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A Lie algebraic approach to non-Hermitian Hamiltonians with real spectra

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Abstract
An algebraic technique useful in studying non-Hermitian Hamiltonians with real spectra, is presented. The method is illustrated by explicit application to a family of one-dimensional potentials.

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1. Introduction

The existence of non-Hermitian Hamiltonians with real spectra is one of the interesting problems in theoretical physics [1–16]. Firstly, they are used in various branches of theoretical physics, and secondly, for others it is interesting in itself to understand the reasons for the reality (see, e.g., [17] and references therein).

The understanding of non-Hermitian Hamiltonians H with real spectra has been largely improved since the work of Bender and Boettcher [1] by the realization that their existence is deeply related to the existence of symmetry under the combined transformation of parity P and time reversal T,

$$HPT = PTH. \tag{1}$$

Later, Mostafazadeh [18] has shown that the operator H acting in a Hilbert space $\mathcal{H}$ has a real spectrum if there exists a Hermitian automorphism $\eta : \mathcal{H} \to \mathcal{H}$ such that

$$H^{\dagger}\eta = \eta H \tag{2}$$

or

$$HO = OH_0, \tag{3}$$

where $OO^{\dagger} = \eta$ and H_0 is Hermitian.

In a recent paper, however, Kretschmer and Szymanowski [19] proposed a way which might allow for finding in a systematic way large classes of non-Hermitian Hamiltonians with real spectra. The existence of an operator Ω that intertwines a given non-Hermitian Hamiltonian H acting in $\mathcal{H}$ and a Hermitian one h acting in $\mathcal{L}_2$ ensures the reality of the spectrum of H,

$$H\Omega = \Omega h. \tag{4}$$

We would like to emphasize that Ω is an operator from $\mathcal{L}_2$ to $\mathcal{H}$, whereas O acts in $\mathcal{H}$.

In this paper, we present a technique to construct the class of non-Hermitian Hamiltonians H related to Lie groups G. The key to the construction of H lies in the observation that the relation (4) for such systems is essentially a relation between equivalent representations of G. This suggests the following assumption that the operators H and h must be related to the different realizations of a non-unitary representation of G. Our treatment is based on this chief assumption.

2. Non-Hermitian Hamiltonians with real spectra: a case study for $SO(2,1)$

To gain a better understanding of our approach, we illustrate it for Hamiltonians related to $SO(2,1)$. To this end, a few facts from the representation theory of $SO(2,1)$ are useful [20, 21].

Let $R^{2,1}$ be a three-dimensional pseudo-Euclidean space with bilinear form

$$[\xi,\zeta]=\xi_0\zeta_0-\xi_1\zeta_1-\xi_2\zeta_2. \tag{5}$$

By $SO(2,1)$ we denote the connected component of the group of linear transformations of $R^{2,1}$ preserving the form (5). We consider $SO(2,1)$ as acting on $R^{2,1}$ on the right.

Let us choose in $SO(2,1)$ the one-parameter subgroups $\{g_0(t)\}$, $\{g_1(t)\}$ and $\{g_2(t)\}$, where $g_0(t)$ is the rotation in the 1–2 plane

$$g_0(t)=\begin{pmatrix}1&0&0\\0&\cos t&\sin t\\0&-\sin t&\cos t\end{pmatrix}, \tag{6}$$

while $g_1(t)$ and $g_2(t)$ are the pure Lorentz transformations along the 1 and 2 axes

$$g_1(t)=\begin{pmatrix}\cosh t&\sinh t&0\\\sinh t&\cosh t&0\\0&0&1\end{pmatrix},\qquad g_2(t)=\begin{pmatrix}\cosh t&0&\sinh t\\0&1&0\\\sinh t&0&\cosh t\end{pmatrix}. \tag{7}$$

The tangent matrices $a_i=\frac{dg_i(t)}{\mathrm{d}t}\big|_{t=0}$ form a basis of the Lie algebra of $SO(2,1)$ with commutation relations

$$[a_1,a_2]_-=-a_0,\qquad [a_2,a_0]_-=a_1,\qquad [a_0,a_1]_-=a_2. \tag{8}$$

The unitary irreducible representations (UIRs) of $SO(2,1)$ are known to form three series [20, 21]: principal, supplementary and discrete. It is also known that any UIR of $SO(2,1)$ is equivalent to some sub-representation of an elementary representation of $SO(2,1)$. They occur as unitarizations of elementary representations or as unitarizations of quotients of such representations.

Let us recall some facts about the elementary representations of $SO(2,1)$. The elementary representations T_σ of the group $SO(2,1)$ are labelled by a complex number σ. They can be realized in the Hilbert space $L^2(S)$ with inner product

$$(f_1,f_2)=\frac{1}{2\pi}\int_S f_1^*(n)f_2(n)\,\mathrm{d}n, \tag{9}$$

where $S=\{n=(1,\cos\varphi,\sin\varphi)\}$ denotes the circle of radius 1 and $\mathrm{d}n=\mathrm{d}\varphi$. The representation T_σ is defined by

$$T_\sigma(g)f(n)=|(ng)_0|^\sigma f\left(\frac{ng}{(ng)_0}\right). \tag{10}$$

where $(ng)_0$ is the zero component of the vector ng, i.e.,

$$(ng)_0=g_{00}+n_1g_{10}+n_2g_{20}. \tag{11}$$

The infinitesimal operators $A_i = (\mathrm{d}/\mathrm{d}t)T_\sigma(g_i(t))|_{t=0}, i = 0, 1, 2,$ of the representation T_σ, corresponding to the one-parameter subgroups $g_i(t)$ are given by

$$
\begin{aligned}
A_1 &= \sigma\cos\varphi - \sin\varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}\\
A_2 &= -\sigma\sin\varphi - \cos\varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}\\
A_3 &= \frac{\mathrm{d}}{\mathrm{d}\varphi}.
\end{aligned}
\tag{12}
$$

The Casimir operator

$$C = A_1^2 + A_2^2 - A_0^2 \tag{13}$$

is identically a multiple of the unit

$$C = \sigma(\sigma+1)I. \tag{14}$$

It can be shown that

$$(T_\sigma f_1, T_{-\sigma^*-1} f_2) = (f_1, f_2). \tag{15}$$

Therefore, the representation T_σ is unitary if $\operatorname{Re}\sigma = -\frac{1}{2}$. In this case the infinitesimal operators (12) satisfy the condition

$$A_i^\dagger = -A_i, \qquad i = 0, 1, 2, \tag{16}$$

i.e. the operators

$$J_k = -iA_k, \qquad k = 0, 1, 2 \tag{17}$$

are Hermitian. For $\operatorname{Re}\sigma \neq -\frac{1}{2}$ the representation T_σ is non-unitary (with respect to the inner product (9)) although J_3 is still Hermitian. If we diagonalize J_3 we obtain

$$J_3\psi_m = m\psi_m, \qquad C\psi_m = -\sigma(\sigma+1)\psi_m, \qquad m = 0, \pm1, \pm2, \ldots, \tag{18}$$

with $\psi_m(\varphi) = \exp(\mathrm{i}m\varphi)$.

In order to avoid misunderstanding, we must make a few comments on the operator $T_\sigma(g)$ in (6). As was pointed out above, the representation just described gives three series of unitarizable representations of $SO(2, 1)$. If $\operatorname{Re}\sigma = -\frac{1}{2}$ then the representation $T_\sigma(g)$ is unitary and these representations form the principal series of $SO(2, 1)$. It turns out that when $-1/2 < \sigma < 0$ or $\sigma = -1, -2, -3, \ldots,$ the operator $T_\sigma(g)$ is unitary with respect to an inner product which is different from (9). These representations form the complementary and discrete series of $SO(2, 1)$, respectively.

A key concept in group-theoretical approach is that the Hamiltonian H under study is a function of infinitesimal operators A_i of the representation of some Lie group G,

$$H = \Phi(A_i). \tag{19}$$

Here we want to construct a non-Hermitian Hamiltonian H intertwined with the Hermitian operator h. (The operator h is not necessarily a Hamiltonian for some physical problem, but a Hermitian operator in $\mathcal{L}_2$.) As was mentioned in the introduction, the key to their construction lies in the observation that the relation (4) is essentially an equivalence relation between two representations of G. Therefore, we should look for another realization of the elementary representation and introduce the intertwining operator.

Let us denote by $\mathcal{H}^\sigma$ the space of functions $F(\xi)$ on one sheet hyperboloid Ξ,

$$\xi_0^2 - \xi_1^2 - \xi_2^2 = -1, \tag{20}$$

satisfying the equation

$$\triangle F(\xi) = \sigma(\sigma+1)F(\xi), \qquad \sigma \in \mathbb{C}, \tag{21}$$

where

$$\triangle = -\frac{\partial^2}{\partial\xi_1^2} - \frac{\partial^2}{\partial\xi_2^2} + \wedge(\wedge+1), \qquad \wedge = \xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2} \tag{22}$$

is the Laplace–Beltrami operator on Ξ. Here, we have assumed ξ_1 and ξ_2 as the independent variables on Ξ. The inner product in $\mathcal{H}^\sigma$ is defined by

$$(F_1, F_2) = \int_\Xi F_1^*(\xi)F_2(\xi)\,\mathrm{d}\xi, \tag{23}$$

where $\mathrm{d}\xi = \mathrm{d}\xi_1\,\mathrm{d}\xi_2/|\xi_0|$ is an invariant measure on Ξ. Then the elementary representation of $SO(2,1)$ can be realized in $\mathcal{H}^\sigma$. In this realization the representation is defined by

$$U_\sigma F(\xi) = F(\xi g). \tag{24}$$

The interrelation between representations (10) and (24) is given by

$$\begin{aligned} F(\xi) &= \int_S |[\xi, n]|^{-1-\sigma} f(n)\,\mathrm{d}n \\ &\equiv (Wf)(\xi). \end{aligned} \tag{25}$$

We are now prepared to construct a one-dimensional non-Hermitian Hamiltonian H

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \tag{26}$$

intertwined with the Hermitian operator h

$$h \equiv J_3^2 = -\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}.$$

(We suppose that H is subject to the representation U_σ.) For this purpose instead of the coordinates ξ_1 and ξ_2 we introduce the coordinates x and θ via

$$\xi_1 = \frac{\cos\theta}{\sqrt{1-z(x)^2}}, \qquad \xi_2 = \frac{\sin\theta}{\sqrt{1-z(x)^2}}, \qquad z(x) \in [-1, 1]. \tag{27}$$

Then from (4) it follows that

$$F(x,\theta) = \int_0^{2\pi} \left| \frac{z(x)}{\sqrt{1-z(x)^2}} - \frac{\cos(\varphi-\theta)}{\sqrt{1-z(x)^2}} \right|^{-1-\sigma} f(\varphi)\,\mathrm{d}\varphi. \tag{28}$$

Further, putting $\theta = 0$ we have

$$\begin{aligned} F(x) &= \int_0^{2\pi} k(x,\varphi) f(\varphi)\,\mathrm{d}\varphi \\ &\equiv (\Omega f)(x), \end{aligned} \tag{29}$$

where

$$k(x,\varphi) = \left| \frac{z(x)}{\sqrt{1-z(x)^2}} - \frac{\cos\varphi}{\sqrt{1-z(x)^2}} \right|^{-1-\sigma}.$$

(For the sake of simplicity the restriction of a function F on the line $\theta = 0$ is denoted by the same symbol F.)

According to equation (4) the non-Hermitian Hamiltonian H can be derived by demanding that, for any $f \in \mathcal{L}_2$

$$H\Omega f = \Omega h f$$

or more explicitly

$$\int_0^{2\pi} Hk(x,\varphi)f(\varphi)\,\mathrm{d}\varphi = \int_0^{2\pi} k(x,\varphi)hf(\varphi)\,\mathrm{d}\varphi. \tag{30}$$

It follows from here that

$$Hk(x,\varphi) = hk(x,\varphi). \tag{31}$$

The proof needs to be given only for $\psi_m(\varphi) = \exp(\mathrm{i}m\varphi)$ because the functions ψ_m form a basis in $\mathcal{L}_2$. Using integration by parts we see immediately that

$$\int_0^{2\pi} k(x,\varphi)h\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}\varphi = \int_0^{2\pi} [hk(x,\varphi)]\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}\varphi.$$

Hence the equation

$$\int_0^{2\pi} Hk(x,\varphi)\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}\varphi = \int_0^{2\pi} k(x,\varphi)h\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}\varphi \tag{32}$$

can be reduced to the form

$$\int_0^{2\pi} Hk(x,\varphi)\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}\varphi = \int_0^{2\pi} [hk(x,\varphi)]\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}\varphi,$$

which yields equation (31).

So, the requirement that a non-Hermitian Hamiltonian H is intertwined by the Hermitian operator h implies the fulfilment of equality (31). Then, it is not difficult to see that this equality is satisfied if

$$V(x) = \frac{\sigma(\sigma+1)}{1-z^2}, \tag{33}$$

provided

$$\frac{\dot{z}^2}{1-z^2} = 1. \tag{34}$$

The solutions to the last equation are

$$z(x) = \cos x, \qquad 0 < x < \pi \tag{35}$$

and

$$z(x) = \sin x, \qquad -\pi/2 < x < \pi/2. \tag{36}$$

If we compute $V(x)$ for $z(x) = \cos x$ it becomes

$$V(x) = \frac{\sigma(\sigma+1)}{\sin^2 x}. \tag{37}$$

Hence, the spectrum of the non-Hermitian Hamiltonian

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\sigma(\sigma+1)}{\sin^2 x} \tag{38}$$

is real.

We note that, the intertwined operators are isospectral. Particularly, if ψ_m is the eigenfunction of h with eigenvalue m then $\Psi_m = \Omega\psi_m$ is an eigenfunction of H with the same eigenvalue. So the functions

$$\Psi_m(x) = \int \left|\cot x - \frac{\cos\varphi}{\sin x}\right|^{-1-\sigma} \mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}\varphi,$$

which are not necessarily square integrable, are the eigenfunctions of the Schrödinger equation with Scarf potential [22–24].

Another example is provided by Hamiltonian of the form

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\sigma(\sigma+1)}{\cos^2 x}, \tag{39}$$

which is obtained by substituting

$$z(x) = \sin x, \qquad -\pi/2 < x < \pi/2. \tag{40}$$

In this case

$$\Psi_m(x) = \int \left|\tan x - \frac{\cos\varphi}{\cos x}\right|^{-1-\sigma} \mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}\varphi \tag{41}$$

It is worth pointing out that these potentials can be also derived algebraically by relating the non-Hermitian Hamiltonians H to Casimir operator C of the representation U_σ as

$$Q\,(H - E) = [C - \sigma(\sigma+1)]|_{\mathcal{H}} \tag{42}$$

where Q is some nontrivial operator and $\mathcal{H}$ is an eigensubspace of the compact generator. Historically this method was introduced by Ghirardi [25]. (It should be noted that the potential group approach initiated in [26] is a rediscovery of the technique attributable to Ghirardi.)

If we compute the infinitesimal operators of U_σ corresponding to the one-parameter subgroups $g_i(t)$, $i = 0, 1, 2$, they become

$$A_0 = \xi_2\frac{\partial}{\partial\xi_1} - \xi_1\frac{\partial}{\partial\xi_2}, \qquad A_1 = \xi_0\frac{\partial}{\partial\xi_1}, \qquad A_2 = \xi_0\frac{\partial}{\partial\xi_2},$$

while the Casimir operator C coincides with the Laplace–Beltrami operator on Ξ, i.e., $C = \triangle$. Then the Casimir operator C in the parametrization (27) is given by

$$C = \frac{(1-z^2)^2}{\dot{z}^2}\left[\frac{\partial^2}{\partial x^2} - \left(\frac{z\dot{z}}{1-z^2} + \frac{\ddot{z}}{\dot{z}}\right)\frac{\partial}{\partial x} - \frac{\dot{z}^2}{1-z^2}\frac{\partial^2}{\partial\theta^2}\right], \tag{43}$$

where dots represent derivatives with respect to x, i.e., $\dot{z} = \frac{\mathrm{d}z}{\mathrm{d}x}$, etc.

In order to eliminate the term containing the first derivative we make a similarity transformation

$$A_i' = h^{-1/2}\circ A_i \circ h^{1/2}, \qquad i = 0, 1, 2,$$

where $h = h(x) = \dot{z}/(1-z^2)^{1/2}$ and $\circ$ denotes composition of operators. Then the Casimir operator C transforms into

$$\begin{aligned} C' &= h^{-1/2}\circ C \circ h^{1/2} \\ &= \frac{(1-z^2)^2}{\dot{z}^2}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\dddot{z}}{\dot{z}} - \frac{3}{4}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 + \dot{z}^2\frac{2+z^2}{4(1-z^2)^2} - \frac{\dot{z}^2}{1-z^2}\frac{\partial^2}{\partial\theta^2}\right], \end{aligned}$$

while $A_0' = A_0 = -\frac{\partial}{\partial\theta}$.

Let us denote by C'_m a restriction of C' on a one-dimensional subspace $\mathcal{H}_m$ spanned by function $F'_m(\xi) = \Psi'_m(x)\,\mathrm{e}^{\mathrm{i}m\theta}$ with fixed m. Then C'_m becomes a differential operator in x alone; it is found that

$$C'_m = \frac{(1-z^2)^2}{\dot{z}^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}\frac{\dot{\ddot{z}}}{\dot{z}} - \frac{3}{4}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 + \dot{z}^2\frac{2+z^2}{4(1-z^2)^2} - \frac{m^2\dot{z}^2}{1-z^2}\right]. \tag{44}$$

We now allow the eigenvalues of the Casimir operators of $SO(2)$ and of $SO(2,1)$ to be linear functions of the energy E, i.e.,

$$m^2 = \gamma_1 E + \delta_1 \tag{45}$$

and

$$\sigma(\sigma+1) = \gamma_2 E + \delta_2. \tag{46}$$

On the conditions given above, we have

$$C'_m - \sigma(\sigma+1) = -\left(\frac{1-z^2}{\dot{z}}\right)^2\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{2}\frac{\dot{\ddot{z}}}{\dot{z}} + \frac{3}{4}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 \right.$$
$$\left. + \dot{z}^2\frac{ER(z) + \delta_1(1-z^2) - \delta_2}{(1-z^2)^2}\right], \tag{47}$$

where

$$R(z) = \gamma_1(1-z^2) - \gamma_2. \tag{48}$$

This equation is easily reduced to the form (42) with

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\delta_1(1-z^2)-\delta_2}{R} - \left(\frac{1+2z^2}{2} + \frac{5\gamma_2}{4R}z^2\right)\frac{\gamma_2}{R} \tag{49}$$

and

$$Q = -\left(\frac{1-z^2}{\dot{z}}\right)^2 \tag{50}$$

provided that

$$\dot{z}^2 = \frac{(1-z^2)^2}{R(z)}. \tag{51}$$

The Hamiltonians (49) include as a special case the above-mentioned class of non-Hermitian Hamiltonians with real spectra. Indeed, putting $\gamma_1 = 1$, $\gamma_2 = 0$ and $\delta_1 = 0$ equations (49) and (51) reduce to

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\delta_2}{1-z^2}, \qquad \dot{z}^2 = 1 - z^2,$$

with $\delta_2 = \sigma(\sigma+1)$ and $E = m^2$.

3. Conclusion

In this paper, we have shown how an intertwining operator between two non-unitary representations of a group G can be used to obtain a class of non-Hermitian Hamiltonians with real spectra. We illustrated the method by choosing G to be $SO(2,1)$, which led to Scarf potentials. The question then arises: how does one obtain other potentials within the framework of this approach? There are two possibilities. The first is to use other Lie groups. We note that it is quite natural to generalize the representations T_σ and U_σ for $SO(p,q)$. The second is to use other non-unitary representations of an underlying Lie group. These possibilities will be considered in a forthcoming publication.

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