Journal of Physics: Conference Series

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To cite this article: K L Panchuk *et al* 2018 *J. Phys.: Conf. Ser.* **998** 012021

IOP Conf. Series: Journal of Physics: Conf. Series **998** (2018) 012021 doi:10.1088/1742-6596/998/1/012021

Geometric model of pseudo-distance measurement in satellite location systems

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Abstract. The existing mathematical model of pseudo-distance measurement in satellite location systems does not provide a precise solution of the problem, but rather an approximate one. The existence of such inaccuracy, as well as bias in measurement of distance from satellite to receiver, results in inaccuracy level of several meters. Thereupon, relevance of refinement of the current mathematical model becomes obvious. The solution of the system of quadratic equations used in the current mathematical model is based on linearization. The objective of the paper is refinement of current mathematical model and derivation of analytical solution of the system of equations on its basis. In order to attain the objective, geometric analysis is performed; geometric interpretation of the equations is given. As a result, an equivalent system of equations, which allows analytical solution, is derived. An example of analytical solution implementation is presented. Application of analytical solution algorithm to the problem of pseudo-distance measurement in satellite location systems allows to improve the accuracy such measurements.

1. Introduction

The distance between satellite and receiver is the main sought parameter of satellite coordinate metrology. Due to substantial bias in distance measurement, the acquired values of distance are called pseudo-distances. A pseudo-distance is a value of distance between satellite and receiver at the moment of transmission and reception of radio signal with use of pseudorandom code packages generated on satellite and receiver [1]. The former are generated using satellite clock and the latter – using receiver clock.

Satellite and receiver clock readings diverge, which causes inaccuracy in distance measurement. Additional inaccuracies are caused by delays on the path of the radio signal in the atmosphere (ionosphere and troposphere).

With consideration of these singularities in pseudo-distance measurement theory [1] the quadratic equation for pseudo-distance measurement is acquired. The equation includes four unknown values comprising the coordinates (x_r, y_r, z_r) of receiver location and receiver clock adjustment δt_r alongside the coefficients. In order to define these unknown values at least four satellites are surveyed and a system of four quadratic equations is formed, where each equation corresponds to the distance between the receiver and a satellite.

In order to solve the system of equations, linearization of comprising quadratic equations is performed, which decreases accuracy of the solution. As a result, coordinate metrology has inaccuracy level of several meters [1], and the method itself has mostly navigational application.

2. Problem definition

The objective of this paper is to eliminate linearization of quadratic equations of the system on the basis of geometric interpretation of the equations and the system, which leads to simplification of the existing mathematical model of pseudo-distance measurement and increasing the accuracy of measurements.

3. Theory

1. Pseudo-distance measurement mathematical model simplification

Working equation for pseudo-distance measurement is of the following form [1]:

$$R = (t_r - t_s) \cdot c = \rho + c \cdot (\delta t_r - \delta t_s) + \delta R, \tag{1}$$

where t_r and t_s represent clock readings on receiver and satellite registered at the moment of pseudo-distance measurement; δt_r and δt_s represent clock reading drift relative to standard clock on receiver and satellite at the moment of pseudo-distance measurement; c represents electromagnetic wave propagation velocity in vacuum; ρ represents geometric distance between satellite and receiver at the moment of pseudo-distance measurement; $\delta R = c \cdot \delta t_{atm}$ represents adjustment to the measured pseudo-length caused by the influence of the atmosphere.

Additionally:

1) The value of δt_s is determined with the aid of surveillance stations for each individual satellite and receiver and is contained within navigational packages sent to the customer (receiver). In the absolute method of receiver location the value of δt_s is considered known.

2) The value of δt_{atm} is preliminary calculated on the basis of modeling of delays on the path of the radio signal in ionosphere and troposphere.

3) The value of δt_r is not preliminary determined and is among the unknown parameters.

4) $\rho = \sqrt{(x_s - x_r)^2 + (y_s - y_r)^2 + (z_s - z_r)^2}$, (2)

where (x_s, y_s, z_s) represent the known coordinates of the satellite.

Consequently, the equation (1) includes four unknown values comprising the coordinates (x_r, y_r, z_r) of receiver location and receiver clock adjustment δt_r. In order to define these unknown values at least four satellites are surveyed and a system of four quadratic equations (1) is solved.

As follows from the equation (1), taking into the account four satellites:

$$R_{1i} = R_i + c \cdot \delta t_{si} - c \cdot \delta t_{atm} = \rho + c \cdot \delta t_r, \qquad i = 1,2,3,4.$$

From the latter equation the equation $\rho = R_{1i} - c \cdot \delta t_r$ is derived, from which follows the system of equations

$$(x_r - x_{si})^2 + (y_r - y_{si})^2 + (z_r - z_{si})^2 - (c \cdot \delta t_r - R_{1i})^2 = 0, \qquad i = 1,2,3,4.$$

For further convenience of algebraic transformations we substitute designation:

$$x_r \to x_1, y_r \to x_2, z_r \to x_3, x_{si} \to a_{i1}, y_{si} \to a_{i2}, z_{si} \to a_{i3}, R_{1i} \to a_{i4}, c \cdot \delta t_r \to x_4 .$$

The latter system of equations after designation substitution is as follows:

$$F_i(x_1, x_2, x_3, x_4) = (x_1 - a_{i1})^2 + (x_2 - a_{i2})^2 + (x_3 - a_{i3})^2 - (x_4 - a_{i4})^2 = 0, \quad i = 1,2,3,4. \tag{3}$$

Consider the geometric sense of the system of quadratic equations (3). Assuming that (a_{i1}, a_{i2}, a_{i3}) are the coordinates of a certain known sphere S_i, while $r_i = a_{i4}$ is its radius, there is no difficulty in ascertaining that the equation (3) describes a set (∞^3) of spheres centered at (x_1, x_2, x_3) with radiuses $r_4 = x_4$, tangent to the known sphere S_i.

Consequently, the system of equations (3) describes a finite amount of spheres tangent to four known spheres $S_i(a_{i1}, a_{i2}, a_{i3}, r_i = a_{i4})$, $i = 1,2,3,4$.

Thus, the solution of the system of equations (3) determines a finite number of quadruples of numbers $(x_1, x_2 x_3, x_4)$, that determine the coordinates of the receiver and the value of the corresponding adjustment δt_r.

Let us implement the following algebraic transformations of the system of equations (3):

1. We subtract the second equation of the system from the first:

$$F_1 - F_2 = 2[(a_{21} - a_{11}) \cdot x_1 + (a_{22} - a_{12}) \cdot x_2 + (a_{23} - a_{13}) \cdot x_3 - (a_{24} - a_{14}) \cdot x_4] + (a_{11}^2 + a_{12}^2 + a_{13}^2 - a_{14}^2) - (a_{21}^2 + a_{22}^2 + a_{23}^2 - a_{24}^2) = 0.$$

Further designations are introduced:

$$2(a_{21} - a_{11}) = A_{11};\ 2(a_{22} - a_{12}) = A_{12};\ 2(a_{23} - a_{13}) = A_{13};\ -2(a_{24} - a_{14}) = A_{14};$$
$$({a_{11}}^2 + {a_{12}}^2 + {a_{13}}^2 - {a_{14}}^2) - ({a_{21}}^2 + {a_{22}}^2 + {a_{23}}^2 - {a_{24}}^2) = A_{15}.$$

Consequently, we receive:

$$F_1 - F_2 = A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 + A_{15} = 0.$$

2. We subtract the third equation of the system from the first:

$$F_1 - F_3 = A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + A_{24}x_4 + A_{25} = 0,$$

where $A_{21} = 2(a_{31} - a_{11});\ A_{22} = 2(a_{32} - a_{12});\ A_{23} = 2(a_{33} - a_{13});\ A_{24} = -2(a_{34} - a_{14});$ $A_{25} = ({a_{11}}^2 + {a_{12}}^2 + {a_{13}}^2 - {a_{14}}^2) - ({a_{31}}^2 + {a_{32}}^2 + {a_{33}}^2 - {a_{34}}^2).$

3. Likewise, we subtract the fourth equation of the system from the first:

$$F_1 - F_4 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + A_{34}x_4 + A_{35} = 0,$$

where $A_{31} = 2(a_{41} - a_{11});\ A_{32} = 2(a_{42} - a_{12});\ A_{33} = 2(a_{43} - a_{13});\ A_{34} = -2(a_{44} - a_{14});$ $A_{35} = ({a_{11}}^2 + {a_{12}}^2 + {a_{13}}^2 - {a_{14}}^2) - ({a_{41}}^2 + {a_{42}}^2 + {a_{43}}^2 - {a_{44}}^2).$

The following system of equations is the result of the foregoing transformations:

$$\begin{gathered} F_1 = 0, \\ F_1 - F_2 = 0, \\ F_1 - F_3 = 0, \\ F_1 - F_4 = 0. \end{gathered} \qquad (4)$$

The first equation of the latter system is quadratic, the other three are linear. Thus the initial system of four quadratic equations (3), which is currently used as a mathematical model of pseudo-distance measurement, is replaced by an equivalent system of equations (4), which includes only one quadratic equation. The simplification of the initial system of equations (3) is obvious.

Let us perform the geometric interpretation of the new system of equations (4), which represents the simplified mathematical model of pseudo-distance measurement. The first equation of the system $F_1 = 0$ describes a set (∞^3) of points of hyperquadric of space R^4, which represents a hypercone of revolution with body angle at the vertex equal to 90° and a spherical base($a_{11}, a_{12}, a_{13}, r_1 = |a_{14}|$). Let us call this cone an α-hypercone K_α^3. The other three equations of the system (4) describe hyperplanes P_i^3, $i = 1,2,3$ of space R^4.

Let us analyze the solution of the system (4) on the basis of its geometrical representation. Obviously, the equation of intersection $K_\alpha^3 \cap P_i^3 = K_i^2,\ i = 1,2,3$ takes place. This equation defines two-dimensional quadrics K_1^2, K_2^2, K_3^2. Consider the following pairwise intersections: $K_1^2 \cap K_2^2 = k_{12}^4 = (k_{12}^2 + k_{12(\infty)}^2) \subset P_1^3 \cap P_2^3 = P_{12}^2; K_2^2 \cap K_3^2 = k_{23}^4 = (k_{23}^2 + k_{23(\infty)}^2) \subset P_2^3 \cap P_3^3 = P_{23}^2$. Where $k_{12(\infty)}^2 = k_{23(\infty)}^2 = k_\infty^2$ is ideal conic. As follows from the equations of intersection, k_{12} and k_{23} are curves of second order situated correspondingly in planes P_{12}^2 and P_{23}^2, while the quadrics K_i^2 themselves represent α-cones with the same feature of intersection.

Thus the geometric solution of the system of equations (4) is a pair of points belonging to the line of intersection of the planes P_{12}^2 and P_{23}^2 and to the intersecting conics k_{12} and k_{23} simultaneously.

4. Results of the experiment

The computing experiment initial data is taken up as follows:

$$\begin{gathered} a_{11} = 2;\ a_{12} = 0; a_{13} = 2; a_{14} = 3; \\ a_{21} = 5;\ a_{22} = 6; a_{23} = 5; a_{24} = 4; \\ a_{31} = -3;\ a_{32} = 3; a_{33} = 5; a_{34} = 2; \\ a_{41} = 12;\ a_{42} = 0; a_{43} = 4; a_{44} = 5. \end{gathered}$$

Based on the system of equations (4), the following equations of the projections of curves k_{12} and k_{23} on coordinate plane $(x_1 x_2)$ are obtained:

$$k_{12}: \quad (x_1 - 2)^2 + x_2^2 + (\frac{1}{3}x_1 - \frac{3}{2}x_2 + \frac{29}{4})^2 - (4x_1 + \frac{3}{2}x_2 - \frac{43}{4})^2 = 0,$$

$$k_{23}: \quad (x_1 - 2)^2 + x_2^2 + (x_1 - 3x_2 - \tfrac{5}{4})^2 - (6x_1 + 3x_2 - \tfrac{145}{4})^2 = 0.$$

Rendering of the curves is presented on fig. 1.

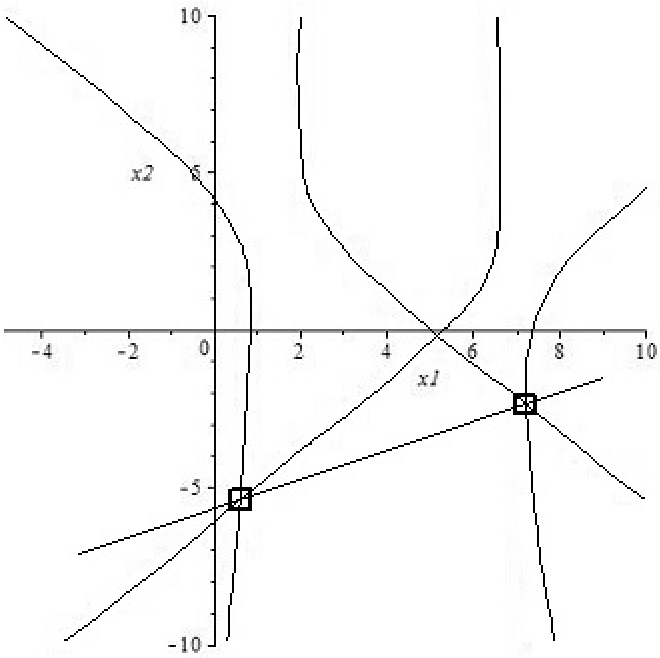

Figure 1. Projections of the conics and projections of their intersection points on the plane $(x_1 x_2)$

The points of intersection of the projections of second order curves k_{12} and k_{23} allow us to pass on to the coordinates of these points in space R^3, and further to rendering of two spheres centered in these points and tangent to the given four spheres (fig. 2, fig. 3).

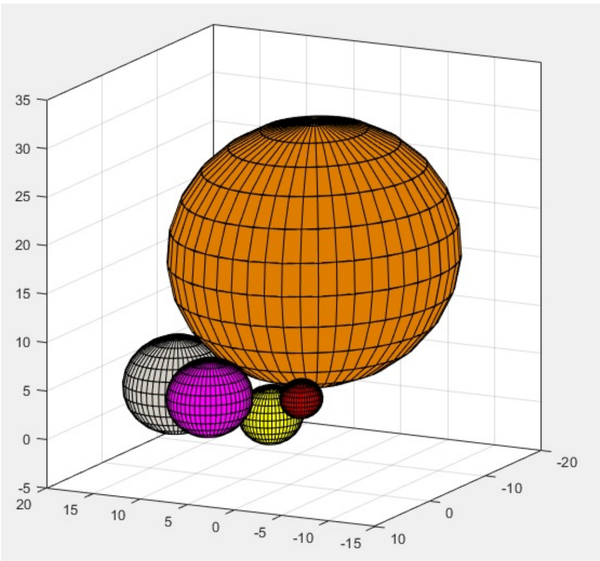

Figure 2. Geometric interpretation the first solution of the system of equations

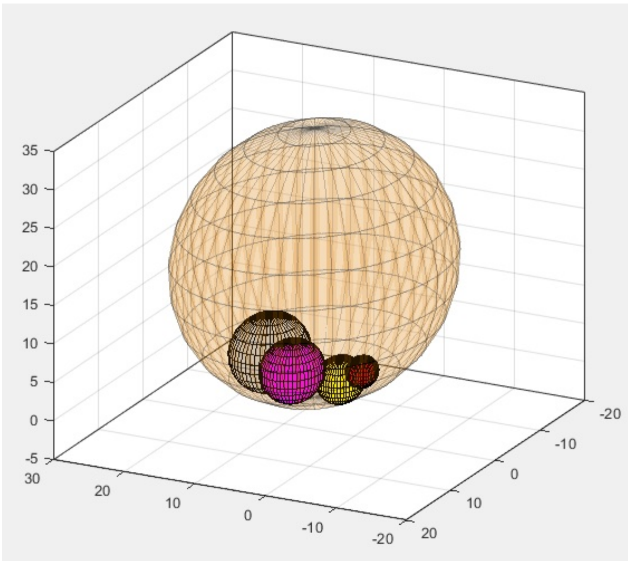

Figure 3. Geometric interpretation of the second solution of the system of equations

The existence of two spheres means that the system of equations (4) has two solutions, only one of which is considered valid based on the conditions of the problem of pseudo-distance measurement. The valid solution represents the coordinates (x_1, x_2, x_3) of the receiver and the value of the adjustment δt_r.

5. Consideration of the results

Computing experiment has confirmed the advantage of the suggested simplified mathematical model of pseudo-distance measurement over the existing mathematical model. The mathematical model suggested in this paper makes it possible to obtain the analytic solution of the system of algebraic equations as opposed to the existing model, the solution of which is based on linearization of the equations. This significantly improves accuracy of pseudo-distance measurement.

6. Conclusion

The results of the study evolve the existing theoretical research aimed on solving the known Fermat problem of finding the set of spheres tangent to the four given spheres [2,3,4]. The development consists in acquiring the analytical solution of the mathematical model of the problem by means of its significant simplification on the basis of geometric representation and interpretation of the equations of the existing model. Application of analytic solution algorithm to the problem of pseudo-distance measurement in satellite location systems allows to improve the accuracy of such measurements.

7. References

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