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Sequences of projective measurements in generalized probabilistic models

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Abstract
We define a simple rule that allows us to describe sequences of projective measurements for a broad class of generalized probabilistic models. This class embraces quantum mechanics and classical probability theory, but, for example, also the hypothetical Popescu–Rohrlich box. For quantum mechanics, the definition yields the established Lüders rule, which is the standard rule for updating the quantum state after a measurement. For the general case, it can be seen as being the least disturbing or most coherent way of performing sequential measurements. We show, as an example, that the Spekkens toy model (Spekkens 2007 *Phys. Rev.* A **75** 032110) provides an instance of our definition. We also demonstrate the possibility of strong postquantum correlations as well as the existence of triple-slit correlations for certain nonquantum toy models.

Keywords: generalized probabilistic theories, Lüders measurement, ordered vector space

1. Introduction

It is a fundamental property of quantum mechanics that any nontrivial measurement disturbs the system that it acts on. This disturbance is responsible for very particular phenomena like the quantum Zeno effect [2, 3], where the time evolution of a system is frozen due to repeated measurements, or the contextual behavior of a quantum system [4], where measurement outcomes depend on the choice of previous compatible measurements. Compared to the classical world scenario, where a measurement—at least in principle—may leave the system unchanged, this quantum property seems to be very particular and at the same time very fundamental.

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The most common formulation of this disturbance is due to Lüders [5, 6] and determines how the state of a system is changed by a measurement: $\rho \mapsto \Pi\rho\Pi/\mathrm{tr}\,(\rho\Pi)$. But this is only one of the many possible state changes that may occur in an experiment. In the most general case the postmeasurement state can be seen as the result of a coherent evolution involving an auxiliary system and a destructive measurement on that auxiliary system. This fundamental result of Ozawa [7, 8], however, does not explain the special role of Lüders' rule. Conversely, Ozawa's result gives a very particular model of a measurement and one might argue that giving up Lüders' rule as a fundamental entity might actually make assumptions on the peculiarities of the measurement process in quantum mechanics that are too strong.

In this work we provide a very small set of assumptions that uniquely single out Lüders' rule within quantum mechanics on the one hand, and on the other hand have many desirable properties when applied to hypothetical nonquantum models. These two aspects have been discussed for a long time [9–12], and some consensus seems to exist that the mathematical concept of a filter is an appropriate approach. We highlight that the axioms that we suggest here are significantly simpler than those that have appeared before, while at the same time they imply more favorable physical properties.

We proceed as follows. The introduction is completed by a detailed reminder on how postmeasurement states are treated in quantum mechanics (cf section 1.1), and a summary of the mathematical framework of ordered vector spaces in section 1.2, enriched with examples in section 1.3. In section 2 we introduce the notion of projective, neutral, and coherent f-compatible maps, the latter of which we propose as a generalized definition of Lüders' rule. We investigate fundamental properties of this definition and give examples; in particular, we study the case of quantum mechanics in section 3.1, a large class of toy models in section 3.2, and the n-slit experiment in section 3.3. We conclude with a discussion of our findings in section 4.

1.1. Quantum instruments

Before we start to formulate the behavior of measurement sequences in generalized probabilistic models, let us first recall the established formalism in quantum mechanics [8].

We consider a situation where, first, an observable A and, then, an observable B are measured. (In order to simplify the discussion, we assume that A and B each have a pure point spectrum.) The system subject to the measurements is initially described by a density operator ρ and the measurement of A is assumed to have yielded the result a. With the spectral decomposition as $A = \sum_a a\Pi_a$, according to Lüders [5, 6], the expected value of B is given by

$$\langle B|A = a\rangle_\rho = \mathrm{tr}\left[\Pi_a\rho\Pi_a \mathrm{B}\right]\Big/\mathrm{tr}\left(\rho\Pi_a\right) = \mathrm{tr}\left[\rho\phi_a(\mathrm{B})\right]\Big/\mathrm{tr}\left[\rho\phi_a(\mathbb{1})\right]. \tag{1}$$

For the second equality we introduced the map $\phi_a\colon X \mapsto \Pi_a X\Pi_a$, so that it becomes manifest that the conditioned expectation value on the l.h.s. arises directly from the laws of conditional probabilities and the quantum instrument $\mathcal{I}_\mathrm{L}\colon a \mapsto \phi_a$. (In the literature, the notion of a Lüders instrument has been established, but it covers a broader set of instruments than those that follow Lüders' rule.)

The situation described in equation (1) can be further formalized. With the spectral decomposition $B = \sum b\mathrm{P}_b$, the probability of firstly getting the outcome a and then getting the outcome b is

$$\mathbb{P}_\omega\left(\Pi_a \triangleright \mathrm{P}_b\right) = \omega\left[\phi_a\left(\mathrm{P}_b\right)\right], \tag{2}$$

where ω: $X \mapsto \operatorname{tr}(\rho X)$ is a way of writing the quantum state and $\Pi_a \triangleright \mathrm{P}_b$ is the event 'Π_a then P_b'.

Depending on the experimental implementation, the actual instrument $\mathcal{I}'$ will deviate from the instrument that has been described by Lüders. But there is confidence that $\mathcal{I}_\mathrm{L}$ can be approximated to an arbitrary precision, since on a formal level [7] one can implement $\mathcal{I}_\mathrm{L}$ by virtue of an ancilla system in a pure state, an entangling unitary, between the probe and the ancilla system, and a destructive measurement solely on the ancilla system. This shows that $\mathcal{I}_\mathrm{L}$ can be implemented as an immediate consequence of

(i) independent pure state preparation, $\rho \mapsto \rho \otimes |\psi\rangle\langle\psi|$,
(ii) unitary evolution,
(iii) Born's rule, $\mathbb{P}_\omega(A = a) = \omega(\Pi_a)$.

However, any instrument can be implemented with the ingredients (i)–(iii). The question that drives our subsequent analysis is that of which of the properties of the instrument $\mathcal{I}_\mathrm{L}$ corresponding to Lüders' rule are most characteristic. Within the framework of quantum mechanics there would be a variety of possible characteristics that single out Lüders' rule and, without comparing to other possibilities, it would be difficult to argue in favor of one or another. Our approach is to broaden the mathematical concepts, so that not only can quantum mechanics be described but also a wider set of generalized probabilistic models is covered.

1.2. Positivity and generalized probabilistic models

Quantum events as well as classical events can be mathematically described by ordered vector spaces. This is based on the observation that the main characteristics of either theory are dominated by the notion of positivity. In particular in quantum mechanics, the (mixed) states are given by maps ω: $X \mapsto \operatorname{tr}(\rho X)$ which obey $\omega(\mathbb{1}) = 1$ and $\omega(F) \geqslant 0$ for all positive semidefinite operators F. Conversely, a generalized measurement in quantum mechanics is a family of positive semidefinite operators (F_a) with $\sum_a F_a = \mathbb{1}$. The operators F_a are then called effects. This positivity structure is largely motivated from the probabilistic interpretation $\mathbb{P}_\omega(F_a) = \omega(F_a)$. The class of models which follows a similar interpretation is captured by the mathematical concept of an ordered vector space. In turn, the set of models that can be fitted into this mathematical concept contains instances that are in conflict with the predictions of quantum mechanics [13, 14]. For this reason, these models are called generalized probabilistic models.

We now discuss the mathematical concepts related to ordered vector spaces, while in section 1.3 we present explicit examples. For a more verbose introduction to the mathematical concepts, we particularly recommend the introduction of [15] and the books by Alfsen [16] and Paulsen [17]. A real order unit vector space is a triple (V, V^+, e) such that

(i) V is a real vector space (not necessarily finite dimensional),
(ii) $V^+ \subset V$ is a cone, i.e., $V^+ + V^+ = V^+ = \mathbb{R}^+ V^+$ and $V^+ \cap -V^+ = \{0\}$,
(iii) $e \in V^+$ is an order unit, i.e., for any $x \in V$ there is an $r \in \mathbb{R}^+$ such that $re + x \in V^+$.

We wrote $\mathbb{R}^+$ for the set of nonnegative reals. It follows [15] that $V^+ - V^+ = V$. For two elements $x, y \in V$, the condition $x - y \in V^+$ defines a partial order and one writes $x \geqslant y$.

The order unit e is Archimedean provided that for any $x \in V$ the property $x + \mathbb{R}^+ e \subset V^+ \cup \{x\}$ implies $x \in V^+$. This property in some sense requires that V^+ is 'closed'. While we use this property merely for technical reasons, we also note that an order unit vector space can always be modified in such a way that it has an Archimedean order unit. This Archimedeanization [15] works by constructing the 'closure' of the cone and identifying operationally indistinguishable elements. These operations are physically benign and hence we only consider Archimedean order unit vector (AOU) spaces.

We continue to fix notation. Within the dual space $V^* = \{\alpha\colon V \to \mathbb{R} \mid \alpha \text{ is linear}\}$, the set

$$\mathcal{S}\left(V, V^+, e\right) = \left\{\omega \in V^* \mid \omega(e) = 1 \text{ and } \omega\left(V^+\right) \subset \mathbb{R}^+\right\} \tag{3}$$

is the convex set of states and the definition

$$\|x\| = \inf\left\{r \in \mathbb{R}^+ \mid -re \leqslant x \leqslant re\right\} \tag{4}$$

provides the order norm of $x \in V$. It is convenient to define the set of effects, i.e., the convex set of positive elements bounded by e,

$$V_e^+ = V^+ \bigcap \left(e - V^+\right), \tag{5}$$

and to write for the normalized representatives of the extremal rays of V^+ the symbol

$$\partial^+ V^+ = \left\{f \in V^+ \middle| \|f\| = 1 \text{ and } \left(0 \leqslant g \leqslant f \text{ implies } g \in \mathbb{R}^+ f\right)\right\}. \tag{6}$$

We occasionally construct V^+ from a finite set $\mathcal{A} \subset V$ of extremal rays via

$$\operatorname{cone}\mathcal{A} = \left\{x \in V \middle| x = \sum\nolimits_{a \in \mathcal{A}} r_a\, a, \text{ where all } r_a \in \mathbb{R}^+\right\}. \tag{7}$$

For two AOU spaces (V, V^+, e) and (W, W^+, e'), a linear map $\phi\colon V \to W$ is positive provided that it maps positive elements to positive elements, $\phi(V^+) \subset W^+$. (When we let ϕ be a map, we always imply that ϕ is linear.) If $\phi(e) = e'$, then ϕ is unital. The spaces are order isomorphic if there exists a positive unital bijection $\psi\colon V \to W$ such that its inverse is also positive.

Proposition 1. *We recall three results from [15].*

(i) $f \in V^+$ if and only if $\omega(f) \geqslant 0$ for all $\omega \in \mathcal{S}$.
(ii) If $f \in V^+$, then there exists a state $\omega \in \mathcal{S}$ such that $\omega(f) = \|f\|$.
(iii) For $x \in V$ we have $-\|x\|e \leqslant x \leqslant \|x\|e$.

In principle one is free to choose the AOU space (V, V^+, e) or the states $\mathcal{S} \subset U$ with some embedding vector space U as the fundamental object. If $\mathcal{S}$ is fundamental, then [10] we can define V to be the space of affine functions on U, let $V^+ = \{\xi \in V \mid \xi(\mathcal{S}) \subset \mathbb{R}^+\}$, and choose e with $e(\mathcal{S}) = \{1\}$. Since we do not want to make any particular point out of which space is fundamental, we may assume that V is reflexive, $V = V^{**}$. By virtue of Proposition 1 (i), this would imply that (V, V^+, e) and $[V^{**}, (V^{**})^+, e^{**}]$ are order isomorphic.

1.3. Examples of ordered vector spaces

The reason that AOU spaces are considered to be a good framework in which to describe generalized probabilistic models is that classical events and quantum events can be described by means of AOU spaces [18, 19]. For a recent introduction to the physical interpretation, we refer the reader to [20].

Classical events. A set of discrete classical events—e.g. the outcomes when rolling dice—defines a so-called AOU lattice. It is the n-fold Cartesian product of $(\mathbb{R}, \mathbb{R}^+, 1)$, where n is the number of outcomes. The set of states is given by the maps $\mathbf{v} \mapsto \mathbf{p} \cdot \mathbf{v}$ with $\mathbf{p}_k \geqslant 0$ for all k, and $\sum_k \mathbf{p}_k = 1$. The order norm reads $\|\mathbf{v}\| = \max_k |\mathbf{v}_k|$, turning V into the Banach space ℓ_n^∞.

Quantum events. For quantum mechanics, we choose the bounded self-adjoint operators as vector space V and we identify V^+ as the set of positive semidefinite operators. With the choice $e = \mathbb{1}$ this forms an AOU space; cf Theorem 1.95 in [21]. The set of quantum effects is V_e^+. The quantum states can be represented by the maps $X \mapsto \operatorname{tr}(\rho X)$ where ρ is positive semidefinite with $\operatorname{tr} \rho = 1$. (For infinite-dimensional Hilbert spaces, however, not all functionals in $\mathcal{S}$ can be written in this way.) The order norm $\|X\|$ yields the operator norm of X and the extremal set $\partial^+ V^+$ is exactly the set of rank-1 projections.

Dichotomic norm cones. A simple class of examples is constructed as $V = \mathbb{R} \times \mathbb{R}^d$, $V^+ = \{(t, \mathbf{x}) | t \geqslant \|\mathbf{x}\|\}$, and $e = (1, \mathbf{0})$, where $\|\mathbf{x}\|$ is a norm in $\mathbb{R}^d$. Such cones only allow dichotomic observables in the sense that $e - \partial^+ V^+ = \partial^+ V^+$. However, several interesting cases are instances of this example: the event space of tossing a coin (classical bit, $d = 1$ and $\|\mathbf{x}\| = |\mathbf{x}_1|$), the local part of a Popescu–Rohrlich box [13] (generalized bit [22], $d = 2$ and $\|\mathbf{x}\| = |\mathbf{x}_1| + |\mathbf{x}_2|$), the quantum mechanical two-level system (quantum bit, $d = 3$ and $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$), and 'hyperbits' [23] which generalize the quantum bit by allowing for $d > 3$ while keeping the Euclidean norm. The states for a dichotomic norm cone are the maps $(t, \mathbf{x}) \mapsto t + \mathbf{w} \cdot \mathbf{x}$ with $\|\mathbf{w}\|_* \leqslant 1$, where $\|\mathbf{w}\|_* \equiv \sup\{\mathbf{w} \cdot \mathbf{y} | \|\mathbf{y}\| \leqslant 1\}$ is the dual norm. The order norm is also easy to evaluate: $\|(t, \mathbf{x})\| = |t| + \|\mathbf{x}\|$.

A pathological example. We define $V^+ = \operatorname{cone}\{a_1, a_2, \ldots, a_6\}$ where $a_1, \ldots, a_4$ is a basis of V, $a_5 = a_1 - a_3 + a_4$, and $a_6 = a_2 + a_3 - a_4$. The order unit is chosen to be $e = a_1 + a_2 + \frac{1}{2}(a_3 + a_4)$. This case is pathological in the sense that there is no way to write $e = \sum_{v \in \mathcal{A}} v$ for any $\mathcal{A} \subset \{a_1, \ldots, a_6\} = \partial^+ V^+$.

2. Sequential measurements

We now discuss sequential measurements for such generalized probabilistic models for which the measurement effects can be squeezed into an AOU space (V, V^+, e). That is, any measurement can be described by a family of effects $(f_k) \subset V_e^+$ with $\sum_k f_k = e$—this is in analogy to the generalized measurements that occur in quantum mechanics. Following the discussion in section 1.1, we consider the situation where a sequence of two measurements has been performed and the consecutive outcomes $f, g \in V_e^+$ have occurred. What is the prediction for the probability $\mathbb{P}_\omega(f \triangleright g)$ for the event $f \triangleright g$, given that the system was in a state $\omega \in \mathcal{S}$?

This probability will clearly depend on the actual implementation of the first measurement and this implementation is readily summarized by a map $\phi: V \to V$, so $\mathbb{P}_\omega(f \triangleright g) = \omega[\phi(g)]$. This implies that ϕ is positive and, for consistency, we assume that $\phi(e) = f$, i.e., the all-embracing outcome e occurs with unit probability given that previously the outcome f has occurred. We also assumed that ϕ is linear, so performing with probability p a measurement with outcome g and with probability $1 - p$ a measurement with outcome h obeys $\mathbb{P}[f \triangleright pg + (1 - p)h)] = p\mathbb{P}(f \triangleright g) + (1 - p)\mathbb{P}(f \triangleright h)$. A positive map ϕ with $\phi(e) = f$ is called f-compatible [8].

In principle, any choice of an f-compatible map[1] may be suitable for describing $f \triangleright g$. Here we are concerned with the projective measurements which generalize Lüders' rule. The following notions capture important properties of Lüders' rule.

[1] In quantum mechanics we would be restricted to completely positive maps, but this subtlety can be ignored for the discussion here.

Definition 2. Let ϕ be an f-compatible map for $f \in V_e^+$, i.e., $\phi(e) = f$ and $\phi(V^+) \subset V^+$.

(i) ϕ is projective if $\phi \circ \phi = \phi$.
(ii) ϕ is neutral if $\omega \circ \phi = \omega$ for any $\omega \in \mathcal{S}$ with $\omega(f) = 1$.
(iii) ϕ is coherent if $\phi(g) = g$ for any $g \in V^+$ with $g \leqslant f$.

One might be tempted to use f-compatible projections for defining a generalization of Lüders' rule. For an extremal element, $f \in \partial^+ V^+$, such a map is of the form $\phi = f\omega$, where $\omega \in \mathcal{S}$ is a state with $\omega(f) = 1$ (the existence of such a state is due to Proposition 1 (ii)). In quantum mechanics this already yields uniquely Lüders' rule for rank-1 projections. Furthermore, any family $(f_k) \subset V_e^+$ with $\sum f_k \leqslant e$ and f_k-compatible projections ϕ_k enjoys perfect repeatability, $\phi_k \circ \phi_\ell = \delta_{k,\ell}\phi_k$, utilizing the Kronecker symbol $\delta_{k,\ell}$. This holds, since for $k \neq \ell$ and any $h \in V_e^+$ we have $0 \leqslant \phi_k \phi_\ell h \leqslant \phi_k \phi_\ell e = \phi_k f_\ell = -\phi_k(e - f_k - f_\ell) \leqslant 0$.

Unfortunately, projectivity does not sufficiently fix the choices for ϕ. For example, $\phi = e\omega$ is an e-compatible projection, but any subsequent measurement will solely depend on the arbitrary choice of $\omega \in \mathcal{S}$. Previously [9–12], filters have been considered as a possible extensions of Lüders' rule to generalized probabilistic models. A filter is a neutral f-compatible projection, but it is only called a filter if there also exists a neutral f-compatible projection for $e - f$. Here, we study a different extension of Lüders' rule, namely the coherent Lüders rules.

Definition 3. A coherent Lüders rule (CLR) for $f \in V_e^+$ is a coherent f-compatible map.

We occasionally write $f^\sharp$ for a CLR of f, although this map is not necessarily uniquely defined by the above condition.

A possible interpretation behind the definition of coherence is that the relation $g \leqslant f$ indicates that the outcome g always provides finer information than f in the sense that, independently of the state ω of the system, g is always less likely to be triggered than f. Thus getting firstly the coarse-grained information f and then the fine-grained information g is assumed not to influence g. Hence f preserves all the 'coherences' of g. We also refer the reader to proposition 5, proposition 6, the example of a triple-slit experiment in section 3.3, and the discussion in section 4 for further reasoning in favor of this definition. In section 2.3 it is also shown that neutral f-compatible projections and coherent f-compatible maps are different concepts.

2.1. Basic properties of coherent Lüders rules

There are several equivalent ways of expressing definition 3.

Lemma 4. *For a positive map ϕ and an effect $f \in V_e^+$, the following statements are equivalent.*

(i) $\phi(e) = f$ and $\phi(g) = g$ for all $0 \leqslant g \leqslant f$.
(ii) $\phi(e) \leqslant f$ and $\phi(g) \geqslant g$ for all $0 \leqslant g \leqslant f$.
(iii) $a \leqslant \phi(g) \leqslant f\|g\|$ for all $g \in V^+$ whenever $0 \leqslant a \leqslant f$ and $a \leqslant g$.
(iv) $a \leqslant \phi(g) \leqslant f$ for all $g \in V_e^+$ whenever $0 \leqslant a \leqslant f$ and $a \leqslant g$.

Proof. In order to see that (i) implies (iii), note that $\phi(g) = \phi(g - a) + a \geqslant a$. Furthermore, $f\|g\| - \phi(g) \geqslant 0$ follows immediately when considering $\phi(\|g\|e - g) \geqslant 0$ and by fact that $\|g\|e \geqslant g$ holds since e is Archimedean.

Obviously (iii) implies (iv), since for $g \in V_e^+$ we have $\|g\| \leqslant 1$.

Statement (ii) follows from (iv) on letting $g_{\text{(iv)}} = e$ (yielding $\phi(e) \leqslant f$) and choosing $g_{\text{(iv)}} = g_{\text{(ii)}} = a$ (yielding $\phi(g_{\text{(ii)}}) \geqslant g_{\text{(ii)}}$).

We finally show that (i) follows from (ii). We first use that $\phi(e - f) \geqslant 0$ and thus $f \geqslant \phi(e) \geqslant \phi(f) \geqslant f$, i.e., $\phi(e) = f = \phi(f)$. Then $\phi(g) - g \leqslant \phi(f) - f \equiv 0$, where the inequality follows from $f - g \leqslant \phi(f - g)$, which is due to $0 \leqslant f - g \leqslant f$. But $\phi(g) \leqslant g$ can only be compatible with $\phi(g) \geqslant g$ when $\phi(g) = g$. □

Note that with statement (iv) of this lemma, we have $\phi(h) = f$ for $f \leqslant h \leqslant e$, on letting $a = f$ and $g = h$.

From a physical perspective, a CLR for f describes exactly such a measurement that does not disturb any other subsequent measurement with outcome f.

Proposition 5. *Let $C \supset (V^+ \otimes \mathcal{S})$ be some cone of positive maps and let ϕ be an f-compatible map for $f \in V_e^+$. Then ϕ is coherent if and only if $\phi \circ \psi = \psi$ holds for all f-compatible maps $\psi \in C$.*

Proof. If ψ is f-compatible, then $\psi(h) \leqslant \psi(e) = f$ for any $h \in V_e^+$. It follows that $\phi \circ \psi = \psi$ if ϕ is a CLR. For the converse we consider $\psi = (f - g)\omega + g\sigma \in C$ with $0 \leqslant g \leqslant f$ and $\omega, \sigma \in \mathcal{S}$. This map is clearly f-compatible and we define $\Delta \equiv \phi \circ \psi - \psi = [\phi(f) - f]\omega + [\phi(g) - g](\sigma - \omega)$. From $\Delta(e) = 0$ we obtain $\phi(f) = f$ and, assuming $\sigma \neq \omega$, also $\phi(g) = g$ must hold. Hence ϕ is coherent. □

A CLR in particular obeys repeatability and compatibility.

Proposition 6. *Let $f^\sharp$ and $g^\sharp$ be two CLRs for $f, g \in V_e^+$, respectively. We have:*

(i) $f^\sharp$ is projective;
(ii) if $g \leqslant f$, then $f^\sharp g = g^\sharp f$;
(iii) if $g \leqslant f$ and $g^\sharp$ is unique for g, then $f^\sharp g^\sharp = g^\sharp f^\sharp$.

Proof. We implicitly use lemma 4 (iv). Then $f^\sharp h \leqslant f$ for any $h \in V_e^+$ and hence $f^\sharp(f^\sharp h) = f^\sharp h$. If $g \leqslant f$ then immediately $f^\sharp g = g = g^\sharp f$ (cf also the remark after lemma 4). If the CLR for g is unique, then $f^\sharp g^\sharp = g^\sharp f^\sharp$, since $f^\sharp g^\sharp = g^\sharp$ and on the other hand $g^\sharp f^\sharp$ is a valid CLR for g. □

We mention that the property of being neutral or coherent is robust under sections. A section [24] is a positive unital injection τ from (W, W^+, e') to (V, V^+, e) such that there exists a positive surjection $\tau'\colon V \to W$ with $\tau' \circ \tau = id_W$. If ϕ is a neutral/coherent $\tau(f)$-compatible map, then $\tau' \circ \phi \circ \tau$ is a neutral/coherent f-compatible map. An important instance of this observation is the embedding of the classical events into quantum events via $\tau\colon \mathbf{v} \mapsto \mathrm{diag}(\mathbf{v})$. In contrast, general $\tau(f)$-compatible projections do not always induce f-compatible projections.

2.2. Conditions on elements with a coherent Lüders rule

Not all $f \in V_e^+$ admit a CLR, as we see next. But the CLR for e is the identity mapping, while for 0 it is the zero mapping. On the other hand, if f is extremal, $f \in \partial^+ V^+$, then any f-compatible projection is a CLR. For the general situation we have:

Proposition 7. *For $f \in V_e^+$ consider the following statements.*

(i) f admits a CLR.
(ii) $g \leqslant f\|g\|$ for all $0 \leqslant g \leqslant f$.
(iii) $0 \leqslant g \leqslant f$ and $g \leqslant e - f$ only for $g = 0$.

Then (i) implies (ii) and (ii) implies (iii).

Proof. Statement (ii) is a direct consequence of lemma 4 (iii), $g = f^\sharp g \leqslant f\|g\|$. For the second part we consider $0 \leqslant g \leqslant f \leqslant e - g$. Then $0 \leqslant g \leqslant f\|g\| \leqslant \|g\|(e - g)$ and therefore $e\|g\|/(\|g\| + 1) \geqslant g$, which contradicts $\|g\| \equiv \inf\{r \in \mathbb{R}^+ | re \geqslant g\}$ unless $\|g\| = 0$. By the Archimedean property the assertion follows. □

From part (ii) of this proposition it immediately follows that if $f = \sum_k p_k f_k$ with $(f_k) \subset \partial^+ V^+$ and real numbers $p_k > 0$, then already $f_k \leqslant f$. But one cannot conclude that there exists a decomposition of f into extremal elements with unit weights; cf the pathological example form section 1.3 with $f = e$. This pathological space also provides an example where (iii) does not imply (ii). The counterexample works with $f = e - a_1 - a_2 \equiv (a_3 + a_4)/2$, which obeys (iii). But $f - pa_3 \geqslant 0$ only for $p \leqslant \frac{1}{2}$ in contradiction to (ii). At the moment it remains unclear whether (ii) implies (i), even though it does not seem plausible for this to hold. On the other hand, for quantum mechanics, already statement (iii) can only hold if F is a projection since $0 \leqslant \sqrt{F}(\mathbb{1} - F)\sqrt{F} \equiv F - F^2 \leqslant F$ and $0 \leqslant (\mathbb{1} - F)^2 \equiv \mathbb{1} - 2F + F^2$, i.e., $F - F^2 \leqslant \mathbb{1} - F$. By assumption we then have $F - F^2 = 0$ and hence F is a projection.

2.3. Neutral maps

Neutral f-compatible projections have been suggested previously [9–12] as an extension of Lüders' rule to generalized probabilistic models. For the moment we call them neutral Lüders rules (NLRs). If f and $e - f$ allow an NLR, then an NLR for f is a filter. We observe:

1. Some elements do not have an NLR, despite being extremal. Consider the dichotomic norm cone (cf section 1.3) with $\|\mathbf{x}\| = \sum |\mathbf{x}_i|$ and $d \geqslant 2$. In this case, there exists no neutral map ϕ for any of the extremal elements $f \in \partial^+ V^+$ since states with $\omega(f) = 1$ are not unique but on the other hand $\phi = f\omega$ must hold for ϕ to be an f-compatible projection.

2. Some elements with an NLR do not have a CLR. An example occurs in the pathological example from section 1.3 for the effect $f = e - a_1 - a_2$. As demonstrated at the end of section 2.2, this element does not have a CLR. But the only state with $\omega(f) = 1$ is $\omega(a_k) = (0, 0, 1, 1, 0, 0)_k$ and hence $f\omega$ is an NLR for f. One can also construct an NLR for the complement $f_\neg = e - f$, showing that $f\omega$ is a filter. The NLR for $f_\neg$ is not unique, but a possible representative is given by $a_1\omega_1 + a_2\omega_2$ with $\omega_i(a_k) = \delta_{i,k} + \delta_{i+4,k}$.

3. Applications

3.1. Quantum mechanics

In quantum mechanics, $F \in V_e^+$ admits a CLR if and only if it is a projection. We have shown necessity in section 2.2 and in order to show sufficiency we assume that F is a projection and that $F^\sharp(X) = FXF$. It remains to show that G=FGF for any $0 \leqslant G \leqslant F$. Although this is a simple and well-known relation, we shall spend a few lines in showing it. We write $F_\neg = \mathbb{1} - F$. Then $0 \leqslant F_\neg(F - G)F_\neg = -F_\neg G F_\neg \leqslant 0$ and thus $F_\neg G = F_\neg GF$. But

$0 \leqslant (F + \lambda F_\neg) G (F + \lambda F_\neg) = FGF + \lambda (F_\neg GF + FGF_\neg)$ for all $\lambda \in \mathbb{R}$ implies $F_\neg GF = -FGF_\neg$, i.e., $G = FGF$.

The rule $F^\sharp: X \mapsto FXF$ is unique, as we demonstrate by construction. Assume that $G \in V_e^+$. Then $0 \leqslant F(\mathbb{1} - G)F = F - FGF$, implying $F^\sharp(FGF) = FGF$, and $0 \leqslant F^\sharp[F_\neg(\mathbb{1} - G)F_\neg] = -F^\sharp(F_\neg GF_\neg)$, which yields $F^\sharp(F_\neg GF_\neg) = 0$. With $G'_\lambda \equiv (F + \lambda F_\neg)G(F + \lambda F_\neg) \geqslant 0$, we have

$$F^\sharp\left(G'_\lambda\right) = FGF + \lambda A \geqslant 0, \text{ where } A = F^\sharp\left(F_\neg GF + FGF_\neg\right), \tag{8}$$

for all $\lambda \in \mathbb{R}$. This implies again $A = 0$ and hence $F^\sharp(G) \equiv F^\sharp(G'_1) = FGF$.

We mention that we did not assume that $F^\sharp$ is completely positive but nevertheless obtained the intended quantum mechanical Lüders rule.

3.2. Dichotomic norm cones

We consider, as a second example, the dichotomic norm cones of section 1.3. For these AOU spaces the set of effects admitting a CLR is given by $\{0, e\} \cup \partial^+ V^+$; cf appendix A. This shows that dichotomic norm cones form a very convenient toy model for which basically the assumption of an f-compatible projection alone leads to a reasonable Lüders rule. Any extremal element $f \in \partial^+ V^+$ is of the form $f = (\frac{1}{2}, \mathbf{f})$ with $\|\mathbf{f}\| = \frac{1}{2}$ and the corresponding CLR thus reads

$$f^\sharp: (t, \mathbf{x}) \mapsto (t + \mathbf{f}' \cdot \mathbf{x})f, \text{ with } \mathbf{f}' \cdot \mathbf{f} = \frac{1}{2}, \text{ and } \|\mathbf{f}'\|_* = 1. \tag{9}$$

Since the set of CLRs for a given effect f is convex, it follows that if $\|\mathbf{x}\|$ is a p-norm with $1 < p < \infty$, then the CLR is unique. This is due to the fact that then the dual norm $\|\mathbf{x}\|_*$ is the $[p/(p-1)]$-norm, the unit sphere of which only has convex subsets with a single vector. On the other hand, for the Manhattan norm, $p = 1$, and e.g. $\mathbf{f} = (\frac{1}{2}, 0, \ldots, 0)$, the available choices are any of the $\mathbf{f}' = (1, \xi_2, \ldots, \xi_d)$ with arbitrary coefficients $-1 \leqslant \xi_k \leqslant 1$.

We compute, as an example, the effective 'observable' for a sequential measurement of two dichotomic observables $A = a - a_\neg$ and $B = b - b_\neg$ with $a_\neg = e - a$ and $b_\neg = e - b$. That is, with the notation $A\sharp = a^\sharp - a_\neg^\sharp$, we aim at $A\sharp B$. For simplicity, we assume that in $a^\sharp$ and $a_\neg^\sharp$ we have $\mathbf{a}_\neg{}' = -\mathbf{a}'$, which holds surely when both CLRs are unique. Writing $b = (\beta, \mathbf{b})$ yields

$$A\sharp B = (2\beta - 1)A + 2(\mathbf{a}' \cdot \mathbf{b})e. \tag{10}$$

If $\beta = \frac{1}{2}$, e.g., because b is extremal, then the expected value $\langle A\sharp B\rangle_\omega \equiv \omega(A\sharp B)$ does not depend on the prepared state ω. For the case of the Euclidean norm, $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$, and $B\sharp$ defined analogously to $A\sharp$, we find, in addition, $A\sharp B = B\sharp A$. Both aspects have been observed already for qubits [25], corresponding to the dichotomic norm cone with $d = 3$ and the Euclidean norm.

3.2.1. Strong postquantum correlations. Dichotomic norm cones can exhibit strong nonquantum behavior. We consider, as an example, the simplest correlation term $\langle \mathrm{LG}' \rangle$:

$$\langle \mathrm{LG}' \rangle_\omega = \omega(A\sharp B + B - A). \tag{11}$$

For so-called macro-realistic systems (which are in our language CLR measurements on the classical events), the constraint $\langle \mathrm{LG}' \rangle \leqslant 1$ is valid [26], while for quantum mechanics the bound $\langle \mathrm{LG}' \rangle \leqslant \frac{3}{2}$ is in order [27]. Note that the quantum mechanical bound only holds for

CLRs [28]. For dichotomic norm cones and assuming again that we always have $\mathbf{a}' = -\mathbf{a}_\neg'$, we obtain the sharp bound (cf appendix B)

$$\langle \mathrm{LG}' \rangle \leqslant 2\|\mathbf{b} - \mathbf{a}\| + 2\mathbf{a}' \cdot \mathbf{b}, \quad \text{where } \|\mathbf{b}\| = \frac{1}{2}. \tag{12}$$

In the case of the Manhattan norm, $\|\mathbf{x}\| = \sum |\mathbf{x}_k|$, and $d = 2$, we find that the r.h.s. of this inequality can easily reach 3 on choosing $\mathbf{a} = (\frac{1}{2}, 0)$, $\mathbf{b} = (0, \frac{1}{2})$, and $\mathbf{a}' = (1, 1)$.

3.2.2. Spekkens' toy model. We finally mention that the Spekkens toy model [1] implements a CLR. In this model, there are six extremal elements $\partial^+ V^+ = \{a_{\pm 1}, a_{\pm 2}, a_{\pm 3}\}$ given by $a_i = (\frac{1}{2}, \mathbf{a}_{(i)})$, with

$$\mathbf{a}_{(\pm 1)} = \left(\pm \frac{1}{2}, 0, 0\right), \mathbf{a}_{(\pm 2)} = \left(0, \pm \frac{1}{2}, 0\right) \text{ and } \mathbf{a}_{(\pm 3)} = \left(0, 0, \pm \frac{1}{2}\right). \tag{13}$$

These elements form observables $A_k = a_{+k} - a_{-k}$ and hence $e = a_{+k} + a_{-k} \equiv (1, \mathbf{0})$. In this way, Spekkens' toy model is the dichotomic norm cone with $d = 3$ and the Manhattan norm. Spekkens also introduced a state update rule for this model, which is such that

$$\mathbb{P}\left(a_i \triangleright a_j\right) = \mathbb{P}(a_i) \begin{cases} 1 & i = j, \\ 0 & i = -j, \\ \frac{1}{2} & \text{else}. \end{cases} \tag{14}$$

This update rule corresponds to the CLR defined in equation (9) with the choice $\mathbf{a}'_{(i)} = 2\mathbf{a}_{(i)}$.

3.3. The triple-slit experiment

While the double-slit experiment is a prime example showing a quantum effect, within quantum mechanics there are no higher order interference terms, as has been found by Sorkin [29]. This absence was also verified in experiments [30]. Recently, the triple-slit experiment has been investigated as an instance of sequential measurements in the context of generalized probabilistic models [12] and the (im)possibility of triple-slit correlations in such models was discussed in e.g. [31, 32].

In an n-slit experiment with slits labeled with $\mathcal{N} = \{1, 2, \ldots, n\}$, detecting that the particle passed through any of the slits $\alpha \subset \mathcal{N}$ plays the role of the first measurement, described by a map ϕ_α. The measurement of the interference pattern on the screen is hence the second measurement. Each possible combination of open slits α may have its particular interference pattern as long as the integrated intensity is independent of whether the slits are opened individually or jointly, so $\phi_\alpha(e) = \sum_{k \in \alpha} \phi_{\{k\}}(e)$. Clearly, the total intensity is bounded by unity, so $\phi_{\mathcal{N}}(e) \in V_e^+$.

We now briefly discuss the assumption that ϕ_α is coherent for the effect $\phi_\alpha(e)$ and hence is a CLR. Assume that the probability for an effect g depends only on the integrated intensity that arrives through the slits α, i.e., $\phi_\alpha(e) \equiv \sum_{k \in \alpha} \phi_{\{k\}}(e) \geqslant g$. In this case, the coherence assumption $\phi_\alpha(g) = g$ ensures that putting the simultaneously opened slits α in front of a measurement with outcome g does not change that outcome.

We recursively define (in general nonpositive) maps η_α via

$$\phi_\alpha = \sum_{\beta\subset\alpha} \eta_\beta. \tag{15}$$

Then those maps η_α are exactly the interference terms $I_{|\alpha|}(\alpha)$ as defined by Sorkin [29], adapted to the language chosen here. We try in equation (15) to write the map on the l.h.s. in terms of the lower order correlations. The difference between the actual map ϕ_α and this lower order sum is then defined as η_α.

In a quantum mechanical n-slit experiment the slits are described by projections Π_k obeying $\sum \Pi_k \leqslant \mathbb{1}$. We let $\Pi_\alpha = \sum_{k\in\alpha} \Pi_k$ and therefore

$$\phi_\alpha\colon X \mapsto \Pi_\alpha X \Pi_\alpha \equiv \sum_{\beta\subset\alpha:|\beta|\leqslant 2} \eta_\beta(X), \tag{16}$$

that is, $\eta_\beta = 0$ whenever $|\beta| > 2$. That is, in quantum mechanics all interference terms above the second order vanish. We mention that in general this absence only occurs if the quantum instrument follows Lüders' rule; as a counterexample, $\phi_\alpha\colon X \mapsto \sqrt{A_\alpha} X \sqrt{A_\alpha}$ with $A_\alpha = \sum_k A_k$ and $A_1 = \mathbb{1}/2$, $A_2 = |0\rangle\langle 0|/2$, $A_3 = |1\rangle\langle 1|/2$ may serve. Such measurements, however, may fail to have a proper physical interpretation as a triple-slit experiment, since the operators A_k may act nonlocally.

For generalized probabilistic models, though, we can easily have higher order correlations. Consider the AOU space with $V^+ = \operatorname{cone}\{a_1, \ldots, a_5\}$, where $a_1, \ldots, a_4$ is a basis of V, $a_5 = a_1 + a_2 + a_3 - a_4$, and $e = a_1 + a_2 + a_3 \equiv a_4 + a_5$. We choose $\phi_\alpha(e) = \sum_{k\in\alpha} a_k$ for $\alpha \subset \{1, 2, 3\} \equiv \mathcal{N}$. A brief calculation yields, for $\alpha \subsetneq \mathcal{N}$,

$$\phi_\alpha = \sum_{k\in\alpha} a_k \omega_k^\alpha, \tag{17}$$

where the maps ω_k^α are arbitrary states with $\omega_k^\alpha(a_k) = 1$. Since those states are not unique, we can use this freedom e.g. to achieve commutativity, $\phi_\alpha \circ \phi_\beta = \phi_\beta \circ \phi_\alpha$, or to get vanishing double-slit correlations, $\eta_{\{k,\ell\}} = 0$. In contrast, the map for the triple slit is the identity mapping, $\phi_{\mathcal{N}} = e^\sharp \equiv id$. From equation (17) we see that $a_4 \notin \eta_\alpha(V)$ except for $\alpha = \mathcal{N}$, i.e., nonvanishing triple-slit correlations occur.

4. Discussion

An important property of quantum systems is that the measurement necessarily changes the state of the system—or in a Heisenberg type of picture, that the description of a measurement depends on previous measurements that have been performed. How this change occurs in general depends on the actual implementation of the measurement. In quantum mechanics, however, the change induced by projective measurements according to Lüders is the least disturbing and least biased implementation of a projective measurement. We rederived this rule in quantum mechanics (cf section 3.1) solely from the coherence assumption stated in definition 3. This definition of coherent Lüders rules (CLRs) can be applied to a wide class of hypothetical nonquantum models, namely the generalized probabilistic models which can be described by means of Archimedean ordered vector spaces.

We showed in proposition 5 that CLRs are exactly those maps which do not disturb any subsequent and possibly more 'noisy' implementation of the same measurement. We also showed that familiar results of repeatability and compatibility hold (proposition 6; cf also [9, 11]).

In quantum mechanics, Lüders' rule is directly linked to and singles out the projection operators, which in turn play a key role in e.g. spectral theory. (Celebrated results for a generalized spectral theory [10, 33, 34] are, however, linked to neutral maps.) We find that for extremal measurement effects (a generalization of rank-1 projections in quantum mechanics) a CLR always exists, while necessary conditions for existence have been given in proposition 7. Also, in certain pathological cases, the CLR is not unique. This ambiguity might be unsatisfactory, but for quantum mechanics and classical mechanics the conditions of being a CLR are sufficient for achieving uniqueness, so adding any further condition is rather speculative.

Finally we demonstrated in section 3.2 that CLRs already occurred earlier in Spekkens' toy model [1] and that this toy model can now be seen as an instance of a much wider class of models with a natural notion of sequential measurements. For those models it is e.g. straightforward to compute the upper limit for the Leggett–Garg inequality in equation (12). We discussed, as a last instance, in section 3.3 the triple-slit experiment, finding that generalized probabilistic models with a CLR can easily have substantial triple-slit correlations, while it is an important prediction of quantum mechanics that they are absent.

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Appendix A. Elements with a coherent Lüders rule in dichotomic norm cones

In a dichotomic norm cone (cf section 1.3), the set of effects admitting a CLR is given by $\{0, e\} \cup \partial^+ V^+$, as stated in section 3.2. For $f = (t, \mathbf{f}) \in V_e^+$ we have $\|f\| = 1$ if and only if $t = 1 - \|\mathbf{f}\|$ and $\|\mathbf{f}\| \leqslant \frac{1}{2}$. Assume now that f admits a CLR, but $0 \neq f \neq e$. By virtue of proposition 7 (ii) it follows that $\|f\| = 1$ and $\|\mathbf{f}\| = \frac{1}{2}$. The first statement is obtained by choosing $g = f$ and the second statement through the choice $0 \leqslant g = (1 - 2\|\mathbf{f}\|)e = f - (\|\mathbf{f}\|, \mathbf{f}) \leqslant f$. If now $a \in \partial^+ V^+$ and $p > 0$ such that $pa \leqslant f$, then also $a \leqslant f$. This inequality now reads as $\frac{1}{2} - \frac{1}{2} \geqslant \|\mathbf{f} - \mathbf{a}\|$ and therefore $f = a$.

Appendix B. Obtaining equation (12)

Under the result $A \sharp B = (2\beta - 1)A + 2(\mathbf{a}' \cdot \mathbf{b})e$ (equation (10)), we bound the correlation term $\langle \mathrm{LG}' \rangle_\omega = \omega(A \sharp B + B - A)$ (equation (11)) for dichotomic norm cones, assuming that $A = a - a_\neg = (0, 2\mathbf{a})$ and $B = b - b_\neg = (2\beta - 1, 2\mathbf{b})$. Writing $\omega = (1, \mathbf{w})$, this yields, for $\mathbf{b} \neq \mathbf{0}$,

$$\frac{1}{2}\langle \mathrm{LG}' \rangle_\omega = \mathbf{a}' \cdot \mathbf{b} + \mathbf{w} \cdot (\mathbf{b} - \mathbf{a}) + (2\beta - 1)\left(\mathbf{w} \cdot \mathbf{a} + \frac{1}{2}\right)$$
$$\leqslant \|\mathbf{b}\| \left[\mathbf{a}' \cdot \underline{\mathbf{b}} + \|\underline{\mathbf{b}} - 2\mathbf{a}\| - 1\right] + \frac{1}{2} \qquad \text{(B.1)}$$

with $\underline{\mathbf{b}} = \mathbf{b}/\|\mathbf{b}\|$. The inequality is due to $\beta \leqslant 1 - \|\mathbf{b}\|$, $\|\mathbf{w}\|_* \leqslant 1$, and $\mathbf{w} \cdot \mathbf{a} \geqslant -\frac{1}{2}$. The bound is sharp if $\beta = 1 - \|\mathbf{b}\|$ and $\mathbf{w} \cdot (\underline{\mathbf{b}} - 2\mathbf{a}) = \|\underline{\mathbf{b}} - 2\mathbf{a}\|$. Using the conditions from equation (9), we have $\|\underline{\mathbf{b}} - 2\mathbf{a}\| \geqslant -\mathbf{a}' \cdot (\underline{\mathbf{b}} - 2\mathbf{a}) = 1 - \mathbf{a}' \cdot \underline{\mathbf{b}}$ and hence the term in square brackets is never negative. This makes the choice $\|\mathbf{b}\| = \frac{1}{2}$ optimal and we arrive at the sharp bound of equation (12).

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