\delta^4},\ \frac{x^{13} P_{27}}{\delta^5},\ \frac{x^{12} P_{28}}{\delta^5}\Big], \tag{65}$$

where

$$\delta = (x-1)\,(x-3)\,(x+24)\,(2\,x+15)\,(7\,x+60)\,(2\,x+3)$$
$$\times (x+15)\,(4\,x+15)\,(x+9)\,(x+5)\,(x+4)\cdot x^5, \tag{66}$$

and where the polynomial P_n in (65) are too large to be displayed here. The polynomials $P_{28},\ P_{27},\ P_{25}, P_{22},\ P_{18}$ are of degree 49, polynomials $P_{26},\ P_{24},\ P_{21}, P_{17}$ are of degree 38, P_{13} is of degree 37, polynomials $P_{23},\ P_{20},\ P_{16},\ P_{12},\ P_7$ are of degree 27, polynomials $P_{19},\ P_{15},\ P_{11},\ P_6$ are of degree 16, and $P_{14},\ P_{10}, P_5$ are of degree 5. Furthermore we have some equalities like $P_{14} = -P_{10} = P_5,\ P_{11} = -2\cdot P_{15} = -2/3\cdot P_6$.

Having this rational solution at our disposal, we can, *now*, find the rational solutions for the exterior square of the equivalent operators:

$$G_2^{(n)}\cdot G_8^{6\mathrm{Dfcc}} = G_8^{(n)}\cdot D_x^n. \tag{67}$$

Recalling (52), the rational solution for $n=2$ reads p_5/δ. The differential Galois group of $G_8^{6\mathrm{Dfcc}}$ is included in (and probably equal to) $Sp(8,\ C)$.

Remark 3.4. The same calculations can be performed on all the linear differential operators we have encountered in lattice statistical mechanics [8, 11–17, 19, 20]: all the examples we have tested give operators whose irreducible factors are actually equivalent to their adjoint.

Remark 3.5. The remarkable decompositions (41) and (61), encountered with $G_6^{5\mathrm{Dfcc}}$ and $G_8^{6\mathrm{Dfcc}}$ can easily be generalized. In fact, one can systematically introduce the *even* order operators

$$M_{2p}^{(n,\,2p-n)} = L_{2p-n}\cdot a(x)\cdot L_n + \frac{\lambda}{a(x)}, \tag{68}$$

or, after rescaling

$$\tilde{M}_{2p}^{(n,\,2p-n)} = a(x)\cdot L_{2p-n}\cdot a(x)\cdot L_n + \lambda, \tag{69}$$

where the L_m's are *self-adjoint operators* of order m. They are, naturally, homomorphic to their adjoint, with intertwiners corresponding to these decompositions (68) and (69): $a(x)\cdot L_n\cdot M_{2p}^{(n,\,2p-n)} = adjoint(M_{2p}^{(n,\,2p-n)})\cdot a(x)\cdot L_n$, etc. Experimentally we have seen (for instance with our two previous lattice Green functions examples of order 6 and 8, see (38) and (39) for (41), and (49) and (50) for (61)), that this corresponds to *two different types* of operators: the operators with *even* n, for which the *exterior square* of an equivalent operator (or of the corresponding differential system) will have a rational solution (yielding a symplectic differential Galois group), and the operators with *odd* n for which the *symmetric square* of the corresponding differential system will have a rational solution (yielding an orthogonal differential Galois group).

4. Focus on order-4 differential operators: Calabi–Yau conditions

It has been underlined by Guttmann that these lattice Green functions are (most of the time) solutions of Calabi–Yau ODEs, or higher order Calabi–Yau ODEs [2, 3]. The definition of Calabi–Yau ODEs, and some large lists of Calabi–Yau ODEs, can be found in [4, 5, 40, 44]. Calabi–Yau ODEs are defined by several constraints, some are natural like being MUM, others (like some cyclotomic constraints) are essentially introduced, in a classification perspective like [5] to provide hopefully exhaustive lists of Calabi–Yau ODEs, some are related to the concept of '*modularity*', requiring the *integrality* of various series like the nome or the Yukawa coupling. Therefore, in the definition of Calabi–Yau ODEs, there is some 'mix' between analytic and differential constraints, and constraints of a more arithmetic[28], or algebraic geometry character. Among all these constraints defining the Calabi–Yau ODEs, the most important one is the so-called '*Calabi–Yau condition*'. Let us consider a (monic) order-4 linear differential operator:

$$\Omega_4 = D_x^4 + a_3(x) \cdot D_x^3 + a_2(x) \cdot D_x^2 + a_1(x) \cdot D_x + a_0(x). \tag{70}$$

The exterior square of (70), $Ext^2(\Omega_4)$, reads

$$C_6(x) \cdot Ext^2(\Omega_4) = \sum_{n=0}^{6} C_n(x) \cdot D_x^n, \tag{71}$$

where the $C_i(x)$'s are polynomial expressions of $a_3(x)$, $a_2(x)$, $a_1(x)$, $a_0(x)$ and of their derivatives (up to the third derivative).

The vanishing condition $C_6(x) = 0$, which reads

$$8\, a_1(x) + a_3(x)^3 - 4 \cdot a_3(x) \cdot a_2(x) + 6 \cdot a_3(x) \cdot \frac{\mathrm{d}a_3(x)}{\mathrm{d}x} - 8 \cdot \frac{\mathrm{d}a_2(x)}{\mathrm{d}x} + 4 \cdot \frac{d^2 a_3(x)}{\mathrm{d}x^2} = 0, \tag{72}$$

is satisfied if, and only if, the exterior square is of order 5, instead of the order 6 one expects generically. It is called '*Calabi–Yau condition*' by some authors [6] and is one of the conditions for the ODE to be a *Picard–Fuchs equation* of a family of Calabi–Yau manifolds (see (11) in [45]). This Calabi–Yau condition (72) is actually *preserved by pullbacks, but not by operator equivalence*. Note that this Calabi–Yau condition (72) is *preserved by the adjoint transformation*. This is a straight consequence of the following *conjugation relation between the exterior square of an order*-4 operator Ω_4 *and the exterior square of its adjoint* ($W(x)$ denotes the Wronskian of Ω_4):

$$W(x) \cdot Ext^2(adjoint(\Omega_4)) = Ext^2(\Omega_4) \cdot W(x). \tag{73}$$

Do note that such a Calabi–Yau condition is actually independent of $a_0(x)$ in (70). Also note that all the order-4 operators M_4 that can be written as the sum of the symmetric-cube of an order-2 operator, and a function, $M_4 = Sym^3(M_2) + f(x)$, automatically verify the Calabi–Yau condition (72). This gives a practical way to quickly provide examples of order-4 operators satisfying the Calabi–Yau condition (72).

Of course similar Calabi–Yau conditions can be introduced for higher order operators, imposing, for order-N operators, that their *exterior squares* are of order *less than* the generic $N \cdot (N-1)/2$ order. These higher order *Calabi–Yau conditions* actually correspond to *self-adjoint operators*[29].

[28] In order to disentangle these various constraints see [18, 19].

[29] The Calabi–Yau condition (72) is equivalent to say that an order-4 operator is conjugated to its adjoint [22].

Furthermore similar 'Calabi–Yau conditions' can be introduced for *symmetric squares instead of exterior squares*, imposing, for order-N operators, that their symmetric squares are of order *less than* the generic $N \cdot (N+1)/2$ order. For an order-3 operator written in a monic form

$$\Omega_3 = D_x^3 + a_2(x) \cdot D_x^2 + a_1(x) \cdot D_x + a_0(x), \tag{74}$$

the 'symmetric Calabi–Yau condition' reads[30]:

$$4\, a_2(x)^3 - 18\, a_1(x)\, a_2(x) + 9 \cdot \frac{\mathrm{d}^2 a_2(x)}{\mathrm{d}x^2} + 18 \cdot a_2(x) \cdot \frac{\mathrm{d}a_2(x)}{\mathrm{d}x} + 54\, a_0(x) - 27\, \frac{\mathrm{d}a_1(x)}{\mathrm{d}x} = 0. \tag{75}$$

Operators satisfying this 'symmetric Calabi–Yau condition' actually correspond to the situation described in (3.3). If their Wronskian $W(\Omega_3)$ is a Nth root of a rational function they are conjugated to their adjoint ($f(x) = W(\Omega_3)^{2/3}$):

$$\Omega_3 \cdot f(x) = f(x) \cdot adjoint(\Omega_3) \qquad \text{with:} \qquad a_2(x) = -\frac{3}{2}\, \frac{1}{f(x)}\, \frac{\mathrm{d}f(x)}{\mathrm{d}x}. \tag{76}$$

In order to disentangle the main focus of this paper, namely the *algebraic-differential structures*, from other structures of more analytical, or arithmetic, character (series integrality [18, 19], MUM property, etc), we concentrate, in this section, on (mostly order-4) linear differential operators satisfying the Calabi–Yau condition (72), or *homomorphic to operators satisfying* (72).

4.1. Weak and strong Calabi–Yau conditions

If one considers an operator that is homomorphic to an operator with a *rational Wronskian*, satisfying the Calabi–Yau condition (72), with intertwiners that are of order greater or equal to 1[31], the exterior square of that operator actually *has a rational solution*. Unfortunately, in contrast with (72), the condition for an order-4 operator to be such that its exterior square has a rational solution, *cannot be written directly and explicitly on its coefficients* $a_n(x)$ (see (70)). We will call 'weak Calabi–Yau condition' this condition that the exterior square of an operator has a rational solution, the Calabi–Yau condition (72) being the 'strong' Calabi–Yau condition. Note that the weak Calabi–Yau condition is *preserved by the adjoint transformation*. This is also a straight consequence of (73).

As far as classifications of Calabi–Yau operators are concerned [4, 5, 40, 44], an operator non-trivially[32] homomorphic to a 'Calabi–Yau operator' is certainly as interesting for physics as these 'Calabi–Yau operators', and an operator non-trivially homomorphic to an operator verifying the 'strong' Calabi–Yau condition (72), or satisfying the 'weak Calabi–Yau condition' is certainly as interesting as an operator verifying the 'strong' Calabi–Yau condition.

Let us explore the relation between the 'weak Calabi–Yau condition' and the 'strong Calabi–Yau condition'.

[30] The 'symmetric Calabi–Yau condition' for order-4 operators can be found but is drastically larger than (75).
[31] We must exclude intertwiners of order zero (namely functions): in that case, it is a straightforward calculation to see that the operators are conjugated by a function, both operators satisfying the Calabi–Yau condition (72).
[32] With intertwiners of order greater or equal to 1.

4.2. A decomposition of operators equivalent to operators satisfying the Calabi–Yau condition

Let us consider an order-4 operator Ω_4, of Wronskian $w(x) = u(x)^2$, which satisfies the Calabi–Yau condition (72). Let us consider a monic order-4 operator $\tilde{\Omega}_4$ which is (non-trivially) homomorphic (equivalent in the sense of the equivalence of operators [30]) to the order-4 operator Ω_4 satisfying the Calabi–Yau condition (72). This amounts to saying that there exist two intertwiners, U_3 and L_3, of order less or equal to 3[33], such that:

$$\tilde{\Omega}_4 \cdot U_3 = L_3 \cdot \Omega_4. \qquad (77)$$

It is shown in appendix A that the order-4 operator $\tilde{\Omega}_4$ *can, in fact, be written in terms of a remarkable decomposition*

$$\tilde{\Omega}_4 = Z_2^s \cdot \frac{1}{A_0} \cdot A_2 + A_0, \qquad (78)$$

where Z_2^s and A_2 are two *self-adjoint operators*, A_0 being a function. appendix A shows how to get Z_2^s, A_2 and A_0 in such a decomposition: they can simply be obtained as the intertwiners of $\tilde{\Omega}_4$ with its adjoints (use (A.10), (A.15), (A.13) in appendix A). Experimentally we have checked that an operator (nontrivially) homomorphic to an operator of the form (78) (see (A.2)) can always be decomposed in a form (78): *the decomposition* (78) *is preserved by operator equivalence*.

Byproduct. As a byproduct one finds out that the left and right intertwiners of an order-4 operator satisfying the weak Calabi–Yau condition are *necessarily of order* 2. Note, however, that this is *not* true for the intertwiners of an order-4 operator satisfying the *symmetric* weak Calabi–Yau condition which are of *odd orders* (see (88) in the section 4.4 on the anisotropic simple cubic lattice Green function).

4.3. Rational solutions for the exterior square of operators satisfying the weak Calabi–Yau condition

We have the following general result. Any order-4 linear differential operator of the form[34]

$$M_4 = L_2 \cdot c_0(x) \cdot M_2 + \frac{\lambda}{c_0(x)}, \qquad (79)$$

where L_2 and M_2 are two (general) self-adjoint operators

$$L_2 = \alpha_2(x) \cdot D_x^2 + \frac{\mathrm{d}\,\alpha_2(x)}{\mathrm{d}x} \cdot D_x + \alpha_0(x), \qquad (80)$$

$$M_2 = \beta_2(x) \cdot D_x^2 + \frac{\mathrm{d}\,\beta_2(x)}{\mathrm{d}x} \cdot D_x + \beta_0(x), \qquad (81)$$

is such that *its exterior square has* $1/\beta_2(x)$ *as a solution* (up to an overall multiplicative constant).

Byproduct. Thus the *exterior square* of $\tilde{\Omega}_4$ has a *rational solution*, which is the inverse of the head polynomial of the second order self-adjoint operator A_2 in the decomposition (78).

[33] Higher order intertwiners can always be reduced to intertwiners with an order less, or equal, to 3.
[34] Note that one can always restrict to $\lambda = 1$ rescaling $c_0(x)$ into $\lambda^{1/2} \cdot c_0(x)$.

To sum-up. The operators, non-trivially homomorphic to operators satisfying the (strong) Calabi–Yau condition (72), necessarily satisfy the 'weak Calabi–Yau condition': their exterior square have a *rational solution*. Furthermore this rational solution corresponds to the Wronskian of a self-adjoint order-2 operator L_2 of a remarkable decomposition (79). Decomposition (79) is the most general form of an order-4 operator satisfying the 'weak Calabi–Yau condition'.

Conversely. This naturally raises the reciprocal question. Is any order-4 operator satisfying the 'weak Calabi–Yau condition' (its exterior square has a rational solution) non-trivially homomorphic to an operator satisfying the (strong) Calabi–Yau condition (72)? In view of the remarkable decomposition (79), we can also ask the following questions. Is any order-4 operator satisfying the 'weak Calabi–Yau condition' necessarily of the form (79), i.e. homomorphic to its adjoint with *order*-2 intertwiners? Is any order-4 operator of the form (79) homomorphic to an operator satisfying the (strong) Calabi–Yau condition (72)? These questions will be revisited in a forthcoming publication[35]. The reason why these questions are difficult to answer in general, beyond specific examples, comes from the fact that such a reduction by operator equivalence of operators satisfying the weak Calabi–Yau condition to operators satisfying the strong Calabi–Yau condition, is *not unique* (see, for instance, appendix O.4 in [19], where an infinite number of equivalent operators satisfy the Calabi–Yau condition (72)).

4.4. The lattice Green function of the anisotropic simple cubic lattice

At this step it is important to recall the results of Delves and Joyce [48, 49] for the lattice Green function of the *anisotropic simple cubic lattice*, generalizing the results displayed in section (3.1). The lattice Green function of that anisotropic lattice is solution of an order-4 operator (see (14) in [49]), depending on an anisotropy parameter α. This order-4 operator reads in terms of the homogeneous derivative $\theta = x \cdot D_x$:

$$\begin{aligned} G_4^{\rm asc} = \, & 24 \cdot \theta^3 \cdot (\theta - 1) - 4 \cdot x \cdot \theta \cdot P_1(\theta) + 2 \cdot x^2 \cdot (2\,\theta + 1) \cdot P_2(\theta) \\ & - A \cdot x^3 \cdot (2\,\theta + 3)\,(2\,\theta + 1) \cdot P_3(\theta) \\ & + 5 \cdot (A + 4) \cdot A^3 \cdot x^4 \cdot (2\,\theta + 5)\,(2\,\theta + 3)\,(2\,\theta + 1)\,(\theta + 1), \end{aligned} \tag{82}$$

where $A = \alpha^2 - 4$ and

$$P_1(\theta) = 6 \cdot (2\,\theta + 1)\,(10\,\theta^2 + 10\,\theta + 3) + A \cdot (28\,\theta^3 + 7\,\theta^2 + 16\,\theta + 3),$$

$$\begin{aligned} P_2(\theta) = \, & 12 \cdot (4\,\theta + 5)\,(2\,\theta + 3)\,(4\,\theta + 3) + 2\,A \cdot (172\,\theta^3 + 252\,\theta^2 + 234\,\theta + 81) \\ & + 3\,A^2 \cdot (16\,\theta^3 + 21\,\theta^2 + 18\,\theta + 6), \end{aligned}$$

$$P_3(\theta) = 40 \cdot (4\,\theta + 3)\,(4\,\theta + 1) + 12\,A \cdot (22\,\theta^2 + 29\,\theta + 12) + A^2 \cdot (36\,\theta^2 + 57\,\theta + 31).$$

Operator (82) is *globally nilpotent* [8, 39].

This order-4 operator $G_4^{\rm asc}$ is *not* MUM. It has two solutions analytic at $x = 0$

$$\begin{aligned} & 1 + \tfrac{1}{2}\,(\alpha^2 + 2) \cdot x + \tfrac{3}{8}\,(\alpha^4 + 8\,\alpha^2 + 6) \cdot x^2 \\ & \quad + \tfrac{5}{16}\,(\alpha^6 + 18\,\alpha^4 + 54\,\alpha^2 + 20) \cdot x^3 + \cdots, \\ & \quad x + \tfrac{3}{8}\,(3\,\alpha^2 + 11) \cdot x^2 + \tfrac{5}{48}\,(11\,\alpha^4 + 119\,\alpha^2 + 146) \cdot x^3 + \cdots \end{aligned}$$

[35] If one switches to a representation in terms of *differential systems*, such a system with Galois group $Sp(4, \mathbb{C})$ can always be reduced, via a 'gauge-like' transformation [46, 47], to a system with a *Hamiltonian* matrix M. Such a system is such that the exterior square system associated with a 6×6 matrix, has a constant solution, namely [0, 1, 0, 0, 1, 0] (see [46, 47]). Switching back to the operator representation, one actually finds that this operator is homomorphic to its adjoint with order-2 intertwiners (themselves homomorphic to their adjoints). Consequently they can always be decomposed into a form (68).

together with a solution with a log and a solution with a $\log^2$. The first analytic solution is *globally bounded* [18, 19] for generic rational values of α, or even rational values of A: for $A = p/q$ the rescaling $x \rightarrow 4\,q \cdot x$ changes this series into a series with *integer coefficients*. The second analytic solution (83) is *not globally bounded* for generic rational values of α, but becomes globally bounded for $\alpha = \pm 1$: with a rescaling $x \rightarrow 4\,x$, the series becomes a series with *integer coefficients*.

The exterior square of $G_4^{\rm asc}$ (depending on the parameter α) is of order 6 with no rational (or hyperexponential [24]) solutions. The *symmetric square* of $G_4^{\rm asc}$ *is of order* 9, instead of the order 10 one could expect. In other words, $G_4^{\rm asc}$ *verifies the symmetric Calabi–Yau condition* for order-4 operators (see (75) above for order-3 symmetric condition).

If, as above, we introduce an order-4 operator $\tilde{G}_4^{\rm asc}$ equivalent to $G_4^{\rm asc}$

$$S_1^{\rm asc} \cdot G_4^{\rm asc} = \tilde{G}_4^{\rm asc} \cdot D_x, \tag{83}$$

the *symmetric square* of that equivalent order-4 operator has a *rational solution* $r(x)$:

$$r(x) = \frac{(\alpha^2-4)\cdot\, x+3}{x^2\cdot\,(1-\alpha^2\cdot\, x)\cdot\,(1-\,(\alpha-2)^2\cdot\, x)\cdot\,(1-(\alpha+2)^2\cdot\, x)}. \tag{84}$$

The order-4 operator (82) can be decomposed in terms of two self-adjoint operators of order 1 and 3, $Y_1^{(s)}$ and $Y_3^{(s)}$, namely

$$G_4^{\rm asc} = Y_1^{(s)}\cdot\,\rho(x)\cdot\, Y_3^{(s)} + \frac{8\cdot\,(\alpha^2-1)^2}{\rho(x)}, \qquad \rho(x) = \frac{((\alpha^2-4)\cdot\, x+3)^4}{(5\,(\alpha^2-4)\cdot\, x-3)^3}, \tag{85}$$

where:
$$\rho(x)\cdot\, Y_1^{(s)} = 2\cdot((\alpha^2-4)\cdot\, x+3)\,\,(5\,\,(\alpha^2-4)\cdot\, x-3)\cdot\, D_x \\ +(\alpha^2-4)\,(5\,(\alpha^2-4)\cdot\, x+69), \tag{86}$$

$Y_3^{(s)}$ being slightly more involved. One more time, and similarly to what has been done for $G_6^{\rm 5Dfcc}$ and $G_8^{\rm 6Dfcc}$ (see (42) and (62)), one can rewrite (85) as

$$\rho(x)\cdot\, Y_1^{(s)}\cdot\,\rho(x)\cdot\, Y_3^{(s)} = -8\cdot\,(\alpha^2-1)^2+\rho(x)\cdot\, G_4^{\rm asc}, \tag{87}$$

which means that the two intertwiners $\rho(x)\cdot\, Y_1^{(s)}$ and $\rho(x)\cdot\, Y_3^{(s)}$ are *inverse of each other modulo the operator* $\rho(x)\cdot\, G_4^{\rm asc}$.

The order-4 operator (82) is homomorphic to its adjoint with the intertwining relations:

$$Y_3^{(s)}\cdot\,\rho(x)\cdot\, G_4^{\rm asc} = adjoint\big(G_4^{\rm asc}\big)\cdot\,\rho(x)\cdot\, Y_3^{(s)}, \\ G_4^{\rm asc}\cdot\,\rho(x)\cdot\, Y_1^{(s)} = Y_1^{(s)}\cdot\,\rho(x)\cdot adjoint\big(G_4^{\rm asc}\big). \tag{88}$$

If one denotes W the Wronskian of $G_4^{\rm asc}$ one has the relation $r(x)^{20} = W^8\cdot\,\rho(x)^5$ $(5\,\,(\alpha^2-4)\cdot\, x-3)^7$.

Recalling the example of the order-6 lattice Green operator $G_6^{\rm 5Dfcc}$, one sees that the fact that it is the *symmetric square* (and not the exterior square) of that order-4 operator which has a rational solution, is related to the *odd order* of the two intertwiners. This anisotropic example shows that all the differential algebra structures we display in this paper *can be generalized, mutatis mutandis, to problems with more than one variable* (see also [50]).

5. Exceptional differential Galois groups

Recently a set of Calabi–Yau type operators whose differential Galois group is $G_2(C)$, the *exceptional*[36] subgroup [51] of $SO(7)$, were explicitly given [28, 29]. These examples read (see page 18 section 4.3 of [28]

$$E_1 = \theta^7 - 128 \cdot x \cdot (48\,\theta^4 + 96\,\theta^3 + 124\,\theta^2 + 76\,\theta + 21)\,(2\,\theta + 1)^3 \\ +4194\,304 \cdot x^2 \cdot (\theta + 1) \cdot (12\,\theta^2 + 24\,\theta + 23) \cdot (2\,\theta + 1)^2 \cdot (2\,\theta + 3)^2 \\ -34\,359\,738\,368 \cdot x^3 \cdot (2\,\theta + 5)^2 \cdot (2\,\theta + 1)^2 \cdot (2\,\theta + 3)^3, \tag{89}$$

and:

$$E_2 = \theta^7 - 128 \cdot x \cdot (8\,\theta^4 + 16\,\theta^3 + 20\,\theta^2 + 12\,\theta + 3)\,(2\,\theta + 1)^3 \\ +1048\,576\,x^2 \cdot (2\,\theta + 1)^2 \cdot (2\,\theta + 3)^2 \cdot (\theta + 1)^3,$$
$$E_3 = \theta^7 - 3^3 \cdot x \cdot (81\,\theta^4 + 162\,\theta^3 + 198\,\theta^2 + 117\,\theta + 28)\,(2\,\theta + 1)\,(3\,\theta + 1)\,(3\,\theta + 2) \\ +3^{12}\,x^2 \cdot (3\,\theta + 5) \cdot (3\,\theta + 1) \cdot (\theta + 1) \cdot (3\,\theta + 2)^2 \cdot (3\,\theta + 4)^2,$$
$$E_4 = \theta^7 - 2^7 \cdot x \cdot (128\,\theta^4 + 256\,\theta^3 + 304\,\theta^2 + 176\,\theta + 39)\,(4\,\theta + 1)\,(4\,\theta + 3)\,(2\,\theta + 1) \\ +2^{26}\,x^2 \cdot (4\,\theta + 7)\,(4\,\theta + 5)\,(4\,\theta + 3)\,(4\,\theta + 1)\,(2\,\theta + 1)\,(2\,\theta + 3)\,\,(\theta + 1),$$
$$E_5 = \theta^7 - 2^7 3^3 \cdot x \cdot (648\,\theta^4 + 1296\,\theta^3 + 1476\,\theta^2 + 828\,\theta + 155)\,(6\,\theta + 5) \\ \times(6\,\theta + 1)\,(2\,\theta + 1) + 2^{20}3^{12}\,x^2 \cdot (6\,\theta + 11)\,(6\,\theta + 7)\,(6\,\theta + 5)\,(6\,\theta + 1) \\ \times(3\,\theta + 5)\,(3\,\theta + 1)\,(\theta + 1). \tag{90}$$

Note that for the five E_n, their conjugate $x^{-1/2} \cdot E_n \cdot x^{1/2}$ are *self-adjoint operators*, and of course $x^{-1} \cdot E_n$ are *also self-adjoint operators*.

The solution-series, analytic at $x = 0$, of these order-7 operators E_n are actually series with *integer coefficients*. These order-7 operators are MUM and are *globally nilpotent*. The corresponding nomes (called 'special coordinates' in [28]) defined as $q^{(n)} = \exp(y_1^{(n)}/y_0^{(n)}) = x \cdot \exp(\tilde{y}_1^{(n)}/y_0^{(n)})$, as well as the various *Yukawa couplings* [18, 19] of these order-7 operators, correspond to series with *integer coefficients*.

Note that, after performing the following rescalings $x \rightarrow x/4096,\ x/19\,683,\ x/262\,144,\ x/80\,621\,568$ on the E_n's for $n = 2, 3, 4, 5$, these four rescaled E_n's have now the same Wronskian: $(1-x)^{-7} \cdot x^{-21}$. The homogeneous derivative being invariant by these rescalings the previous rescalings just amount to modifying the coefficients in front of the x^n's in the previous definitions, for instance:

$$E_2 \quad \longrightarrow \qquad \hat{E}_2 = 2^5 \cdot \theta^7 - x \cdot (8\,\theta^4 + 16\,\theta^3 + 20\,\theta^2 + 12\,\theta + 3)\,(2\,\theta + 1)^3 \\ +2\,x^2 \cdot (2\,\theta + 1)^2 \cdot (2\,\theta + 3)^2 \cdot (\theta + 1)^3.$$

After these rescalings these rescaled operators $\hat{E}_i$ for $i = 2, \ldots, 5$, have now their singularities in $0,\ 1$ and ∞.

The exterior squares of these order-7 operators $\hat{E}_n$ are of order 14 instead of the order 21 one could expect generically. The *exterior cube* of these order-7 operators are of order 27 (instead of order 35). The symmetric squares of these order-7 operators are of order 27, instead of the order 28 one could expect generically (see (91)).

Note that any operator whose symmetric square is of order less than the generic expected order (here 28) is such that its solutions verify a quadratic relation. For instance, the seven formal solutions of the order-7 operator $\hat{E}_1$ verify the simple quadratic identity:

$$2\,S_1\,S_7 - 2\,S_6\,S_2 + 2\,S_5\,S_3 - S_4^2 = 0 \tag{91}$$

where S_7 is the solution analytic at $x = O$, where $S_6 = S_7 \cdot \ln(x)^2 + T_6 \cdot \ln(x) + R5, \ldots.$

[36] The compact form of G_2, *subgroup of* $SO(7)$, can be described as the automorphism group of the octonion algebra.

Let us consider the operators $E_n^{(m)}$ non-trivially homomorphic to the $\hat{E}_n$'s:

$$E_n^{(m)} \cdot D_x^m = L_m \cdot \hat{E}_n, \tag{92}$$

where L_m is an order-m operator. For $m = 1$ the *exterior squares* of the $E_n^{(m)}$'s are of order 21 (as expected generically), but the symmetric squares of the $E_n^{(m)}$'s are still of order 27. The exterior cube of the $E_n^{(m)}$'s are of order 34 instead of the order 35 expected generically.

For $m = 2$ the symmetric squares of the $E_n^{(m)}$'s are still of order 27, however for $m = 3$ the symmetric squares of the $E_n^{(m)}$'s is of the order 28 expected generically. The *exterior cube* of the $E_n^{(m)}$'s are of order 35, as expected generically.

The exterior squares of the $E_n^{(1)}$'s are a *direct sum* of an order-14 operator and an order-7 operator

$$Ext^2\big(E_n^{(1)}\big) = L_{14}^{(n)} \oplus L_7^{(n)}, \tag{93}$$

where the order-7 operators are simply conjugated to the $\hat{E}_n$'s:

$$L_7^{(1)} = \frac{1}{(1-x)^{3/2}\,x^3} \cdot \hat{E}_1 \cdot (1-x)^{3/2}\,x^3, \qquad L_7^{(n)} = \frac{1}{(1-x)\,x^3} \cdot \hat{E}_n \cdot (1-x)\,x^3.$$

The *symmetric squares* of the $E_n^{(3)}$'s are a *direct sum* of an order-27 operator and an order-1 operator

$$Sym^2\big(E_n^{(3)}\big) = L_{27}^{(n)} \oplus L_1^{(n)}, \tag{94}$$

where the order-1 operators $L_1^{(n)}$ have the following *rational* solutions r_n:

$$r_1 = \frac{1}{(x-1)^3 \cdot x^6}, \qquad r_n = \frac{1}{(x-1)^2 \cdot x^6} \qquad n = 2, \ldots, 5. \tag{95}$$

The *exterior cubes* of the $E_n^{(2)}$'s are actually a *direct sum* of an order-27 operator $M_{27}^{(n)}$, an order-7 operator $M_7^{(n)}$ and an order-1 $M_1^{(n)}$ operator

$$Ext^3\big(E_n^{(2)}\big) = M_{27}^{(n)} \oplus M_7^{(n)} \oplus M_1^{(n)}, \tag{96}$$

where the order-1 operators $M_1^{(n)}$ have an *algebraic* solution for $E_1^{(2)}$:

$$a_1 = \frac{1}{(x-1)^{9/2} \cdot x^9}, \tag{97}$$

and the following *rational* solutions for the exterior cube of the other $E_n^{(2)}$'s:

$$\rho_n = \frac{1}{(x-1)^3 \cdot x^9}, \qquad n = 2, \ldots, 5. \tag{98}$$

Remark 5.1. The emergence of the exceptional group G_2 corresponds to the appearance of rational (or square root of rational[37]) solutions for the *symmetric square and exterior cube* of these operators. This is reminiscent (see page 320 of chapter 9 of [7]) of the (non-Fuchsian) order-7 operator $D_x^7 - x \cdot D_x - 1/2$, which has the differential Galois group G_2, namely the *exceptional* subgroup [51] of $SO(7)$. If we had only rational (or square root of rational) solutions of the symmetric square of the operators we would have $SO(7)$ differential Galois groups: the appearance of the rational (or square root of rational) solutions for the *exterior cube* of these operators explains the emergence of this exceptional subgroup of $SO(7)$.

[37] In that case, the group is not connected but its Lie algebra is still $\mathfrak{g}_2$, i.e the connected component of the group which contains the identity is G_2.

Remark 5.2. Throughout the paper, we see the systematic emergence of decompositions as *direct sums* (see for instance (93), (94), (96)) instead of just a factorization, each time we find a rational solution for some symmetric or exterior power. This should not be seen as a surprise. Indeed, the linear differential operator $E_n^{(m)}$ is *irreducible*. This implies that its differential Galois group is *reductive*, i.e. all its *representations are semi-simple* (i.e. decompose as a *direct sum of irreducible representations*): see section 2.2, specially discussion before lemma 2.3 in [31]. In practice, this means that if we perform any construction like Sym^m, Ext^r, etc, and if the corresponding operator factors, then it decomposes as an LCLM of *irreducible* operators (because the differential Galois group acts on the solution space of this operator).

Remark 5.3. All these results on symmetric squares, exterior squares and exterior cubes of (equivalent) order-7 operators have *not* been obtained using Maple's DEtools commands 'ratsols' and 'expsols', the corresponding algorithms being *not powerful enough* to cope with such examples of too large order. Furthermore, if one switches to differential systems representations using the packages in [42], one finds[38] again that the corresponding algorithms[39] are *not powerful enough* to cope with such examples of too large order. One needs to go a step further, *switching to θ-systems*, a method that yields systematically simple poles.

To search for rational solutions of differential systems (for regular systems like these), one (roughly) needs to find a transformation that transforms them into systems with simple poles. Then one finds the exponents, and then one reduces to polynomial solutions. Switching to θ-systems, one automatically has simple poles: this bypasses the first reduction step (which can be costly on big systems).

Using the Tensor Construction package, the command to be used is Theta-companion-system or full-theta-companion-system. One needs to perform a 'reduction at ∞' in order to find polynomial solutions. Doing all these tricks, one finally finds these results almost immediately for the symmetric squares, and for the exterior cubes.

5.1. Three-parameter family of order-7 operators with exceptional Galois groups

The order-7 operators $\hat{E}_i$ for $i = 2 \ldots 5$ can also be seen as special cases of an order-7 operator $\Omega_{a,b,c}$ depending on three parameters (see appendix C, see also operator P_1 in section 5.1 of [29]).

Let us denote again $\tilde{\Omega}_{a,b,c}^{(m)}$ the order-7 linear differential operator homomorphic to $\Omega_{a,b,c}$ with a D_x^m intertwiner. The operator $\Omega_{a,b,c}$ is generically irreducible. Again this implies *direct sum decompositions* for any construction Sym^m, Ext^r.

The *symmetric square* of $\tilde{\Omega}_{a,b,c}^{(3)}$ is actually a *direct sum* of an order-27 operator and an order-1 operator, L_1, with the rational solution $1/(x-1)^2/x^6$

$$Sym^2\big(\tilde{\Omega}_{a,b,c}^{(3)}\big) = L_{27} \oplus L_1 \tag{99}$$

and the *exterior cube* of $\tilde{\Omega}_{a,b,c}^{(2)}$ is actually a *direct sum* of an order-27 operator $M_{27}^{(n)}$, an order-7 operator $M_7^{(n)}$, and an order-1 $M_1^{(n)}$ operator which has the rational solution $1/(x-1)^3/x^9$:

$$Ext^3\big(\tilde{\Omega}_{a,b,c}^{(2)}\big) = M_{27}^{(n)} \oplus M_7^{(n)} \oplus M_1^{(n)}. \tag{100}$$

[38] Use the commands with(Tensor Constructions); with(Integrable Connections); then companion-system(*), exterior-power-system(*,N), symmetric-power-system(*,N), Rational Solutions([*],[x]), Hyperexponential Solutions([*],[x]).
[39] We try to promote, in this paper the idea that switching to differential systems is a more intrinsic and powerful method than working on the operators (seen at first sight by physicists, as simpler). With these examples we see that even switching to differential systems is not enough: one needs to switch to θ-systems.

Furthermore one has[40]

$$Ext^2\big(\tilde{\Omega}^{(1)}_{a,\,b,\,c}\big) = L_{14} \oplus L_7, \qquad \text{where:}$$

$$L_7 = \frac{1}{(x-1)\cdot x^3} \cdot \Omega_{a,\,b,\,c} \cdot (x-1)\cdot x^3. \tag{101}$$

This three-parameter operator probably also has the exceptional group $G_2(C)$ as its differential Galois group.

6. Comments and speculations: diagonal of rational functions

Let us recall that the (minimal) linear differential operators for the $\chi^{(n)}$'s, the n-particle contributions of the magnetic susceptibility of the square Ising model, are not irreducible, but *factor into many irreducible operators* of various orders [11–14, 52] (2, 3, 4, ...). For all the factors for which the calculations can be performed[41] we have seen that these irreducible factors are *actually homomorphic to their adjoint*. Thus, the interesting question is to see whether all the factors of these (minimal) operators for the $\chi^{(n)}$'s are homomorphic to their adjoint, i.e. have a 'special' differential Galois group, possibly as a consequence of the fact that the $\chi^{(n)}$'s are *diagonals of rational functions* (see [18, 19] for a definition).

In this paper we underline selected linear differential operators having selected differential structures (special differential Galois groups, namely orthogonal or symplectic) characterized in a differential algebra way (homomorphisms to their adjoint, rational, or hyperexponential [41], solutions of their exterior or symmetric powers). The idea is to disentangle these selected geometrical properties from other selected structures of a more arithmetic properties (globally bounded series solutions [18, 19]), both kinds of selected properties occurring simultaneously with the concept of 'modularity'. It is important to understand the relationship between these two kinds of properties. Operators with selected differential Galois groups do not necessarily correspond to globally bounded solution series [18, 19]. It is thus natural to see whether operators with globally bounded solution series [18, 19] necessarily correspond to selected differential Galois groups. This question being probably too difficult to address, let us ask the following question: if a linear differential operator has solutions that are *diagonals of rational functions*[42], does it necessarily correspond to selected differential Galois groups, or, more simply, are such operators homomorphic to their adjoint (possibly in an algebraic extension)? Note that we have accumulated a rather large number of operators with solutions that are *Hadamard products* [55, 56] of *algebraic functions* (and are thus simple examples of diagonals of rational functions [18, 19]). They all have been seen to be homomorphic to their adjoint (sometimes up to algebraic extensions). Let us recall that *diagonals of rational functions* are (most of the time transcendental) functions that are the *simplest extensions of algebraic functions* [18, 19] (modulo each prime, they are algebraic functions). It is worth noting that linear differential operators with algebraic solutions are always homomorphic to their adjoint (up to an algebraic extension). It is thus tempting to ask whether (the factors of minimal) differential operators with solutions that are *diagonal of rational functions* are necessarily *homomorphic to their adjoint* (possibly in an algebraic extension). In order to gather some evidence on this question, we consider a set of simple but, hopefully, sufficiently

[40] Note that these results (99), (100), (101), are obtained for *arbitrary values of the three parameters* a, b, c, of $\Omega_{a,b,c}$.
[41] There are factors of order 12 or 23, that are too large to see, by brute-force calculations, if they are homomorphic to their adjoint, or such that their exterior or symmetric square could have a rational solution.
[42] Diagonals of rational functions are necessarily solutions of linear differential operators (see Lipshitz [53, 54]).

generic[43], diagonals of rational functions, find the minimal operators that annihilate them, and check wether the factors of these operators could all be homomorphic to their adjoint (up to algebraic extensions).

6.1. Diagonal of rational function: a heuristic simple example of an arbitrary number of variables

Let us first consider one of the simplest example of diagonal of rational functions of N variables, namely the diagonal of the rational function

$$S_N = Diag\left(\frac{1}{1 - x_1 - x_2 \ \cdots - x_N}\right) = \sum_{k=0}^{\infty} \frac{(k\,N)!}{(k!)^N} \cdot x^k. \qquad (102)$$

The series S_N are solutions of the order $N-1$ *self-adjoint* linear differential operators L_{N-1}:

$$x \cdot L_{N-1} = N \cdot x \cdot (N\theta + 1) \cdot (N\theta + 2) \cdots (N\theta + N - 1) - \theta^{N-1}, \qquad (103)$$

which makes crystal clear that these operators are hypergeometric operators with ${}_{N-1}F_{N-2}$ solutions. For instance, for L_4, we recover the ${}_4F_3$ hypergeometric solution occurring in the paper of Candelas *et al* [57]:

$${}_4F_3\left(\left[\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}\right], [1, 1, 1], 5^5 \cdot x\right). \qquad (104)$$

For arbitrary values of N we get, for L_{N-1}, the ${}_{N-1}F_{N-2}$ hypergeometric solution:

$${}_{N-1}F_{N-2}\left(\left[\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right], [1, 1, \ldots, 1], N^N \cdot x\right). \qquad (105)$$

It is worth noting, for larger values of N, that the L_{N-1} operators are such that, not only the series-solution, associated with (105), is a globally bounded series [18, 19] (with the N^N factor in the argument of the hypergeometric function it is even a series with *integer* coefficients), but that the series for the nome and *all the Yukawa couplings*, are all series with *integer* coefficients, thus corresponding to a *modularity* of the operators. These results can be seen to be a consequence of [58] which gives the special parameters of generalized hypergeometric equations leading to mirror maps with integral Taylor coefficients at $x = 0$. For instance, the nome $q(L_{N-1})$ of the first L_{N-1}'s read:

$$q(L_3) = x + 104\,x^2 + 15\,188\,x^3 + 2585\,184\,x^4 + 480\,222\,434\,x^5 + \cdots$$
$$q(L_4) = x + 770\,x^2 + 1014\,275\,x^3 + 1703\,916\,750\,x^4 + 3286\,569\,025\,625\,x^5 + \cdots$$

and all the Yukawa series, including the 'higher order Yukawa couplings[44]', K_n, are globally bounded, and, even, series with integer coefficients. We actually found the following relations between the Yukawa couplings. For L_4, one has $K_4 = K_3^2$, for L_5, one has $K_4 = K_3^3$ and $K_5 = K_3^5$. For L_6, one has $K_5 = K_4^3$ and $K_6 = K_4^3$. For L_7, the relations read $K_7 = R_7^7$, $K_6 = R_7^5$ and $K_5 = R_7^3$, where R_7 denotes the ratio K_4/K_3. For L_6, one has $K_5 = K_4^3$ and $K_6 = K_4^3$. For L_8, the relations read $K_8 = R_8^4$, $K_7 = R_8^3$ and $K_6 = R_8^2$, where R_8 denotes the ratio K_5/K_3.

The first Yukawa couplings K_3, for L_4 and L_5, read

$$K_3(L_4) = 1 + 575\,x + 1418\,125\,x^2 + 3798\,200\,625\,x^3 + 10\,597\,067\,934\,375\,x^4 + \ldots,$$

$$K_3(L_5) = 1 + 10\,080\,x + 357\,073\,920\,x^2 + 13\,943\,124\,679\,680\,x^3 + \cdots.$$

[43] We try to avoid operators with hypergeometric or Hadamard product solutions.
[44] See appendix C, and especially C.2, in [18, 19] for the definition of these 'higher order' Yukawa couplings K_n.

For L_6 and L_7 one has *two* independent Yukawa couplings. For instance for L_6 these two Yukawa couplings read:

$$K_3(L_6) = 1 + 10\,097\,920\,x + 381\,994\,497\,763\,200\,x^2 + 16\,633\,254\,043\,776\,570\,088\,000\,x^3 \\ +775\,506\,882\,960\,998\,615\,640\,344\,320\,000\,x^4 + \ldots,$$
$$K_4(L_6) = 1 + 37\,273\,810\,x + 1993\,144\,925\,004\,100\,x^2 + 110\,716\,785\,445\,910\,533\,561\,000\,x^3 \\ +6240\,527\,867\,851\,744\,863\,088\,075\,810\,000\,x^4 + \cdots.$$

For L_8 and L_9, one has *three* independent Yukawa couplings; for L_{10} and L_{11}, one has *four* independent Yukawa couplings, etc.

For an *odd* integer N, the *exterior square* of the $(N-1)$-order operator (103) is of order $(N-1)\,(N-2)/2 - 1$ instead of $(N-1)\,(N-2)/2$. For N an *even* integer the *symmetric square* of the $(N-1)$-order operator (103) is of order $N\,(N-1)/2 - 1$ instead of $N\,(N-1)/2$. Similarly to (45) or (67), introducing an equivalent operator $\tilde{L}^n_{N-1}$, such that $S^n \cdot L_{N-1} = \tilde{L}^n_{N-1} \cdot D^n_x$, the exterior or symmetric square of that equivalent operator has, for a well-suited value of n, the rational solution $1/x^{N-2}/(N^N\,x - 1)$.

6.2. Diagonal of rational function: heuristic simple examples of three variables

Increasing the degree of the rational functions, an example corresponds to the diagonal of

$$R(x,\, y,\, z) = \frac{1 - 7\,x + 2\,y\,z}{1 + 3\,x\,z - 5\,y^3}. \tag{106}$$

Its diagonal is the series

$$1 - 540\,x^3 + 510\,300\,x^6 - 541\,282\,500\,x^9 + 604\,514\,137\,500\,x^{12} + \cdots$$

which is an *algebraic function* solution of an order-3 operator (homomorphic to its adjoint).

Another example corresponds to the diagonal of

$$R(x,\, y,\, z) = \frac{1 - x}{1 - 3\,x + z - 5\,y^2}. \tag{107}$$

which also corresponds to the series expansion of a hypergeometric function:

$$\frac{2}{15} + \frac{13}{15} \cdot {}_4F_3\left(\left[\tfrac{1}{5},\, \tfrac{2}{5},\, \tfrac{3}{5},\, \tfrac{4}{5}\right],\, \left[\tfrac{1}{2},\, \tfrac{1}{2},\, 1\right],\, 5^6 \cdot \left(\tfrac{3}{4} \cdot x\right)^2\right) \\ = 1 + 1170\,x^2 + 5528\,250\,x^4 + 33\,202\,669\,500\,x^6 + \cdots. \tag{108}$$

The corresponding order-5 operator is a *direct sum* $D_x \oplus M_4$ where the order-4 operator M_4, which annihilates the ${}_4F_3$ in (108), is homomorphic to its adjoint and its exterior square has a rational solution namely $1/x/(9 \cdot 5^6\,x^2 - 16)$. The order-4 operator M_4 is of the form (79), namely:

$$M_4 = L_2 \cdot c_0(x) \cdot M_2 + \frac{9}{5^4} \cdot \frac{1}{c_0(x)}, \qquad c_0(x) = x^2 \cdot \left(x^2 - \frac{16}{9 \cdot 5^6}\right), \tag{109}$$

where L_2 and M_2 are two order-2 *self-adjoint* operators[45].

These cases, reducible to algebraic or hypergeometric situations, are still too simple to be representative of the 'generic' situation.

[45] M_2 is the product of two order-1 operators the right factor having the polynomial solution $x^2\,(9 \cdot 5^6\,x^2 - 16)$.

6.3. Towards a 'generic' diagonal of rational function example

Trying to avoid these too simple cases[46] reducible to hypergeometric functions (or Hadamard product of algebraic functions), we have considered the operator annihilating the diagonal of a rational function of three variables, hopefully involved enough, with no symmetry between the three variables, to be seen as a 'generic' diagonal of a rational function.

6.3.1. Towards a 'generic' diagonal of rational function: a first example. The rational function we have considered reads:

$$R(x,\, y,\, z) \,=\, \frac{1}{1-3\,x-5\,y-7\,z+x\,y+2\,y\,z^2+3\,x^2\,z^2}. \tag{110}$$

The diagonal of this rational function reads[47]:

$$S_0^{(0)} \,=\, Diag(R(x,\, y,\, z)) \,=\, 1+616\,x+947\,175\,x^2+1812\,651\,820\,x^3+\cdots \tag{111}$$

The minimal order operator that annihilates the diagonal of this rational function (110) is a *rather large order*-6 linear differential operator[48]. Again, this operator is too large to check that it is *homomorphic to its adjoint*. We can, however, check that its *exterior square* is of order 15. Switching to the associated differential theta-system, we have been able to see that it is actually *homomorphic to its adjoint*: one actually finds the *exterior square* of the associated differential system has a rational solution (but not its symmetric square). The differential Galois group thus has a *symplectic structure*.

In fact this operator is *not* MUM. It has four solutions, analytic at $x\,=\,0$, namely $S_0^{(0)}$ given by (111) and

$$S_0^{(1)} \,=\, x-\frac{947\,569\,825\,302\,083\,891\,091\,227\,422\,045}{3191\,686\,441\,638\,931\,584\,990\,008\,514}\,x^4 - \frac{13\,038\,344\,513\,942\,350\,315\,758\,249\,091\,274\,688\,499}{19\,626\,034\,561\,464\,639\,086\,279\,672\,353\,532}\,x^5+\cdots,$$

$$S_0^{(2)} \,=\, x^2+\frac{60}{7^3}\,x^4-\frac{576}{7^4}\,x^5+\cdots,$$

$$S_0^{(3)} \,=\, x^3+\frac{30\,608\,172\,563\,777\,847\,511\,388\,970\,395}{14\,474\,768\,442\,806\,945\,963\,673\,508}\,x^4 + \frac{6637\,738\,302\,888\,023\,001\,730\,565\,011\,179\,544\,651}{1401\,859\,611\,533\,188\,506\,162\,833\,739\,538}\,x^5+\cdots \tag{112}$$

the last series $S_0^{(3)}$ being *not globally bounded*. The two other solutions have a log (but no $\log^2$, $\log^3, \ldots$):

$$S_1^{(0)} \,=\, S_0^{(0)}\,\cdot\,\ln(x)+T_0^{(0)}, \qquad S_1^{(2)} \,=\, S_0^{(2)}\,\cdot\,\ln(x)+T_0^{(2)}, \tag{113}$$

the two series $T_0^{(0)}$ and $T_0^{(2)}$ being analytic at $x=0$, for instance:

$$T_0^{(0)} \,=\, \frac{1769\,904\,090\,259\,426\,475\,015\,551\,868\,948\,047\,756\,831\,494\,229\,112\,489}{6347\,493\,572\,699\,380\,825\,284\,454\,014\,187\,955\,842\,800}\,x^4 + \frac{21\,577\,983\,707\,661\,117\,706\,708\,514\,436\,988\,691\,858\,431\,632\,715\,744\,973\,527\,227\,853}{21\,340\,441\,599\,994\,994\,868\,198\,204\,433\,731\,283\,524\,323\,434\,200}\,x^5 +\cdots. \tag{114}$$

[46] When the operators annihilating diagonal of rational functions are of order 2, one often finds modular forms, the corresponding nome being seen to be a globally bounded series [18, 19]. A set of examples of diagonal of Szego's rational functions can be found in [59].

[47] Use the maple command mtaylor(F, $[x, y, z]$, terms), to get the series in three variables, then take the diagonal. Other method, in Mathematica install the risc package Riscergosum [60], and in Holonomic Functions' use the command Find Creative Telescoping.

[48] We thank Alin Bostan for providing this order-6 operator from a creative telescopic code.

With this example that is not MUM, we exclude any simple modularity property for the operator, where the series for the nome, Yukawa couplings, etc would be globally bounded. Diagonal of rational functions do not necessarily yield modularity.

6.3.2. Towards a 'generic' diagonal of rational function: a second example. Let us consider another simpler example with the diagonal of another rational function of three variables:

$$R(x,\, y,\, z) \; = \; \frac{1-x-y+xyz}{1-x-y-xy-\, y^2 z^3}. \qquad (115)$$

The diagonal of this rational function reads:

$$S_0^{(0)} = Diag(R(x,\, y,\, z)) = 1+x+10\,x^3+32\,x^4+966\,x^6+\cdots. \qquad (116)$$

It is solution of an *order*-5 operator L_5 which factors as $L_5 \; = \; L_4 \cdot D_x$, where L_4 is an *irreducible* order-4 operator that is *not*MUM. The *exterior square* of L_4 is an order-6 operator with a *rational function*[49] *solution* $R(x)$, corresponding to the *direct sum* decomposition:

$$Ext^2(L_4) = L_5 \,\oplus\, \left(D_x - \frac{\mathrm{d}\ln(R(x))}{\mathrm{d}x}\right), \qquad R(x) = \frac{p_{10}(x)}{x^2\cdot\, p_6(x)^2}, \qquad (117)$$

$$\begin{aligned} p_{10}(x) &= 11\,008\,x^{10}+165\,760\,x^9-637\,392\,x^8+383\,388\,x^7+196\,287\,x^6 \\ &\quad -281\,004\,x^5-66\,582\,x^4-45\,360\,x^3+15\,660\,x^2-810\,x+162, \\ p_6(x) &= 1024\,x^6-9909\,x^3+54. \end{aligned} \qquad (118)$$

This order-4 operator L_4 is non-trivially homomorphic to its adjoint, with order-2 intertwiners and is actually of the form (79):

$$\begin{aligned} L_4 &= L_2\cdot\, c_0(x)\cdot\, M_2+\frac{1305}{29\,584}\cdot\frac{1}{c_0(x)}, \qquad c_0(x)=\frac{145}{473\,344}\cdot\frac{p_{10}(x)^2}{p_{16}(x)}, \\ p_{16}(x) &= 37\,120\,x^{16}+1255\,680\,x^{15}\;-4887\,0560\,x^{14}+594\,756\,560\,x^{13}\;-31\,084\,335\,x^{12} \\ &\quad +2785\,358\,960\,x^{11}+4430\,975\,954\,x^{10}-8858\,296\,096\,x^9\;-1107\,376\,429\,x^8 \\ &\quad -369\,545\,240\,x^7+4215\,494\,304\,x^6-1487\,095\,128\,x^5\;-466\,418\,052\,x^4 \\ &\quad +228\,523\,680\,x^3-21\,096\,612\,x^2+2737\,800\,x-717\,336, \end{aligned} \qquad (119)$$

where L_2 and M_2 are two self-adjoint operators, their Wronskian reading respectively

$$\frac{x^2\cdot\, p_{10}(x)\cdot\, p_6(x)^2}{p_{16}(x)}, \qquad \frac{p_{10}(x)}{x^2\cdot\, p_6(x)^2}. \qquad (120)$$

The operator L_4 is not MUM: it has *three* solution analytic at $x=0$, namely the derivative of diagonal (116)

$$\frac{\mathrm{d}S_0^{(0)}}{\mathrm{d}x} = 1+30\,x^2+128\,x^3+5796\,x^5+22\,344\,x^6+1083\,060\,x^8+\,\ldots, \qquad (121)$$

and the two series-solutions

$$\begin{aligned} & x-\tfrac{595}{1107}\,x^2+\tfrac{3515\,617}{1225\,449}\,x^3+\tfrac{227\,188\,435}{1225\,449}\,x^4-\tfrac{2520\,602}{15\,129}\,x^5+\tfrac{8346\,429\,274}{11\,029\,041}\,x^6+\ldots, \\ & x^2-\tfrac{595}{1107}\,x^3+\tfrac{4523}{1107}\,x^4+\tfrac{37\,758}{205}\,x^5-\tfrac{1412\,590}{9963}\,x^6+\,\cdots. \end{aligned} \qquad (122)$$

[49] It may be tempting to imagine a relation between these two rational functions, namely $R(x,\, y,\, z)$ and $R(x)$. There is no such relation. A given series like (116) can be seen as the diagonal of an infinite number of rational functions [18, 19]. Furthermore the solution of the exterior square of an operator of the form (119), namely $L\,c_0(x)\,M+A/c_0(x)$ (L and M self-adjoint), depends on M *whatever* L is.

One also has one formal series solution with a log, namely

$$S_1(x) + \ln(x) \cdot \frac{dS_0^{(0)}}{dx}, \qquad (123)$$

where $S_1(x)$ is a series analytic at $x = 0$:

$$S_1(x) = \frac{1}{5\,x} + \frac{3083}{2214} + \frac{5222\,887}{4084\,830}\,x + \frac{956\,031\,447\,781}{22\,609\,534\,050}\,x^2 + \frac{661\,652\,916\,345\,161}{2502\,875\,419\,335}\,x^3 + \cdots.$$

The last series in (122), as well as $S_1(x)$, are *not globally bounded* series [18, 19].

Remark 6.1. The series (122) is, of course, also a diagonal of a rational function:

$$x \cdot \frac{dS_0^{(0)}}{dx} = Diag\left(u \cdot \frac{\partial R(x,\, y,\, z)}{\partial u}\right) \qquad \text{where:} \qquad u = x, y \text{ or } z.$$

Remark 6.2. With this order-5 example, we see that the minimal order operator L_5, that annihilates the diagonal of a rational function, is not necessarily irreducible. Let us recall the results of [18, 19] where we have shown that the $\tilde{\chi}^{(n)}$'s of the susceptibility of the square Ising model are *actually diagonals of rational functions*. The corresponding (globally nilpotent) linear differential operators annihilating the $\tilde{\chi}^{(n)}$'s are not irreducible, on the contrary they factor into many linear differential operators, of various orders [8, 11–14, 17] (1, 2, 3, ..., 12, 23, ...). The interesting property we must focus on, is not that the minimal order linear differential operators annihilating the $\tilde{\chi}^{(n)}$'s are homomorphic to their adjoint, but that *all their factors* could be homomorphic to their adjoint. It is the differential Galois group of *all these factors* that we expect to be 'special'.

To sum up. One may consider the following conjecture: all the irreducible factors of the minimal order linear differential operator annihilating a diagonal of a rational function should be homomorphic to their adjoint (possibly on an algebraic extension).

Remark 6.3. Let us recall that the series of the hypergeometric function considered in [18, 19]

$${}_3F_2\left(\left[\tfrac{1}{9},\, \tfrac{4}{9},\, \tfrac{5}{9}\right],\, \left[\tfrac{1}{3},\, 1\right],\, 729\,x\right) = 1 + 60\,x + 20\,475\,x^2 + 9373\,650\,x^3 + \ \cdots \qquad (124)$$

still remains a series with *integer* coefficients such that one cannot prove that it is the diagonal of a rational function, or discard that option. The minimal order operator annihilating this series is an order-3 operator L_3 which is *not*[50] homomorphic to its adjoint. If our conjecture above was correct, this would be a way to show that the series (124) *cannot be the diagonal of a rational function*.

7. Conclusion

Selected differential Galois groups correspond to symmetric square or exterior square, and possibly higher powers (as seen with order-7 operators with exceptional differential Galois groups of section (5)) of operators, or equivalent operators, having rational solutions (or Nth root of rational solutions, i.e. hyperexponential [24, 41] solutions). We have focused, in this paper, on a concept of 'Special Geometry' corresponding to operators *homomorphic to their adjoint*.

[50] It is not even homomorphic up to algebraic extensions. The order-2 intertwining operator M_2 such that $M_2 \cdot L_3 = adjoint(L_3) \cdot adjoint(M_2)$ has *transcendental* coefficients.

In [22, 28] Bogner has been able, from the very existence of underlying Calabi–Yau varieties, to show that the Calabi–Yau differential operators are actually conjugated to their adjoint (Poincaré pairing). If one does not assume strong hypotheses like this one, it is not simple to disentangle the differential algebra structures (corresponding to selected differential Galois groups) we have addressed in this paper, and more 'arithmetic' concepts associated to the notion of *modularity*, and the *integrality*, or *globally bounded* properties [18, 19] of the various series occurring with these differential operators (solution of the operator, the nome, the Yukawa couplings, ...). Recalling section 6, it is clear that the concept of 'Special Geometry', which we address in this paper, does not necessarily yield[51] arithmetic properties like the globally bounded [18, 19] character of various series associated with the operators. Conversely, we know that globally bounded series [18, 19] do not necessarily correspond to holonomic functions (see the example of the *non-holonomic* susceptibility of the Ising model and its series with *integer* coefficients [52]). Along such 'modularity' line, the idea that operators annihilating *diagonals of rational functions* should always correspond to a modularity property that the corresponding nome and all the Yukawa's [18, 19] are globally bounded series, has been ruled out (see, for instance, the example of subsection 6.3.1). From a mathematics viewpoint, there is still a lot of work to be performed to clarify the relations between these various neighboring concepts around the notion of 'modularity'. In that respect, it is useful to keep in mind all the simple examples[52] of section 6. From a physics viewpoint, one would like to identify, more specifically, what kind of 'Special Geometry' we encounter (Calabi–Yau, selected hypergeometric functions up to pull-backs [20], etc).

In this paper, the emergence of selected differential Galois groups has been seen, in a down-to-earth physicist's viewpoint, as *differential algebra* properties: one calculates various exterior, or symmetric, powers, and looks (up to operator equivalence) for their rational solutions (or hyperexponential [24, 41] solutions), and *one calculates the homomorphisms of an operator with its adjoint*. We have shown that quite involved lattice Green operators of order 6 and 8 are non trivially homomorphic to their adjoint, and that this yields the non-trivial decompositions (41) and (61), where their intertwiners emerge in a crystal clear way (see also (85) in section 4.4). Such decompositions enable to understand why the lattice Green operator (30) has a differential Galois group included in the orthogonal group $O(6, \mathbb{C})$ instead of the symplectic $Sp(6, \mathbb{C})$ differential Galois group, that one might expect for an order-6 operator: the intertwiners are of *odd orders*.

Decompositions such as (68), (69) can be generalized for linear differential operators of *any even* order. In fact, one can actually use the decompositions (68), (69) as an *ansatz* to provide linear differential operators of *any even* order, that will *automatically* have selected differential Galois groups.

With these lattice Green operators, we see that the simple generalization of the *Calabi–Yau condition* (72) for operators of order $N > 4$ (namely the condition that their exterior square is of order less than the generic $N \cdot (N-1)/2$ order), is a *too restrictive concept for physics*. These lattice Green operators do not satisfy such higher-order generalization of the Calabi–Yau condition (72), but should be seen as higher-order generalization of a 'weak Calabi–Yau condition' (see section 4.1) which amounts to saying that their exterior or symmetric squares have rational solutions, and that they are non-trivially homomorphic to their adjoint.

For order-4 operators, Calabi–Yau operators are defined, among several other conditions (see Almkvist *et al* [5]), essentially by the *Calabi–Yau condition* (72). It is, however, quite clear that any equivalent operator (in the sense of the equivalence of operator, i.e. homomorphic to

[51] Katz's book [7] provides examples of *self-adjoint* operators with special differential Galois groups that are *not even Fuchsian* (see also one of our first (hypergeometric) examples (2).

[52] For instance the order-6 and order-8 operators $G_6^{5\mathrm{Dfcc}}$ and $G_8^{6\mathrm{Dfcc}}$ of sections 3.6 and 3.7 are not MUM.

the Calabi–Yau operator), is also a selected operator interesting for physics. We have shown, in this paper, that any order-4 operator, non-trivially homomorphic to an irreducible operator satisfying the *Calabi–Yau condition* (72), has the following properties: it is *homomorphic to its adjoint* with *order*-2 intertwiners, it has a simple decomposition (78), and its exterior square necessarily has a rational solution. Conversely, showing that 'irreducible order-4 operators whose exterior squares have a rational solution, or, even, have a decomposition (78)' are necessarily equivalent to irreducible operators satisfying the *Calabi–Yau condition* (72) is a difficult question.

To illustrate the differential algebra structures corresponding to higher-order symmetric or exterior powers, we have also analysed some families of order-7 self-adjoint operators with exceptional differential Galois groups, where one sees, very clearly, the emergence of rational solutions for symmetric square and *exterior cube* of equivalent operators. Finally, since the Derived From Geometry n-fold integrals ('Periods') occurring in physics are often *diagonals of rational functions* [18, 19], we have also addressed many examples of (minimal order) operators annihilating diagonals of rational functions, and remarked that they have *irreducible factors homomorphic to their adjoint*.

The n-fold integrals we encounter in theoretical physics are solutions of *Picard–Fuchs* differential equations, or in a more modern mathematical language [61, 62], variation of Hodge structures[53] and Gauss–Manin systems [22, 29, 63, 64]. According to mathematicians, one should necessarily have for such variation of Hodge structures, a '*polarization*'[54] providing a '*duality*' which would send differential operators into their adjoint[55]. In our physical examples, one seems to systematically inherit this 'duality' on *each factor* of the minimal order operator, each irreducible factor being homomorphic to its adjoint. Along this line, section 6 strongly suggests to consider the conjecture that *(minimal) operators annihilating diagonal of rational functions always factor into irreducible operators homomorphic to their adjoint*, maybe on algebraic extensions, these factors thus corresponding to 'special' differential Galois groups.

This paper tries to promote the idea that, before deciphering the obfuscation of mathematicians on this subject, physicists should, in a down-to-earth way, use all the differential algebra tools[56] they have at their disposal, checking systematically if the linear differential operators they work on, have factors which are homomorphic to their adjoint, or are such that, up to operator equivalence, their exterior (resp. symmetric) square have a rational solution. The emergence of this 'duality' on all the irreducible factors of a large class of differential operators of physics needs to be understood.

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[53] Corresponding to the integrands in the n-fold integrals, namely one-parameter families of algebraic varieties [67].
[54] Which is a non-degenerated bilinear map (dual to an intersection mapping, see also the Poincaré duality [65, 66]).
[55] The Picard–Fuchs linear differential operators associated with a family of smooth projective manifolds are homomorphic to their adjoint. This can be seen using the Poincaré duality [68].
[56] DEtools in Maple.

Appendix A. A decomposition of operators equivalent to operators satisfying the Calabi–Yau condition

Let us again consider an order-4 operator Ω_4, of section (4.2) of Wronskian $w(x) = u(x)^2$, which satisfies the Calabi–Yau condition (72), and the monic order-4 operator $\tilde{\Omega}_4$ equivalent to the order-4 operator Ω_4. This amounts to saying that there exist two (at most) order-3 intertwiners U_3 and L_3

$$U_3 = b_3(x) \cdot D_x^3 + b_2(x) \cdot D_x^2 + b_1(x) \cdot D_x + b_0(x), \tag{A.1}$$

such that:

$$\tilde{\Omega}_4 \cdot U_3 = L_3 \cdot \Omega_4. \tag{A.2}$$

We choose L_3 such that $\tilde{\Omega}_4$ is monic ($\tilde{\Omega}_4 = D_x^4 + \cdots$). Of course, we also have the (adjoint) relation:

$$adjoint(U_3) \cdot adjoint(\tilde{\Omega}_4) = adjoint(\Omega_4) \cdot adjoint(L_3). \tag{A.3}$$

Furthermore, any operator satisfying the Calabi–Yau condition (72), is homomorphic to its adjoint [22], up to a conjugation by the square root of its Wronskian:

$$u(x) \cdot adjoint(\Omega_4) = \Omega_4 \cdot u(x). \tag{A.4}$$

Combining (A.2), (A.3) and (A.4) one straightforwardly deduces:

$$\tilde{\Omega}_4 \cdot Y_6 = adjoint(Y_6) \cdot adjoint(\tilde{\Omega}_4), \tag{A.5}$$

where the order-6 operator Y_6 reads:

$$Y_6 = U_3 \cdot u(x) \cdot adjoint(L_3). \tag{A.6}$$

Let us introduce the two operators N_2 and Z_2 corresponding to the Euclidean division of Y_6 by $adjoint(\tilde{\Omega}_4)$:

$$Y_6 = N_2 \cdot adjoint(\tilde{\Omega}_4) + Z_2. \tag{A.7}$$

N_2 is of course an order-2 operator, but, noticeably, Z_2 is *also* an *order*-2 operator instead of an order-3 operator one could expect generically.

Furthermore, and noticeably, N_2 is an order-2 *self-adjoint* operator such that:

$$\begin{aligned} \frac{1}{b_3(x)} \cdot N_2 \cdot \frac{1}{b_3(x)} &= u(x) \cdot \left(D_x^2 - \frac{\mathrm{d}\ln(1/u(x))}{\mathrm{d}x} \cdot D_x\right) \\ &\quad - u(x) \cdot \left(\frac{\mathrm{d}b_2(x)}{\mathrm{d}x} + b_2(x)^2 - 2\, b_1(x)\right) \\ &\quad - u(x) - \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + \frac{2}{x} \cdot \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^2. \end{aligned} \tag{A.8}$$

A consequence of the self-adjoint character of N_2 is that one also has the 'adjoint' relation[57] of (A.7):

$$adjoint(Y_6) = \tilde{\Omega}_4 \cdot N_2 + adjoint(Z_2). \tag{A.9}$$

Combining (A.5), (A.7) and (A.9) one deduces the following homomorphisms of $\tilde{\Omega}_4$ with its adjoint, with an *order*-2 intertwiner:

$$\tilde{\Omega}_4 \cdot Z_2 = adjoint(Z_2) \cdot adjoint(\tilde{\Omega}_4), \tag{A.10}$$

[57] One uses the fact that the adjoint of the sum of an order-6 and an order-2 operator is the sum of these adjoints.

Let us now perform the euclidean division of $adjoint(\tilde{\Omega}_4)$ by Z_2:

$$adjoint(\tilde{\Omega}_4) = A_2 \cdot Z_2 + A_0 \tag{A.11}$$

where A_2 is an order-2 operator and, surprisingly, A_0 is not an order-1 operator, *but a function* (order zero). Of course (and using the fact that the adjoint of two even order operators is the sum of the adjoints) one also has the 'adjoint relation' of (A.11), namely

$$\tilde{\Omega}_4 = adjoint(Z_2) \cdot adjoint(A_2) + A_0. \tag{A.12}$$

In fact, and noticeably A_2 is a *self-adjoint operator*. Combining (A.10), (A.11) and (A.12), one immediately deduces that Z_2 is *conjugated to its adjoint*, or equivalently, that the following order-2 operator Z_2^s *is self-adjoint*:

$$Z_2^s = A_0 \cdot Z_2 = adjoint(Z_2) \cdot A_0. \tag{A.13}$$

One finds out that the order-4 operator $\tilde{\Omega}_4$ can, in fact, be written in terms of a remarkable decomposition with two *order*-2 *self-adjoint operators*:

$$\tilde{\Omega}_4 = Z_2^s \cdot \frac{1}{A_0} \cdot A_2 + A_0. \tag{A.14}$$

One then deduces the homomorphisms of $\tilde{\Omega}_4$ with its adjoint:

$$A_2 \cdot \frac{1}{A_0} \cdot \tilde{\Omega}_4 = adjoint(\tilde{\Omega}_4) \cdot \frac{1}{A_0} \cdot A_2, \quad \tilde{\Omega}_4 \cdot \frac{1}{A_0} \cdot Z_2^s = Z_2^s \cdot \frac{1}{A_0} \cdot adjoint(\tilde{\Omega}_4).$$

Appendix B. Strong Calabi–Yau conditions versus self-adjoint conditions on higher order operators

An operator of order 5 is self-adjoint if it is of the form:

$$\begin{aligned} L_5 = a_5(x) \cdot D_x^5 + \frac{5}{2} \cdot \frac{da_5(x)}{dx} \cdot D_x^4 + a_3(x) \cdot D_x^3 \\ + \left(\frac{3}{2}\,\frac{da_3(x)}{dx} - \frac{5}{2}\,\frac{d^3a_5(x)}{dx^3}\right) \cdot D_x^2 + a_1(x) \cdot D_x \\ + \left(\frac{1}{2} \cdot \frac{da_1(x)}{dx} + \frac{1}{2} \cdot \frac{d^5a_5(x)}{dx^5} - \frac{1}{4} \cdot \frac{d^3a_3(x)}{dx^3}\right). \end{aligned} \tag{B.1}$$

Its *symmetric square is of order* 14 instead of the order 15 one could expect generically. In other words this (exactly) self-adjoint operator, or an order-5 operator conjugated of (B.1) by an arbitrary function, satisfies the symmetric Calabi–Yau condition (that its symmetric square is of order 14).

It is straightforward to verify that an operator conjugated of a self-adjoint operator of order N verifies, for *any* even order N, the generalization to order N of the order-4 Calabi–Yau condition (72) and for *any* odd order N, the generalization to order N of the order-3 *symmetric Calabi–Yau condition* (75).

Of course the reciprocal, which is true for order-3 and 4 operators (see (76)), is not true for higher orders. For instance, let us introduce the order-5 operator M_5 non-trivially homomorphic to the self-adjoint operator (B.1):

$$\begin{aligned} M_5 \cdot D_x = \frac{1}{a_5(x)} \cdot \left(D_x - \frac{1}{W(x)} \cdot \frac{dW(x)}{dx}\right) \cdot L_5, \\ \text{where:} \qquad W(x) = 2\,\frac{da_1(x)}{dx} - \frac{d^3a_3(x)}{dx^3} + 2\,\frac{d^5a_5(x)}{dx^5}. \end{aligned} \tag{B.2}$$

This operator also verifies the order-5 *symmetric Calabi–Yau condition* (75): its symmetric square is also of order 14. This result generalizes with M_5

$$M_5 \cdot (D_x + \rho(x)) = \frac{1}{a_5(x)} \cdot (D_x - z(x)) \cdot L_5, \tag{B.3}$$

where $z(x)$ is a quite involved rational expression of $a_1(x)$, $a_3(x)$, $a_5(x)$, $\rho(x)$ and their derivatives.

These last results can easily be generalized. For instance for the order-9 and order-10 self-adjoint operators L_9, L_{10} the corresponding equivalent operators M_9, M_{10} obtained from the LCLM of L_9 or L_{10} with an *order*-3 operator verify respectively the order-9 *symmetric Calabi–Yau condition* (namely the symmetric square of M_9 is of order 44 instead of 45) and the order-10 Calabi–Yau condition, (namely that the symmetric and exterior squares of M_9 and M_{10} are of order 44 instead of 45), and so on.

Appendix C. Exceptional Galois groups: three parameter operators

Let us consider the following order-7 operator[58] depending on *three* parameters a, c, d, (here σ denotes $b^2 + b\,c + c^2$):

$$\begin{aligned}\Omega_{a,b,c} = \theta \cdot (\theta^2 - b^2) \cdot (\theta^2 - c^2) \cdot (\theta^2 - (b+c)^2)& \\ - x \cdot (2\theta + 1) \cdot (\theta + a) \cdot (\theta + 1 - a) \cdot (\theta \cdot (\theta + 1) \cdot (\theta^2 + \theta + 1 - \sigma)& \\ + 2\,a \cdot (1-a) \cdot (\theta^2 + \theta + 1 - \sigma - a \cdot (1-a)))& \\ + x^2 \cdot (\theta + 1) \cdot (\theta + a) \cdot (\theta + 1 - a) \cdot (\theta + a + 1) \cdot (\theta + (1-a) + 1)& \\ \times (\theta + 2\,a)\,(\theta + 2 \cdot (1-a)).&\end{aligned} \tag{C.1}$$

On this explicit expression one sees obviously that (C.1) is (b, c)-symmetric, $\Omega_{a,b,c} = \Omega_{a,c,b}$ and that it is invariant by the $a \leftrightarrow 1 - a$ involution, $\Omega_{a,b,c} = \Omega_{1-a,b,c}$. Less obviously one notes that $\Omega_{a,b,c}$ and $\Omega_{a+N,b+M,c+P}$ are homomorphic *for any value of the three integers* N, M, P. This operator can easily be turned into a *self-adjoint* operator $\Omega^s_{a,b,c} = x^{-1/2} \cdot \Omega_{a,b,c} \cdot x^{1/2}$ (or the self-adjoint operator $x^{-1} \cdot \Omega_{a,b,c}$).

The previous order-7 rescaled operators in section (5), namely $\hat{E}_i$ for $i = 2 \cdots 5$ can actually be seen as special cases of the rescaled (C.1). For instance $\hat{E}_2 = \Omega_{1/2,0,0}$, $\hat{E}_3 = \Omega_{1/3,0,0}$, $\hat{E}_4 = \Omega_{1/4,0,0}$, $\hat{E}_5 = \Omega_{1/6,0,0}$.

References

[1] Boukraa S, Hassani S, Maillard J-M and Zenine N 2007 Singularities of n-fold integrals of the Ising class and the theory of elliptic curves *J. Phys. A: Math. Theor.* **40** 11713–48
[2] Guttmann A J 2009 Lattice Green functions and Calabi–Yau differential equations *J. Phys. A: Math. Theor.* **42** 232001
[3] Guttmann A J 2010 Lattice Green functions in all dimensions *J. Phys. A: Math. Theor.* **43** 305205
[4] Batyrev V V and van Straten D 1995 Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in toric varieties *Commun. Math. Phys.* **168** 493–533
[5] Almkvist G, van Enckevort C, van Straten D and Zudilin W 2005 Tables of Calabi–Yau equations arXiv:math/0507430v2
[6] Almkvist G and Zudilin W 2006 Differential equations, mirror maps and zeta values *Am. Math. Soc.* **38** 481–515
[7] Katz N M 1990 *Exponential Sums and Differential Equations (Annals of Mathematics Studies* vol 124*)* (Princeton, NJ: Princeton University Press) p 109 (corollary 3. 6. 1.)

[58] See operator P_1 in [29]. Actually using the family P_1 of operator [29], Bogner deduced [22] a set of parameters yielding to the Calabi–Yau operators of section (5).

[8] Bostan A, Boukraa S, Hassani S, Maillard J-M, Weil J-A and Zenine N 2009 Globally nilpotent differential operators and the square Ising model *J. Phys. A: Math. Theor.* **42** 125206

[9] van Hoeij M, Ragot J-F, Ulmer F and Weil J-A 1999 Liouvillian solutions of linear differential equations of order three and higher *J. Symb. Comput.* **28** 589–609

[10] Compoint E 1998 E-fonctions hypergéométriques et groupes de Galois orthogonaux *J. Algebra* **199** 404–29

[11] Bostan A, Boukraa S, Guttmann A J, Hassani S, Jensen I, Maillard J-M and Zenine N 2009 High order Fuchsian equations for the square Ising model: $\tilde{\chi}^{(5)}$ *J. Phys. A: Math. Theor.* **42** 275209

[12] Nickel B, Jensen I, Boukraa S, Guttmann A J, Hassani S, Maillard J-M and Zenine N 2010 Square lattice Ising model $\tilde{\chi}^{(5)}$ ODE in exact arithmetics *J. Phys. A: Math. Theor.* **42** 195205

[13] Boukraa S, Hassani S, Jensen I, Maillard J-M and Zenine N 2010 High order Fuchsian equations for the square lattice Ising model: $\chi^{(6)}$ *J. Phys. A: Math. Theor.* **43** 115201

[14] Zenine N, Boukraa S, Hassani S and Maillard J-M 2005 Square lattice Ising model susceptibility: connection matrices and singular behavior of $\chi^{(3)}$ and $\chi^{(4)}$ *J. Phys. A: Math. Gen.* **38** 9439–74

[15] Bostan A, Boukraa S, Hassani S, Maillard J-M, Weil J-A, Zenine N J and Abarenkova N 2010 Renormalization, isogenies and rational symmetries of differential equations *Adv. Math. Phys.* **2010** 941560

[16] Boukraa S, Hassani S, Maillard J-M, McCoy B M, Weil J-A and Zenine N 2007 The diagonal Ising susceptibility *J. Phys. A: Math. Theor.* **40** 8219–36

[17] Zenine N, Boukraa S, Hassani S and Maillard J-M 2004 The Fuchsian differential equation of the square Ising model $\chi^{(3)}$ susceptibility *J. Phys. A: Math. Gen.* **37** 9651–68

[18] Bostan A, Boukraa S, Christol G, Hassani S and Maillard J-M 2013 Ising n-fold integrals as diagonals of rational functions and integrality of series expansions *J. Phys. A: Math. Theor.* **46** 185202

[19] Bostan A, Boukraa S, Christol G, Hassani S and Maillard J-M 2012 Ising n-fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity arXiv:1211.6031

[20] Bostan A, Boukraa N, Hassani S, van Hoeij M, Maillard J-M, Weil J-A and Zenine N 2011 The Ising model: from elliptic curves to modular forms and Calabi–Yau equations *J. Phys. A: Math. Theor.* **44** 045204

[21] Golyshev V 2007 Classification problems and mirror duality *Surveys in Geometry and Number Theory (LMS Lecture Note Series* vol 338*)* ed N Young (Cambridge: Cambridge University Press) pp 88–121

[22] Bogner M 2012 On differential operators of Calabi–Yau type *Dissertation zur Erlangung des Grades Doktor der Naturwissenschaften* Johannes Gutenberg Universität Mainz

[23] Sanabria Malagon C 2010 Reversible linear differential equations *J. Algebra* **325** 248–68

[24] Singer M F and Ulmer F 1995 Necessary conditions for Liouvillian solutions of (third order) linear differential equations *Appl. Algebra Eng. Commun. Comput.* **6** 1–22

[25] Guttmann A J and Prellberg T 1993 Staircase polygons, elliptic integrals, Heun functions, and lattice Green functions *Phys. Rev.* E **47** R2233

[26] Glasser M L and Guttmann A J 1994 Lattice Green function (at 0) for the 4D hypercubic lattice *J. Phys. A: Math. Gen.* **27** 7011–4

[27] Koutschan C 2013 Lattice Green's Functions of the higher-dimensional face-centered cubic lattices *J. Phys. A: Math. Theor.* **46** 125005

[28] Bogner M 2013 Algebraic characterization of differential operators of Calabi–Yau type arXiv:1304.5434v1 [math.AG]

[29] Dettweiler M and Reiter S 2011 The classification of orthogonally rigid G_2-local systems arXiv:1103.5878

[30] van der Put M and Singer M F 2003 *Galois Theory of Linear Differential Equations* (*Grundlehren der Mathematischen Wissenschaften* vol 328) (Berlin: Springer)

[31] Singer M F 1996 Testing reducibility of linear differential operators: a group theoretic perspective *Appl. Algebra Eng. Commun. Comput.* **7** 77–104

[32] Homomorphisms command in Maple's DEtools package 2004 http://www.math.fsu.edu/~hoeij/files/Hom/

[33] Berman P H and Singer M F 1999 Calculating the Galois group of $L_1(L_2(y)) = 0$, L_1, L_2 completely reducible operators *J. Pure Appl. Algebra* **139** 3–23

[34] Joyce G S 1973 On the simple cubic lattice Green function *Phil. Trans. R. Soc. Lond.* A **273** 583–610

[35] Snow C 1952 *Hypergeometric and Legendre functions with applications to integral equations of potential theory (Natl. Bur. Stand. Appl. Math. Ser.* vol 19*)* (Washington, DC: National Bureau of Standards)
[36] Singer M F 1988 Algebraic relations among solutions of linear differential equations: Fano's theorem *Am. J. Math.* **110** 115–43
[37] Beukers F, Broawnawell W D and Heckman G 1988 Siegel normality *Ann. Math.* **127** 279–308
[38] Broadhurst D 2009 Bessel moments, random walks and Calabi–Yau equations http://carma.newcastle.edu.au/jon/Preprints/Papers/Submitted%20Papers/4step-walks/walk-broadhurst.pdf (unpublished)
[39] Dettweiler M and Reiter S 2006 On Globally nilpotent differential equations *J. Differ. Eqns* **248** 2736–45
[40] Almkvist G 2006 Calabi–Yau differential equations of degree 2 and 3 and Yifan Yang's pullback arXiv:math.AG/0612215
[41] Barkatou M A, Cluzeau T, El Bacha C and Weil J-A 2013 Computing closed form solutions of integrable connections *ISSAC'12: Proc. 36th Int. Symp. on Symbolic and Algebraic Computation* (New York: ACM) pp 43–50 (http://perso.ensil.unilim.fr/∼cluzeau/BCEW.pdf)
[42] Maple tools of the Limoges computer algebra team: a Maple package for handling tensor constructions on linear differential systems www.unilim.fr/pages_perso/jacques-arthur.weil/TensorConstructions.html
A Maple package for computing closed form solutions of integrable connections http://perso.ensil.unilim.fr/ cluzeau/PDS.html
A Maple package for local solutions of linear differential systems http://isolde.sourceforge.net/
[43] Koutschan C 2011 Electronic supplementary material to the article 'Lattice Green's functions of the higher-dimensional face-centered cubic lattices' www.koutschan.de/data/fcc/
[44] Almkvist G 2007 Some binomial identities related to Calabi–Yau differential equations arXiv:math.AG/07032551
[45] Guttmann A J 2009 Lattice Green functions and Calabi–Yau differential equations *J. Phys. A: Math. Gen.* **42** 232001
[46] Aparicio-Monforte A, Compoint E and Weil J-A 2013 A characterization of reduced forms of linear differential systems *J. Pure Appl. Algebra* **217** 1504–16
[47] Aparicio-Monforte A and Weil J-A 2012 A reduced form for linear differential systems and its application to integrability of Hamiltonian systems *J. Symb. Comput.* **47** 192–213
[48] Delves R T and Joyce G S 2001 On the Green function of the anisotropic simple cubic lattice *Ann. Phys.* **291** 71–133
[49] Delves R T and Joyce G S 2001 Exact evaluation of the Green function for the anisotropic simple cubic lattice *J. Phys. A: Math. Theor.* **34** L59–65
[50] Boukraa S, Hassani S and Maillard J-M 2012 Holonomic functions of several complex variables and singularities of anisotropic Ising n-fold integrals *J. Phys. A: Math. Theor.* **45** 494010 (Special issue: lattice models and integrability: in honour of F Y Wu)
[51] Agricola I 2008 Old and new on the exceptional group G_2 *Not. Am. Math. Soc.* **55** 922–9
[52] Boukraa S, Guttmann A J, Hassani S, Jensen I, Maillard J-M, Nickel B and Zenine N 2008 Experimental mathematics on the magnetic susceptibility of the square lattice Ising model *J. Phys. A: Math. Theor.* **41** 45520
[53] Lipshitz L 1988 The diagonal of a D-finite power series is D-finite *J. Algebra* **113** 373–8
[54] Lairez P 2013 Diagonales de fractions rationnelles http://pierre.lairez.fr/objets.html#diagonales_de_fractions_rationnelles
[55] Hadamard J 1892 Essai sur l'étude des fonctions données par leur développement de Taylor *J. Math. Pures Appl.* **4** 101–86
[56] Necer A 1997 Séries formelles et produit de Hadamard *J. Théor. Nombres Bord.* **9** 319–35
[57] Candelas P, de la Ossa X C, Green P S and Parkes L 1991 A pair of Calabi–Yau Manifolds as exactly solvable superconformal theory *Nucl. Phys.* B **359** 21–74
[58] Roques J 2012 On generalised hypergeometric equations and mirror maps www-fourier.ujf-grenoble.fr/jroques/OGHEMM.pdf
[59] Chen S, Kauers M and Singer M F 2012 Telescopers for rational and algebraic functions via residues *Proc. ISSAC 2012* 130–7 (arXiv:1201.1954) (Supplementary material available at www.risc.jku.at/people/mkauers/residues/)
[60] RISCErgoSum is a collection of packages created at the Research Institute for Symbolic Computation (RISC), Linz, Austria www.risc.jku.at/research/combinat/software/ergosum/

[61] Voisin C 1996 Symétrie Miroir *Panoramas et Synthèses 2* (France: Société Mathématique de France)
[62] Steenbrink J and Zucker S 1985 Variation of mixed Hodge structure *Invent. Math.* **80** 489–542
[63] Griffiths P A 1970 Periods of integrals on algebraic manifolds; summary of main results and discussion of open problems *Bull. Am. Meteorol. Soc.* **76** 228–96 (available at http://projecteuclid.org/euclid.bams/1183531477)
[64] Griffiths P A 1984 *Topics in Transcendental Algebraic Geometry* (*Ann. Mathematics Studies* vol 106) (Princeton, NJ: Princeton University Press)
[65] Movasati H 2008 Moduli of polarized Hodge structures *Bull. Braz. Math. Soc., New Ser.* **39** 81–107
[66] Cattani E and Kaplan A 1982 Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure *Invent. Math.* **67** 101–15
[67] Doran C F and Morgan J W 2005 Mirror symmetry and integral variations of Hodge structure underlying one parameter families of Calabi–Yau threefolds arXiv:math.AG/0505272v1
[68] Christol G 2013 private communication