Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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To cite this article: Timur Medvedev *et al* 2018 *J. Phys.: Conf. Ser.* **990** 012009

Dynamics of one model of the fast kinematic dynamo

Timur Medvedev[1], **Vladislav Medvedev**[2] **and Evgeny Zhuzhoma**[2]

[1] National Research University Higher School of Economics, 136 Rodionova Street, Niznhy Novgorod, Russia

[2] National Research University Higher School of Economics, 25/12 Bolshaya Pecherckaya Street, Niznhy Novgorod, Russia

E-mail: `tmedvedev@hse.ru`

Abstract. We suggest a new model of the fast nondissipative kinematic dynamo which describes the phenomenon of exponential growth of the magnetic field caused by the motion of the conducting medium. This phenomenon is known to occur in the evolution of magnetic fields of astrophysical bodies. In the 1970s A.D. Sakharov and Ya.B. Zeldovich proposed a "rope" scheme of this process which in terms of the modern theory of dynamical systems can be described as Smale solenoid. The main disadvantage of this scheme is that it is non-conservative. Our model is a modification of the Sakharov-Zeldovich's model. We apply methods of the theory of dynamical systems to prove that it is free of this fault in the neighborhood of the nonwandering set.

1. Introduction

One of the fundamental problems of the natural sciences is formation and evolution of magnetic fields of astrophysical bodies. Particularly, the theory of the kinematic dynamo studies the evolution of the magnetic fields of electrically conducting fluids [1, 2, 3]. The velocity field $\vec{v}$ of an incompressible conducting medium (fluid, gas or plasma) is supposed to be given while the subject of interest being the magnetic field $\vec{H}$ stretched by the fluid flow in the presence of a low diffusion dissipating the magnetic energy. The kinematic dynamo is described by the following equations

$$\frac{\partial \vec{H}}{\partial t} = \mathrm{rot}\left[\vec{v}\vec{H}\right] + \eta\triangle\vec{H}, \quad \mathrm{div}\,\vec{H} = 0, \quad \mathrm{div}\,\vec{v} = 0,$$

here η is the magnetic diffusivity which is in inverse proportion to Magnetic Reynolds number $R_m = \frac{1}{\eta}$ (see the main notions and definitions in [4, 5, 6]). The literature on the magnetohydrodynamics often uses $\vec{B}$ to describe the magnetic field, where $\vec{B} = \mu\vec{H}$ and μ is the permeability of the medium (for us the difference between $\vec{H}$ and $\vec{B}$ is irrelevant). One of the important aspects of the kinematic dynamo is the fast kinematic dynamo when the motion of the conducting medium causes an exponential growth of the magnetic field for a small magnetic diffusion.

The discrete (in time) version of this problem studies the growth of the magnetic field at moments $t = 1, 2, \ldots$. Instead of the transport of the flow and the continuous diffusion of the magnetic fields one considers the composition of these processes. That is, for a given conservative (volume preserving) diffeomorphism $f : M \to M$ the magnetic field is considered to be first transported to the field $f_*(\vec{\boldsymbol{H}})$ and then to be dissipated as the solution of the equation $\frac{\partial f_*(\vec{\boldsymbol{H}})}{\partial t} = \eta\triangle\left(f_*(\vec{\boldsymbol{H}})\right)$.

A kinematic dynamo is said to be dissipative ("realistic") if $\eta \to +0$ or nondissipative ("idealistic") for $\eta = 0$. For the nondissipative dynamo the magnetic field is "frozen" into the movement of the medium [7] and one usually studies the exponential growth of the energy of this field. According to [8] a fast nondissipative dynamo occurs if the diffeomophism f has a nonzero topological entropy (i.e if f chaotic enough). In the 1970s A.D. Sakharov and Ya.B. Zeldovich proposed a scheme of a so called rope dynamo the idea of which modern 3-dimensional models of the fast dynamo widely exploit [1, 2]. From the point of view of the modern theory of the dynamical systems the construction of Sakharov-Zeldovich is an Ω-stable map of the solid torus into itself suggested by S. Smale [9]. The nonwandering set of this map is a topological solenoid and an expanding attractor, therefore the nonwandering set in the Smale's construction is often called the Smale solenoid or the Smale attractor [10]. Notice that the topological entropy of this map is positive. The main disadvantage of this scheme for the theory of the kinematic dynamo is that the suggested map is not conservative (this subject is well discussed in [1], chapter V). In this paper we suggest a modification of the Sakharov-Zeldovich's scheme without this fault in the a neighborhood of the nonwandering set. Notice that the solid torus is naturally foliated by 2-disks perpendicular to its axis and that the Smale map preserves this disk structure. We modify the scheme in such a way that the solid torus maps into its neighborhood, the disk structure is preserved but the intersection of each disk with its image becomes two symmetric Smale horseshoes [11, 9]. Since there are horseshoes with an arbitrary Jacobian determinant it is possible to construct a map which is conservative in the neighborhood of the nonwandering set. This map can be extended to a diffeomorphism of the 3-sphere S^3 or of the Euclidean space $\mathbb{R}^3$, though such an extension is not guaranteed to be conservative on the entire manifold.

The example we suggest is of its own interest for the theory of the dynamical systems. To describe the nonwandering set we give the symbolic model of the restriction of the diffeomorphism to its wandering set. We show that the nonwandering set is the nontrivial 1-dimensional basic set (see basic notions of the theory of the dynamical systems in [12, 13]) and it is in the class of so called solenoidal sets [14].

2. The main construction

Consider the direct product $K \times [0;1]$, where $K = [-1;+1] \times [-1;+1]$ is the square on the plane $\mathbb{R}^2$ with the Cartesian coordinates (x, y). Let $R_t : \mathbb{R}^2 \to \mathbb{R}^2$ denote the counterclockwise rotation

$$\begin{cases} \bar{x} &= x\cos\pi t - y\sin\pi t \\ \bar{y} &= x\sin\pi t + y\cos\pi t \end{cases}$$

of the plane $\mathbb{R}^2$ through the angle πt. The set $\bigcup\limits_{0\le t\le 1}(t, R_t(K))$ is homeomorphic to $K \times [0;1]$ because R_t is a homeomorphism for each t. Since $R_1(K) = K$, the squares $K \times \{0\}$, $K \times \{1\}$ can be naturally identified by $id : K \times \{1\} \to K \times \{0\}$. Let B be the body $\bigcup\limits_{0\le t\le 1}(t, R_t(K))$ with the squares $K \times \{0\}$ and $K \times \{1\}$ identified by id:

$$\bigcup_{0\le t\le 1}(t, R_t(K)) \,/\, \left(K \times \{1\} \overset{id}{\sim} K \times \{0\}\right) \overset{\text{def}}{=} B.$$

We assume that the identification id reverses the orientation if the initial orientation of the squares $K \times \{0\}$, $K \times \{1\}$ is induced by an arbitrary orientation of the body $\bigcup_{0\le t\le 1}(t, R_t(K))$. The body B is a twisted cylinder shown in Fig. 1, where (a) shows the part of B for the values $0 \le t \le \frac{1}{2}$ while (b) shows it for $\frac{1}{2} \le t \le 1$.

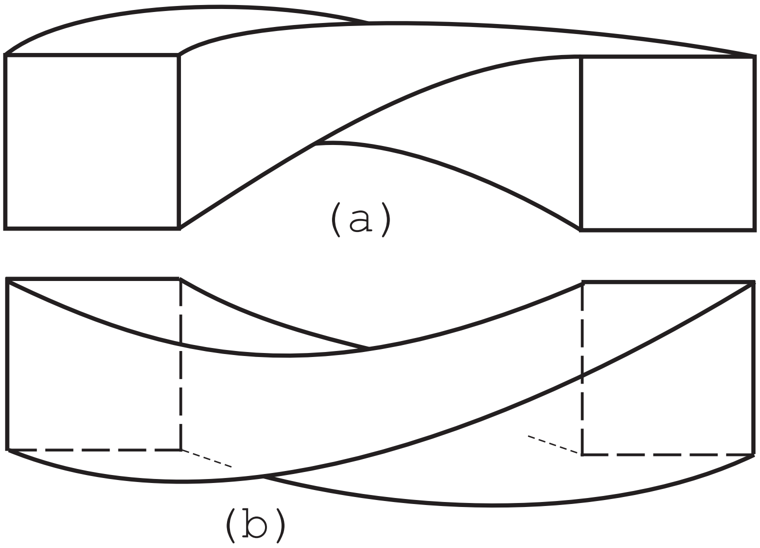

Figure 1. Two halves of the body B. The body is obtained by gluing the halves by the squares.

It is clear that the set

$$\bigcup_{0\leq t<1}(t,(0,0)) \stackrel{\text{def}}{=} S^1_0$$

is a circle on which the quotient map $[0;1]\to[0;1]/(0\sim 1)=S^1$ induces the cyclic coordinate $t \bmod 1$. We say the circle S^1_0 to be the *axis* of the body B. We embed B into $\mathbb{R}^3$ in such a way that the axis B has no knots in $\mathbb{R}^3$ and we consider B to be identical to its embedding. First we are going to construct the desired diffeomorphism $F: B\to f(B)\subset\mathbb{R}^3$ of the body B onto its image in some neighborhood homeomorphic to the solid torus. Notice that the natural projection $K\times[0;1]\to[0;1]$ is a trivial bundle and it induces the locally trivial bundle $p_1: B\to S^1_0$ with the fiber K. Let D_t denote the fiber over $t\in S^1_0$, $D_t=p_1^{-1}(t)$. Evidently D_t can be considered as $R_t(K)$, i.e. as the result of the rotation R_t of the square K.

To define the diffeomorphism F we need to modify the map introduced by Stephen Smale known as the Smale horseshoe [11, 9]. Recall that the classic Smale horseshoe is a diffeomorphism of some disk, containing the square $K=D^2_0$ on the plane $\mathbb{R}^2$, into itself. The diffeomorphism $w: D^2_0=D^2\to\mathbb{R}^2$ of this square is the composition of a contraction along the axis Ox, an expansion along the axis Oy, a bend (the direction of the bend is irrelevant) of the resulting rectangle and, finally, its translation in such a way that the intersection $D^2\cap w(D^2)$ is the union of two disjoint strips which are symmetric with respect to the axis Oy[1] (see Fig. 2(a)). Clearly, the contraction and the expansion could be chosen in such a way that the Jacobian determinant $J(w)$ of w on D^2 equals $\frac{1}{2}$. From now on we suppose these conditions to be satisfied. Denote by $sh_0:\mathbb{R}^2\to\mathbb{R}^2$ the translation $(x;y)\longrightarrow(x+\frac{1}{2};y)$ along the axis Ox and let $w_0=sh_0\circ w: D^2\to\mathbb{R}^2$. Let $S_0:\mathbb{R}^2\to\mathbb{R}^2$ denote the inversion with respect to the origin $(0;0)$, $S_0(x;y)=(-x;-y)$. Again one can pick the contraction, the expansion and the bend such that the following conditions are satisfied (see Fig. 2 (b) on the right):

(i) the intersection $D^2\cap w_0(D^2)$ consists of two disjoint strips;
(ii) the sets $w_0(D^2)$, $S_0\left(w_0(D^2)\right)$ are disjoint,

[1] The horseshoe is sometimes defined as the diffeomorphism of the square which is afterwards extended to the entire plane. It is known [12, 11] that w can be extended to a map of the entire plane $\mathbb{R}^2$ in such a way that this map is the identity outside some neighborhood of D^2.

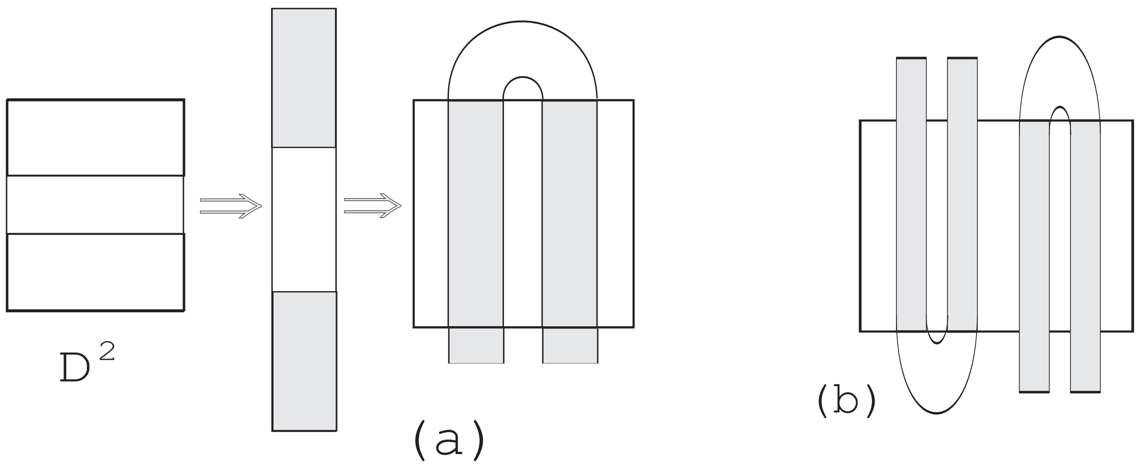

Figure 2. The Smale horseshoes $w(D^2)$ (a) and $w_0(D^2) = sh_0 \circ w(D^2)$ (b).

$$w_0(D^2) \bigcap \left(S_0 \circ w_0(D^2)\right) = \emptyset. \tag{1}$$

The first condition means that the map $sh_0 \circ w \stackrel{\text{def}}{=} w_0$ is a Smale horseshoe whose symmetry line is perpendicular to the axis Ox. The second condition means that the horseshoe $w_0(D^2)$ is disjoint from its reflection by S_0. Notice that $S_0 \circ w_0(D^2)$ is a horseshoe as well.

Consider a neighborhood $S^1 \times B^2$ of the body B homeomorphic to the solid torus; here $B^2 \subset \mathbb{R}^2$ is a disk large enough containing the square K and S^1 is a circle with cyclic coordinate $t \bmod 1$. Below we identify a neighborhood of B with $B^2 \times S^1$. Without loss of generality one assumes S^1 to coincide with the axis S^1_0 of B and that $D_t \subset \{t\} \times B^2$ for each t. Recall that the square D_t is the result of rotation $R_t(K)$ of the square K, therefore we can define a Smale horseshoe on D_t. Let

$$w_{0t} = R_t \circ w_0 \circ R_{-t} : D_t \to \{t\} \times B^2.$$

This map forms the horseshoe in the direction of the line $y = x \cdot \tan \pi t$ only when the symmetry line of the horseshoe $w_{0t}(D^2)$ is perpendicular to the line $y = x \cdot \tan \pi t$.

Let $S^1 = [0;1]/(0 \sim 1)$ be a circle with the natural parametrization $[0;1] \to [0;1]/(0 \sim 1) = S^1$. The map $E_2 : S^1 \to S^1$ of the form $t \to 2t \mod 1$ is an expanding endomorphism of the circle of degree 2 [15]. We now define the map $F : B \to S^1 \times B^2$ in the following way: for every $t \in [0;1)$ and every $z \in D_t$ let

$$(t; z) \longmapsto (E_2(t); R_t \circ w_{0t}(z)), \qquad t \in [0;1), \quad z \in D_t.$$

Notice that from the definition of F if follows that $F(D_t) \subset B_{2t \mod 1}$, (Fig. 3).

Lemma 1 *The map $F : B \to F(B) \subset S^1 \times B^2$ is a diffeomorphism onto its image.*

Proof Assume $F(t_1; z_1) \cap F(t_2; z_2) \neq \emptyset$, then $F(D_{t_1}) \cap F(D_{t_2}) \neq \emptyset$. From the definition of F it follows that $E_2(t_2) = E_2(t_1)$, i.e. $2t_1 \mod 1 = 2t_2$. Since the map w_{0t} is a diffeomorphism to its image one assumes $t_1 \neq t_2$, therefore $t_2 = t_1 + \frac{1}{2}$. Then $F(D_{t_1}) = R_{2t_1} \circ w_0 \circ R_{-t_1}(D_{t_1})$,

$$F(D_{t_2}) = F(D_{t_1+\frac{1}{2}}) = R_{2t_1+1} \circ w_0 \circ R_{-t_1-\frac{1}{2}}(D_{t_1}) = R_1 \circ R_{2t_1} \circ w_0 \circ R_{-t_1-\frac{1}{2}}(D_{t_1}).$$

Since R_1 is the rotation through π, the horseshoes $F(D_{t_1})$ and $S_0 \circ F(D_{t_1})$ must intersect and this contradicts to (1).$\square$

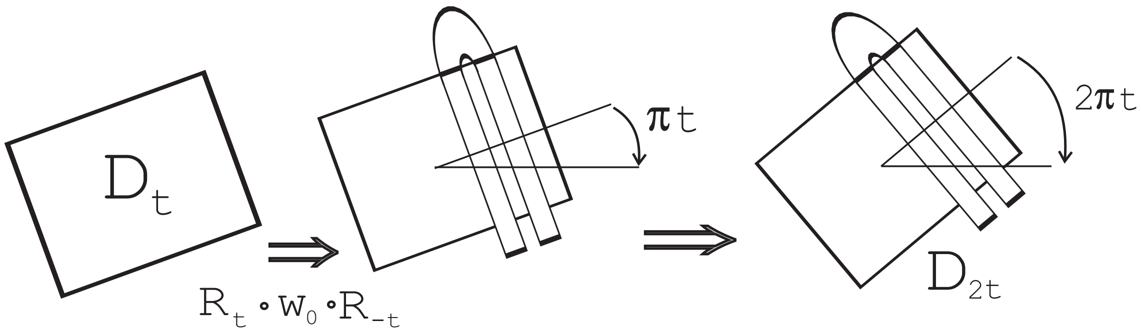

Figure 3. Construction of the map F.

Notice that since the Jacobian determinant $J(w)$ of the map w on D^2 equals $\frac{1}{2}$, the Jacobian determinant of F equals $J(F) = J(w) \cdot DE_2 = \frac{1}{2} \cdot 2 = 1$ and therefore, F is a conservative diffeomorphism to its image. The union $\mathbb{R}^3 \cup \{\infty\}$ of the Euclidean space and the point at infinity $\{\infty\}$ can be identified with the 3-sphere S^3 in the standard way.

Lemma 2 *The map $F : B \to F(B) \subset S^1 \times B^2 \subset \mathbb{R}^3$ can be extended to a diffeomorphism $f : S^3 \to S^3$ which is conservative in some neighborhood of B.*

Proof By construction the circle S^1_0 is the axis of the solid torus $S^1 \times B^2$ and the body B as well as $S^1 \times B^2$ are its tubular neighborhoods. The diffeomorphism of the square in the Smale horseshoe can be extended to a diffeomorphism of a disk large enough (see [12]), therefore F can be extended to a diffeomorphism (which we again denote by F) of the solid torus $S^1 \times B^2$ which preserves the disk structure. Without loss of generality one assumes that F is conservative in some neighborhood of B (otherwise one can consider the square K a bit larger). If follows from the construction that the curves S^1_0 and $F(S^1_0)$ are knot free in $\mathbb{R}^3$. Therefore there is a deformation of S^1_0 to $F(S^1_0)$ which can be easily extended to a deformation of their tubular neighborhoods $\varphi_0 : S^1 \times B^2 \to F(S^1 \times B^2)$. Clearly φ_0 can be made conservative. It follows from [16] that φ can be extended to the desired diffeomorphism $f : S^3 \to S^3$.$\square$

3. The dynamics of the nonwandering set

We say the solid torus B embedded into S^3 to be *basic* and we denote it by $\mathcal{B}$. Let

$$\Omega = \bigcap_{n=-\infty}^{\infty} f^n(\mathcal{B}). \tag{2}$$

The set Ω is invariant with respect to f ([12])and it is not empty because it contains in $D_0 = \{0\} \times D^2 \subset \mathcal{B}$ the invariant nontrivial (0-dimensional) set Ω_0 of the Smale horseshoe ([12, 11, 9]). Let $Diff^1(S^3)$ denote the space of diffeomorphisms of the 3-sphere S^3 with C^1 topology.

Lemma 3 *The set Ω is hyperbolic and the restriction $f|_\Omega$ of f to Ω is of positive topological entropy. Moreover, there is a neighborhood $U(f)$ of f in the space $Diff^1(S^3)$ such that every diffeomorphism $g \in U(f)$ has a hyperbolic invariant set $\Omega_g \subset \mathcal{B}$, the diffeomorphisms $f|_\Omega$, $g|_{\Omega_g}$ are conjugate and the entropy of the restriction $g|_{\Omega_g}$ is positive.*

Proof By construction the Jacobian determinant of the map $f|_\mathcal{B} : S^1 \times D^2 \to \mathbb{R}^3$ is equal to $J(f) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$. Therefore f is hyperbolic (not only on Ω but on $\mathcal{B}$ as well) and it follows that the set Ω is hyperbolic. The restriction $f|_{\Omega_0} : \Omega_0 \to \Omega_0$ has a positive entropy ([12, 13]). Then it follows from [17] that the entropy of the restriction $f|_{\Omega}$ is positive as well. Since hyperbolic sets are stable under C^1-small perturbations, the desired neighborhood $U(f)$ exists because the entropy is invariant with respect to conjugacy.$\square$

Now we study the dynamics of the restriction of the diffeomorphism $f : S^3 \to S^3$ to its nonwandering set belonging to the basic solid torus $\mathcal{B}$. To do this we construct a symbolic model of $f|_{\Omega}$ on the invariant set Ω. Let $t_0 \in S^1$, $0 \leq t_0 < 1$ be fixed. Consider the intersection of Ω and the disk $D_{t_0} = \{t_0\} \times B^2 \subset S^1 \times B^2$.

Following the symbolic model of the classic Smale horseshoe we define two vertical and two horizontal (in the usual sense) strips in the square D_{t_0} in the following way. Recall that the intersection of the square D_{t_0} with its image with respect to the map of the Smale horseshoe w_{0t_0} consists of the two vertical strips,

$$w_{0t_0}(D_{t_0}) \cap D_{t_0} = R_0(t_0) \cup R_1(t_0),$$

where $R_0(t_0)$ (respectively $R_1(t_0)$) is the strip nearest to the center (respectively, the farthest). It follows from the construction that in D_{t_0} there are two disjoint horizontal (perpendicular to $R_0(t_0)$, $R_1(t_0)$) strips (we denote them by $R_0^{(-1)}(t_0)$ and $R_1^{(-1)}(t_0)$) such that

$$w_{0t_0}(R_0^{(-1)}(t_0)) = R_0(t_0) \ \text{ and } \ w_{0t_0}(R_1^{(-1)}(t_0)) = R_1(t_0). \tag{3}$$

We now show that the intersection $f^{-1}(\mathcal{B}) \cap D_{t_0} \cap f(\mathcal{B})$ consists of eight rectangles. By construction the intersection $D_{t_0} \cap f(\mathcal{B}) \cap \mathcal{B} = D_{t_0} \cap f(\mathcal{B})$ consists of four strips. Indeed, there are exactly two points $t_1' = \frac{t_0}{2}$, $t_1'' = \frac{t_0+1}{2} \in S^1$ such that $t_0 = E_2(t_1')$ and $t_0 = E_2(t_1'')$ and $D_{t_0} \cap f(D_\mu) = \emptyset$ for every $\mu \neq t_1', t_1''$, $0 \leq \mu < 1$. Then

$$D_{t_0} \bigcap f(\mathcal{B}) = D_{t_0} \bigcap \left(f(D_{t_1'}) \cup f(D_{t_1''}) \right) = \left(D_{t_0} \cap f(D_{t_1'}) \right) \bigcup \left(D_{t_0} \cap f(D_{t_1''}) \right).$$

Each intersection $D_{t_0} \cap f(D_{t_1'})$, $D_{t_0} \cap f(D_{t_1''})$ consists of two strips. Notice that $D_{t_0} \cap f(D_{t_1'}) = R_0(t_0) \cup R_1(t_0)$. Then by the definition of f

$$f(R_i(t_0)) = R_i(E_2(t_0)), \quad i = 1, 2. \tag{4}$$

Applying (3) with t_0 changed to $E_2(t_0)$ and (4) we get

$$R_0^{(-1)}(t_0) \cap R_1^{(-1)}(t_0) = f^{-1}\left(R_0(E_2(t_0)) \cup R_1(E_2(t_0)) \right).$$

Therefore the intersection $f^{-1}(\mathcal{B}) \cap D_{t_0}$ consists of two horizontal strips $R_0^{(-1)}(t_0) \cap R_1^{(-1)}(t_0)$. Then the intersection $f^{-1}(\mathcal{B}) \cap D_{t_0} \cap f(\mathcal{B})$ consists of 8 rectangles which we say to be rectangles of the first degree. They are the intersection of four vertical strips and two horizontal strips, see Fig. 4 (a).

Similar to the standard process of coding for the classic Smale horseshoe we code the rectangles of first degree in the following way. Recall that $D_{t_0} \cap f(D_{t_1'})$ is two vertical strips and each strip divides the disk. We assign "0" to the strip of $D_{t_0} \cap f(D_{t_1'})$ which is the closest to the coordinate origin and we assign "+1" to the other strip. In the same way we assign "0" and "+1" to the two strips of the intersection $D_{t_0} \cap f(D_{t_1''})$. Notice that $D_{t_0} \cap f(D_{t_1'}) = R_0(t_0) \cup R_1(t_0)$ and we have assigned "0" to the strip $R_0(t_0)$ and we have assigned "+1" to the strip $R_1(t_0)$. We assign $\omega_0' = 0$ and $\omega_0'' = 1$ to the respective horizontal strips $R_0^{(-1)}$ and $R_1^{(-1)}$. Now each rectangle of the first degree has the corresponding block $[(t_0, \omega_0); (t_1, \omega_1)]$ where $t_0 = E_2(t_1)$, $\omega_0 \in \{0; 1\}$, $\omega_1 \in \{0; 1\}$. Denote by $V^{(1)}[(t_0, \omega_0); (t_1, \omega_1)]$ the rectangle with this block. Since

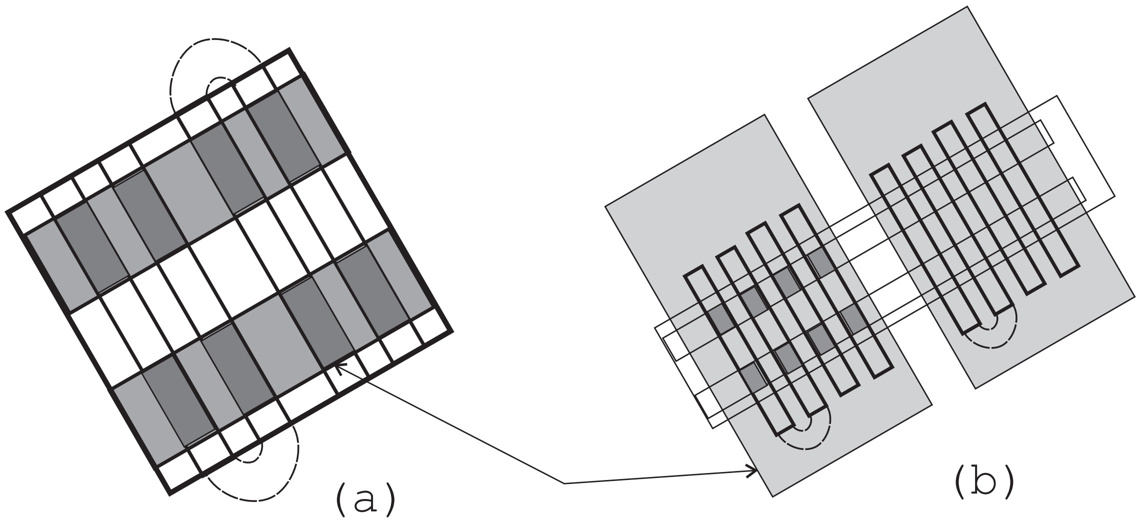

Figure 4. Rectangles of the first degree (a), rectangles of the second degree (b).

the rectangles of the first degree are pairwise disjoint it follows from (2) that an arbitrary point $P \in \Omega \cap D_{t_0}$ belongs to exactly one octagon and $P \in V^{(1)}[(t_0,\omega_0);(t_1,\omega_1)]$ has been assigned the (initial) block $[(t_0,\omega_0);(t_1,\omega_1)]$ of the first degree.

Analogously the intersection

$$f^{-2}(\mathcal{B}) \cap f^{-1}(\mathcal{B}) \cap D_{t_0} \cap f(\mathcal{B}) \cap f^2(\mathcal{B}) = f^{-1}\left[f^{-1}(\mathcal{B}) \cap \mathcal{B}\right] \cap D_{t_0} \cap f\left[\mathcal{B} \cap f(\mathcal{B})\right]$$

consists of 8^2 rectangles which we call the rectangles of the second degree, see Fig. 4 (b). It is easy to see that each rectangle of the first degree contains 8 disjoint rectangles of the second degree. Using the rectangle of the first degree $V^{(1)}[(t_0,\omega_0);(t_1,\omega_1)]$ instead of the disk we analogously assign the block $V^{(2)}[(t_{-1},\omega_{-1});(t_0,\omega_0);(t_1,\omega_1);(t_2,\omega_2)]$ to the rectangle of the second degree $V^{(1)}[(t_0,\omega_0);(t_1,\omega_1)]$, where $t_{-1} = E_2(t_0), t_1 = E_2(t_2), \omega_j \in \{0;1\}, j = -1,0,1,2$. Since a point $P \in \Omega \cap D_{t_0}$ is contained in exactly one octagon of the second degree, say $V^{(2)}[(t_{-1},\omega_{-1});(t_0,\omega_0);(t_1,\omega_1);(t_2,\omega_2)]$, the unique block of the second degree $[(t_{-1},\omega_{-1});(t_0,\omega_0);(t_1,\omega_1);(t_2,\omega_2)]$ is assigned to it. If we continue this procedure then for each point $P \in \Omega \cap D_{t_0}$ we get the bilateral sequence

$$\widehat{P} \stackrel{\text{def}}{=} [\cdots(t_{-n},\omega_{-n});\cdots;(t_{-1},\omega_{-1});\underbrace{(t_0,\omega_0)};(t_1,\omega_1);(t_2,\omega_2);\cdots;(t_n,\omega_n);\cdots],$$

where $\omega_j \in \{0;1\}$, $j \in \mathbb{Z}$, $E_2(t_{i+1}) = t_i$, $i \in \mathbb{Z}$. The underlined pair is conventionally assumed to be at the position 0.

Let $\Sigma_2(E_2)$ denote the set of all sequences of the type

$$[\cdots(t_{-n},\omega_{-n});\cdots;(t_{-1},\omega_{-1});\underbrace{(t_0,\omega_0)};(t_1,\omega_1);(t_2,\omega_2);\cdots;(t_n,\omega_n);\cdots],$$

where $\omega_j \in \{0;1\}$, $j \in \mathbb{Z}$, $E_2(t_{i+1}) = t_i$, $i \in \mathbb{Z}$. Fix a sequence $\widehat{P}^{(0)} \in \Sigma_2(E_2)$, $\widehat{P}^{(0)} = \{(t_i^{(0)},\omega_i^{(0)}\}_{i=-\infty}^{\infty}$ and fix numbers $r \in \mathbb{N}$, $\varepsilon > 0$. A (r,ε)-neighborhood $U_{r,\varepsilon}\left(\widehat{P}^{(0)}\right)$ of the sequence $\widehat{P}^{(0)}$ is the set of sequences $\widehat{P} \in \Sigma_2(E_2)$, $\widehat{P} = \{(t_i,\omega_i)\}_{i=-\infty}^{\infty}$ which satisfy

$$|t_i^{(0)} - t_i| < \varepsilon \text{ for all } -r \le i \le r, \qquad \sum_{i=-\infty}^{\infty} \frac{|\omega_i^{(0)} - \omega_i|}{2^i} < \varepsilon.$$

The set of (r, ε)-neighborhoods generates the topology on $\Sigma_2(E_2)$. Let $\sigma : \Sigma_2(E_2) \to \Sigma_2(E_2)$ denote the map

$$\sigma\left([\cdots; (t_{-1}, \omega_{-1}); \underbrace{(t_0, \omega_0)}; (t_1, \omega_1); \cdots]\right) = [\cdots; \underbrace{(t_{-1}, \omega_{-1})}; (t_0, \omega_0); (t_1, \omega_1); \cdots].$$

Then one proves in the standard way that σ is a homeomorphism.

Lemma 4 *The homeomorphism $\sigma : \Sigma_2(E_2) \to \Sigma_2(E_2)$ is transitive and its set of periodic points is dense.*

Proof The proof is just the compilation of the well known proofs of the similar statements for the classic Smale horseshoe and the Smale solenoid [13, 11, 9], therefore we omit it.□

Theorem 1 *The restriction $f|_\Omega : \Omega \to \Omega$ of the diffeomorphism f to Ω is conjugate to $\sigma : \Sigma_2(E_2) \to \Sigma_2(E_2)$.*

Proof Denote by $\vartheta : \Omega \to \Sigma_2(E_2)$ the map which assigns to a point $P \in \Omega$ its code $\widehat{P} \in \Sigma_2(E_2)$. Since the rectangles of any fixed degree are pairwise disjoint, the map ϑ is well defined and it is single-valued, i.e. if $P_1 \neq P_2$ then $\vartheta(P_1) \neq \vartheta(P_2)$. It is clear that the rectangles of the fixed degree continuously depend (in the Hausdorff topology in the space of compact sets) on the parameter t of the square D_t, therefore ϑ is continuous. Since the space $\Sigma_2(E_2)$ is compact, ϑ is a homeomorphism. It is immediate from the definition of the coding that the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{f|_\Omega} & \Omega \\ \downarrow \vartheta & & \downarrow \vartheta \\ \Sigma_2(E_2) & \xrightarrow{\sigma} & \Sigma_2(E_2) \end{array}$$

is commutative. This means that the maps $f|_\Omega$ and σ are conjugate.□

Corollary 1 *The homeomorphism $f|_\Omega : \Omega \to \Omega$ is transitive and its set of periodic points is dense.*

Consider on $S^1 \times B^2$ a magnetic field $\vec{\boldsymbol{B}}$ of unit vectors tangent to the curves $S^1 \times \{z\}$, $z \in B^2$. One assumes the curves $S^1 \times \{z\}$ to be oriented in the direction of the parameter increase. It is clear that $\vec{\boldsymbol{B}}$ can be extended to the unit vector field (therefore, divergence-free) of the entire sphere S^3. The following result shows that the energy of the magnetic field $\vec{\boldsymbol{B}}$ grows exponentially with the exponent $\mu > 0$.

Theorem 2 *The diffeomorphism $f : S^3 \to S^3$ is fast nondissipative kinematic dynamo with respect to the magnetic field $\vec{\boldsymbol{B}}$.*

Proof Since f stretches the length of the curves $S^1 \times \{z\}$ two-fold, f transforms the field $\vec{\boldsymbol{B}}$ to the field $f_*(\vec{\boldsymbol{B}})$ with the following property: there is a constant $\lambda > 1$ such that the vectors of $f_*(\vec{\boldsymbol{B}})$ are longer by at least factor λ then those of the field $\vec{\boldsymbol{B}}$. The same holds for the lengths of the vectors of the field $f_*^{n+1}(\vec{\boldsymbol{B}})$ with respect to the filed $f_*^n(\vec{\boldsymbol{B}})$. If we ignore the energy dissipation then the energy of the vector field $f_*^n(\vec{\boldsymbol{B}})$ grows exponentially with the exponent $\ln \lambda > 0$. Notice that this result also follows from Lemma 3 and the more general result of the paper [8], in which it is shown that a typical magnetic field is a nondissipative fast dynamo if and only if the diffeomorphism f has a nonzero topological entropy.□

Acknowledgments
This work is supported by the Russian Science Foundation under grant 17-11-01041, except for the proof of Lemma 2 (the extention of the diffeomorphism to a conservative diffeomorphism). The latter proof was obtained within the HSE Basic Research Program (project no. 90) in 2017.

References

[1] Arnold V and Khesin B 1999 *Topological methods in hydrodynamics* (Springer)
[2] Vainstein S and Zeldovich Y 1972 *Uspekhi Phys. Nauk (Russian)* **106** 431–457
[3] Moffatt H 1978 *Magnetic Field Generation in Electrically Conducting Fields* (Cambridge University Press)
[4] Landau L D and Lifshitz E M 1984 *Course of theoretical physics, Vol. 8. Electrodynamics of Continuous Media* (Butterworth-Heinemann)
[5] Alfvén H and Fälthammar C 1963 *Cosmical Electrodynamics: Fundamental Principles* (Oxford)
[6] Childress S and Gilbert A 1995 *Stretch, Twist, Fold: the Fast Dynamo* (Springer-Verlag, Berlin, Heidelberg, NY)
[7] Alfvén H 1943 *Ark. for Astronomi* **29** 1–17
[8] Klapper I and Young L 1995 *Comm. Math. Phys.* **173** 623–646
[9] Smale S 1967 *Bull. Amer. Math. Soc.* **73** 747–817
[10] Katok A and Hasselblatt B 1995 *Introduction to the modern theory of dynamical systems* (Cambridge Univ. Press)
[11] Smale S 1965 Diffeomorphisms with many periodic points *Differential and Combinatorial Topology* (Princeton, NJ) pp 63–80
[12] Anosov D and Solodov V 1995 Hyperbolic sets *Itogi Nauki Tekhniki; Sovremennye Problemy Matematiki, Fundamental'nye Napravleniya, Dynamical Systems 9, VINITI, Akad. Nauk SSSR, Vol. 66, Moscow, (1991), p.12-99 (Russian), English translation in Encyclopaedia of Mathematical Sciences, Dynamical Systems IX* vol 66 (Springer-Verlag, Berlin-Heidelberg)
[13] Robinson C 1999 *Dynamical systems: stability, symbolic dynamics, and chaos* vol 28 (CRC press Boca Raton, FL)
[14] Zhuzhoma E and Isaenkova N 2011 *Sb. Math.* **202** 351–372 ISSN 1064-5616; 1468-4802/e
[15] Shub M 1969 *Amer. J. Math.* **91** 175–199
[16] Jiang B, Ni Y and Wang S 2004 *Trans. Amer. Math. Soc.* **356** 4371–4382 ISSN 00029947 URL `http://www.jstor.org/stable/3844925`
[17] Bowen R 1970 Topological entropy and axiom A *Proc. Sympos. Pure Math* vol 14 pp 23–41