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Abstract.   We suggest to view ultracold atoms in a time-periodically shifted optical lattice as a "dressed matter wave", analogous to a dressed atom in an electromagnetic field. A possible effect lending support to this concept is a transition of ultracold bosonic atoms from a superfluid to a Mott-insulating state in response to appropriate "dressing" achieved through time-periodic lattice modulation. In order to observe this effect in a laboratory experiment, one has to identify conditions allowing for effectively adiabatic motion of a many-body Floquet state.

1. Introduction

The "dressed-atom picture" provides a transparent approach to the dynamics of atoms and molecules in strong electromagnetic fields [1]. In essence, the "dressing" provided by the field may equip the atom or molecule with properties quite different from those of a "bare" one. A hallmark example along these lines is given by the modification of Zeeman hyperfine spectra of atoms interacting with strong radiofrequency fields: As reviewed in Sec. 3 below, in the presence of the dressing field the bare atomic g-factors become multiplied by a certain Bessel function, the argument of which is proportional to the strength of the field, and inversely proportional to its frequency. Thus, experimentally recorded hyperfine spectra depend sensitively on these parameters [2]. Closely related phenomena have been observed, for instance, in radiatively assisted collisions of Rydberg atoms [3].

The appearance of a Bessel function in response to time-periodic forcing is typical for quantum systems that can be viewed as a set of interacting nearest neighbours, such as the angular momentum substates in the case of the g-factor modification. Another striking example has emerged only recently: In experiments with ultracold atoms in time-periodically shifted optical lattices, the hopping matrix elements which quantify the magnitude of the tunnelling contact between states located at adjacent lattice sites differ from those of a bare lattice system again by a Bessel function, and thus can be tuned by adjusting the amplitude or the frequency of the lattice modulation [4, 5]. This finding now suggests an interesting question: The g-factor experiment [2] with single atoms has been instrumental for establishing the dressed-atom picture; could the recent experiments [4, 5] with Bose–Einstein condensates lead to a similar picture of "dressed matter waves"? In other words, can one exploit time-periodic forcing for endowing a macroscopic matter wave with properties it does not have when the forcing is absent?

In order to expand on this question, we proceed as follows: We first recall in Sec. 2 the physics underlying the Bessel-function modification occurring in the dressed-atom picture, using the example of a two-level system interacting with a quantised radiation mode. However, when dealing with cold atoms in time-periodically shifted optical lattices, it is certainly reasonable to describe the time-periodic lattice modulation in terms of an external classical force. Hence,

we discuss in Sec. 3 how the Bessel function appears in that context, employing the Floquet formalism. For fully exploiting the possibilities of control opened up by time-periodic forcing, adiabatic response to slowly changing parameters plays an important role. Therefore, we briefly point out in Sec. 4 how the adiabatic principle works for Floquet states. In Sec. 5 we put all these pieces together and argue that a system of interacting ultracold bosonic atoms in a time-periodically modulated optical lattice can change its state from superfluid to Mott-insulator-like and back, if the modulation is switched on and off in an adiabatic manner, *and* the parameters are chosen judiciously [6, 7, 8]. This scenario, still to be confirmed experimentally, could lend significant support to the notion of dressed matter waves: The dressing achieved through the lattice modulation determines the state of the system. We finally sum up our conclusions in Sec. 6.

2. The dressed two-level system

We start by studying a "two-level atom" interacting with both a static external field and a single mode of a quantised radiation field. Its dynamics are governed by the Hamiltonian

$$H = H_{\text{at}} + H_{\text{int}} + H_{\text{rad}} \, , \tag{1}$$

where

$$H_{\text{at}} = -\frac{J}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

specifies the unperturbed system with energy eigenvalues $\pm J/2$,

$$H_{\text{rad}} = \hbar\omega \left(-\frac{1}{2}\partial_z^2 + \frac{1}{2}z^2 \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

models a radiation mode with frequency ω in terms of a harmonic oscillator with dimensionless oscillator coordinate z, and

$$H_{\text{int}} = \frac{1}{2} \left(K_0 + \gamma z \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

describes the static field of strength K_0 and the coupling to the radiation mode, with a strength specified by a constant γ. When $J = 0$, so that the two "atomic" levels are degenerate, this Hamiltonian (1) obviously is diagonalised by the shifted harmonic-oscillator states

$$\psi_{n,+}(z) = \begin{pmatrix} \varphi_n(z + \gamma/2\hbar\omega) \\ 0 \end{pmatrix} \, , \ \psi_{n,-}(z) = \begin{pmatrix} 0 \\ \varphi_n(z - \gamma/2\hbar\omega) \end{pmatrix} \tag{5}$$

with energies

$$E_{n,\pm}^{(0)} = \hbar\omega \left(n + \frac{1}{2} \right) \pm \frac{K_0}{2} - \frac{\gamma^2}{8\hbar\omega} \, ; \tag{6}$$

the functions $\varphi_n(z)$ denote the familiar eigenfunctions of the harmonic oscillator,

$$\varphi_n(z) = (\sqrt{\pi}\, 2^n\, n!)^{-1/2}\, H_n(z) \exp(-z^2/2) \, . \tag{7}$$

The static field splits the two atomic levels by the amount $\Delta E = K_0$. When this splitting is matched by an integer number of photons, that is, when

$$K_0 = (\ell - n)\hbar\omega \, , \tag{8}$$

the unperturbed states are pairwise degenerate, $E_{n,+}^{(0)} - E_{\ell,-}^{(0)} = 0$. We now study the removal of this degeneracy between an "n-photon state" and an "ℓ-photon state" for nonzero J, assuming $J \ll \hbar\omega$.

Degenerate-state perturbation theory requires to evaluate the matrix elements of the "perturbation" H_{at} in the basis (5), and, hence, to compute the overlap integrals

$$M = \int_{-\infty}^{+\infty} \mathrm{d}z\, \varphi_n(z + \gamma/2\hbar\omega)\, \varphi_\ell(z - \gamma/2\hbar\omega) \,. \tag{9}$$

With the help of the expansion

$$H_n(z + \beta) = \sum_{k=0}^{n} \binom{n}{k} H_k(z)\, (2\beta)^{n-k} \tag{10}$$

this integral can be calculated exactly, yielding

$$M = \exp(-\alpha/2)\, \alpha^{(n-\ell)/2}\, \sqrt{n!\, \ell!} \sum_{k=0}^{\min(n,\ell)} \frac{(-\alpha)^{\ell-k}}{k!\, (n-k)!\, (\ell-k)!} \,, \tag{11}$$

where we have introduced the dimensionless parameter

$$\alpha = 2 \left(\frac{\gamma}{2\hbar\omega} \right)^2 \,. \tag{12}$$

Assuming $\ell \geq n$, and employing the generalised Laguerre polynomials [9]

$$L_n^{(\ell-n)}(x) = \sum_{k=0}^{n} \frac{\ell!\, (-x)^k}{k!\, (n-k)!\, (\ell-n+k)!} \,, \tag{13}$$

this expression (11) takes the form

$$M = (-1)^{\ell-n} \exp(-\alpha/2)\, \alpha^{(\ell-n)/2}\, \sqrt{\frac{n!}{\ell!}}\, L_n^{(\ell-n)}(\alpha) \,. \tag{14}$$

Now we are interested in the limiting case of almost classical fields containing a very large number of photons. In order to maintain the resonance condition (8), we keep the integer $\nu \equiv \ell - n$ fixed while letting n and ℓ tend to infinity. In that limit, one has [9]

$$L_n^{(\ell-n)}(x) \xrightarrow{n \to \infty,\, \ell-n=\nu} \frac{\ell!}{n!}\, \mathrm{e}^{x/2} \left(\frac{n+\ell+1}{2} x \right)^{-(\ell-n)/2} \mathrm{J}_{\ell-n}\!\left(\sqrt{2(n+\ell+1)x} \right) \,, \tag{15}$$

where $\mathrm{J}_\nu(x)$ denotes a Bessel function of integer order ν. Hence, one finally obtains

$$M \longrightarrow (-1)^{\ell-n} \mathrm{J}_{\ell-n}\!\left(\sqrt{2(n+\ell+1)\alpha} \right) \tag{16}$$

in that same limit.

It remains to interpret the peculiar-looking argument of the Bessel function. When placing the field oscillator into a coherent state, its amplitude z_0 is determined by energy considerations: Since, as expressed by the resonance condition (8), the field is exchanging $\ell - n$ photons with the atom, the average field energy is the arithmetic mean of the energy of an n-photon state and that of an ℓ-photon state. This gives

$$\frac{1}{2} \left(n + \frac{1}{2} + \ell + \frac{1}{2} \right) = \frac{1}{2} z_0^2 \,, \tag{17}$$

which, in view of the definition (12), implies

$$\sqrt{2(n+\ell+1)\alpha} = \frac{\gamma z_0}{\hbar\omega} \; . \tag{18}$$

The energy eigenvalues which have been degenerate for $J = 0$, i.e., $E_{n,+}^{(0)} = E_{\ell,-}^{(0)}$, now are shifted by $\pm MJ/2$ for nonzero J. For energies sufficiently high to validate the preceding reasoning, the spectrum of the Hamiltonian (1) thus consists of a sequence of doublets split by $|MJ|$, the doublet centers being separated by $\hbar\omega$. Putting all things together, this means that the "atom" H_{at}, when "dressed" by the interaction $H_{\text{int}} + H_{\text{rad}}$ under the conditions detailed above, behaves like a *noninteracting* system (2) with a modified level splitting determined by the effective J-parameter

$$J_{\text{eff}} = (-1)^\nu \mathrm{J}_\nu\left(\frac{\gamma z_0}{\hbar\omega}\right) J \; . \tag{19}$$

This is the lesson to be learned from the present two-level example: When the driving field can be considered classical, resonant forcing effectuates a modification of the "atomic" level splitting such that the unperturbed splitting is multiplied by a Bessel function with an argument proportional to the driving amplitude divided by $\hbar\omega$. The order ν of this Bessel function is determined by the number of photons in resonance with the transition, according to Eq. (8). In particular, when there is no static field, one has $\nu = 0$ and thus recovers the modification of the splitting by a J_0 Bessel function which also underlies, for instance, the coherent destruction of tunnelling of a single particle in driven symmetric double well potential [10, 11]. Quite recently, this phenomenon has been observed with cold atoms in periodic double-well potentials [12].

3. Elements of Floquet theory

In order to avoid the consideration of a quantised field and to start with a classical driving force right away, we now treat the explicitly time-dependent Hamiltonian

$$H(t) = H_0 + H_1(t) \; , \tag{20}$$

where the time-independent system H_0 corresponds to a spin 1 in a magnetic field B oriented in the x-direction,

$$H_0 = g_1\mu_{\text{B}}B \, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; , \tag{21}$$

with μ_{B} denoting the Bohr magneton, and g_1 the Landé g-factor. The external forcing is given as an additional static magnetic field B_0 and an oscillating field with amplitude B_ω and frequency ω, both directed along the z-axis,

$$H_1(t) = g_1\mu_{\text{B}}[B_0 + B_\omega\cos(\omega t)] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; . \tag{22}$$

Apart from the fact that here the forcing is truly classical, this system closely resembles the previous two-level example (1): The unperturbed system is characterised by "nearest-neighbour coupling", while the forcing is diagonal. Now the Hamiltonian (20) depends periodically on time,

$$H(t) = H(t + T) \; , \tag{23}$$

with period $T = 2\pi/\omega$. Hence, the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)\,|\psi(t)\rangle \tag{24}$$

has Floquet-type solutions [13, 14, 15, 16]

$$|\psi_\alpha(t)\rangle = |u_\alpha(t)\rangle \exp(-\mathrm{i}\varepsilon_\alpha t/\hbar) , \tag{25}$$

where the functions $|u_\alpha(t)\rangle$ inherit the periodic nature of $H(t)$,

$$|u_\alpha(t)\rangle = |u_\alpha(t+T)\rangle . \tag{26}$$

These functions, together with the corresponding quasienergies ε_α, are obtained as solutions to the eigenvalue problem

$$(H(t) - \mathrm{i}\hbar\partial_t)|u_\alpha(t)\rangle\!\rangle = \varepsilon_\alpha |u_\alpha(t)\rangle\!\rangle , \tag{27}$$

which is defined in an *extended Hilbert space* $\mathcal{H} \otimes \mathcal{T}$ of T-periodic functions [17] in which the time t is regarded as a *coordinate* and which, therefore, is equipped with the scalar product

$$\langle\!\langle \cdot | \cdot \rangle\!\rangle \equiv \frac{1}{T}\int_0^T \mathrm{d}t \, \langle \cdot | \cdot \rangle , \tag{28}$$

combining the standard scalar product $\langle \cdot | \cdot \rangle$ for the system's original Hilbert space $\mathcal{H}$ with time-averaging. We stick to the convention of writing $|u_\alpha(t)\rangle$ for a Floquet function viewed in $\mathcal{H}$, but $|u_\alpha(t)\rangle\!\rangle$ when that same function is regarded as an element of the extended space $\mathcal{H} \otimes \mathcal{T}$.

There is one issue implied by the Floquet formalism which requires particular attention. Namely, if $|u_{(n,0)}(t)\rangle\!\rangle$ solves

$$(H(t) - \mathrm{i}\hbar\partial_t)|u_{(n,0)}(t)\rangle\!\rangle = \varepsilon_{(n,0)}|u_{(n,0)}(t)\rangle\!\rangle \tag{29}$$

with quasienergy $\varepsilon_{(n,0)}$, then

$$|u_{(n,m)}(t)\rangle\!\rangle \equiv |u_{(n,0)}(t)\rangle\!\rangle \exp(\mathrm{i}m\omega t) \tag{30}$$

solves

$$(H(t) - \mathrm{i}\hbar\partial_t)|u_{(n,m)}(t)\rangle\!\rangle = \varepsilon_{(n,m)}|u_{(n,m)}(t)\rangle\!\rangle \tag{31}$$

with quasienergy

$$\varepsilon_{(n,m)} = \varepsilon_{(n,0)} + m\hbar\omega , \tag{32}$$

where m is *any* (positive or negative) integer. Hence, the quasienergy spectrum repeats itself periodically on the energy-axis; each "Brillouin zone" of width $\hbar\omega$ contains one respresentative, labelled by m, of the class of eigenvalues belonging to the Floquet state labelled by n. But when following the evolution of a wave function $|\psi(t)\rangle$ in the physical Hilbert space $\mathcal{H}$, only *one* representative from each class is needed, giving an expansion of the form

$$|\psi(t)\rangle = \sum_n c_n |u_{(n,0)}(t)\rangle \exp(-\mathrm{i}\varepsilon_{(n,0)}t/\hbar) \tag{33}$$

with time-independent coefficients c_n.

In order to apply this lore to the spin-1-system (20), we observe that the Floquet basis states

$$|u_{(+,m)}(t)\rangle\!\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \exp\left(-\mathrm{i}\frac{g_1\mu_\mathrm{B}B_\omega}{\hbar\omega}\sin(\omega t) + \mathrm{i}m\omega t\right)$$

$$|u_{(0,m)}(t)\rangle\!\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \exp(\mathrm{i}m\omega t)$$

$$|u_{(-,m)}(t)\rangle\!\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \exp\left(+\mathrm{i}\frac{g_1\mu_\mathrm{B}B_\omega}{\hbar\omega}\sin(\omega t) + \mathrm{i}m\omega t\right) \tag{34}$$

diagonalise the quasienergy operator $H_1(t) - i\hbar\partial_t$ which is obtained when there is no field B, so that the eigenstates of the three-level Hamiltonian (21) are degenerate; the "unperturbed" quasienergies express the Zeeman splitting caused by the other static field B_0:

$$\varepsilon^{(0)}_{(n,m)} = n \cdot g_1 \mu_B B_0 + m\hbar\omega \qquad (n = 0, \pm 1) \ . \tag{35}$$

If the oscillating field is resonant in the sense that

$$g_1 \mu_B B_0 = \nu\hbar\omega \ , \tag{36}$$

then the Floquet functions

$$
\begin{aligned}
|u_1(t)\rangle\!\rangle &\equiv |u_{(+,0)}(t)\rangle\!\rangle \\
|u_2(t)\rangle\!\rangle &\equiv |u_{(0,\nu)}(t)\rangle\!\rangle \\
|u_3(t)\rangle\!\rangle &\equiv |u_{(-,2\nu)}(t)\rangle\!\rangle
\end{aligned}
$$

correspond to the same quasienergy, and thus are degenerate. The removal of this degeneracy for nonvanishing B once again is assessed by degenerate-state perturbation theory, assuming $g_1 \mu_B B \ll \hbar\omega$. In contrast to Sec. 2, now the calculation proceeds in the extended Hilbert space $\mathcal{H} \otimes \mathcal{T}$, and thus invokes the computation of matrix elements $\langle\!\langle u_j | H_0 | u_k \rangle\!\rangle$ with respect to the scalar product (28). But this is what makes the mathematics quite simple: Using the identity

$$e^{iz\sin\omega t} = \sum_{k=-\infty}^{+\infty} e^{ik\omega t} J_k(z) \tag{37}$$

for expanding the unperturbed Floquet functions, time averaging according to the definition (28) serves to filter out one particular term from the sum, determined by the resonance condition (36). Thus, one immediately obtains

$$\langle\!\langle u_j | H_0 | u_k \rangle\!\rangle = (-1)^\nu g_1 \mu_B B \, J_\nu\!\left(\frac{g_1 \mu_B B_\omega}{\hbar\omega}\right) \cdot \frac{1}{\sqrt{2}} \delta_{j,k\pm 1} \ , \tag{38}$$

giving the quasienergies

$$\varepsilon_{(n,m)} = n \cdot (-1)^\nu g_1 \mu_B B \, J_\nu\!\left(\frac{g_1 \mu_B B_\omega}{\hbar\omega}\right) + m\hbar\omega \ . \tag{39}$$

Hence, the effect of the forcing (22) on the system (21) is described by replacing the bare g-factor g_1 by the effective substitute

$$g_{\text{eff}} = (-1)^\nu J_\nu\!\left(\frac{g_1 \mu_B B_\omega}{\hbar\omega}\right) g_1 \ . \tag{40}$$

Evidently, the line of reasoning adopted in this section to treat classical forcing parallels the arguments given in Sec. 2 for a system interacting with a quantised field. But here the argument is considerably more direct, avoiding the analysis referring to "large photon numbers". The price to pay for this simplification is a quasienergy spectrum which is strictly $\hbar\omega$-periodic und thus unbounded from below, whereas the exact quantum mechanical energy spectrum becomes approximately $\hbar\omega$-periodic only for sufficiently high quantum numbers. Nonetheless, for systems subjected to time-periodic classical forcing the Floquet picture combines great conceptual clarity with a fairly succinct computational approach.

4. Adiabatic following of Floquet states

One additional piece of input is required before we can treat ultracold atoms in a periodically shifted optical lattice, namely, the adiabatic response of Floquet states to slowly changing parameters. In order to make the point, let us briefly recapitulate the standard adiabatic theorem of quantum mechanics [18]: The task is solve a time-dependent Schrödinger equation

$$i\hbar\partial_t|\psi(t)\rangle = H^{P(t)}|\psi(t)\rangle \tag{41}$$

with a Hamiltonian $H^{P(t)}$ depending on a parameter $P(t)$ which changes slowly in time. The strategy then is to "freeze" that parameter in a first step, and to consider the family of eigenvalue problems

$$H^P|\varphi_n^P\rangle = E_n^P|\varphi_n^P\rangle \tag{42}$$

for each relevant, fixed value of P. Let us stipulate that the phases of the instantaneous eigenstates $|\varphi_n^P\rangle$ be chosen such that

$$\langle\varphi_n^P|\partial_P\,\varphi_n^P\rangle = 0 \;. \tag{43}$$

If then the system initially, at time $t = 0$, is prepared in a particular eigenstate,

$$|\psi(t=0)\rangle = |\varphi_n^{P(t=0)}\rangle \;, \tag{44}$$

and P is allowed to vary sufficiently slowly, an approximate solution to the Schrödinger equation (41) is given by

$$|\psi(t)\rangle = |\varphi_n^{P(t)}\rangle \exp\left(-\frac{i}{\hbar}\int_0^t dt'\, E_n^{P(t')}\right) \;, \tag{45}$$

provided the parameter variation proceeds smoothly, and $|\varphi_n^P\rangle$ is separated for all P by an energy gap from the other states. Hence, the system stays in the state continuously connected to the one it was originally prepared in, and acquires a "dynamical" phase determined by an integral over the instantaneous energy eigenvalues encountered during its evolution. We remark that it might not be possible to satisfy the phase-fixing condition (43) globally if there is more than one time-dependent parameter; this fact then forces one to explicitly introduce Berry's geometrical phase [19].

When trying to transfer this adiabatic theorem to systems with a Hamiltonian $H^{P(t)}(t)$ which would depend periodically on time if the parameter P where fixed, $H^P(t) = H^P(t+T)$, but which actually exhibits an additional "slow" time-dependence of P, one faces a problem: If one simply "stopped the time" in order to define an instantaneous Hamiltonian, one would not only freeze the parameter P, but also loose the periodic time-dependence. However, it appears much more natural to freeze *only* P, and to maintain the periodic time-dependence on the level of the instantaneous eigenvalue problems. The way to do so, as formulated in Refs. [20, 21], includes a detour to the extended Hilbert space $\mathcal{H}\otimes\mathcal{T}$ introduced in the previous section: Instead of starting from the actual Schrödinger equation

$$i\hbar\partial_t|\psi(t)\rangle = H^{P(t)}(t)|\psi(t)\rangle \;, \tag{46}$$

one first distinguishes two different time variables, a variable τ for the slow, parametric time dependence and a variable t for the fast, oscillating one, and then considers the evolution equation

$$i\hbar\partial_\tau|\Psi(\tau,t)\rangle\!\rangle = (H^{P(\tau)}(t) - i\hbar\partial_t)|\Psi(\tau,t)\rangle\!\rangle \tag{47}$$

in $\mathcal{H} \otimes \mathcal{T}$. If this equation can be solved, one returns to the desired wave function $|\psi(t)\rangle$ evolving in the system's true Hilbert space $\mathcal{H}$ by equating τ and t: One has

$$|\psi(t)\rangle = |\Psi(\tau,t)\rangle\rangle\Big|_{\tau=t} , \qquad (48)$$

since

$$
\begin{aligned}
i\hbar\partial_t|\psi(t)\rangle &= i\hbar\partial_\tau|\Psi(\tau,t)\rangle\rangle\Big|_{\tau=t} + i\hbar\partial_t|\Psi(\tau,t)\rangle\rangle\Big|_{\tau=t} \\
&= (H^{P(\tau)}(t) - i\hbar\partial_t)|\Psi(\tau,t)\rangle\rangle\Big|_{\tau=t} + i\hbar\partial_t|\Psi(\tau,t)\rangle\rangle\Big|_{\tau=t} \\
&= H^{P(t)}(t)|\psi(t)\rangle .
\end{aligned}
\qquad (49)
$$

On the level of the extended evolution equation (47), one can now freeze P by stopping solely the time τ, while leaving the other time t unaffected. This then defines the instantaneous eigenvalue problems in terms of the operators appearing on the right-hand side of Eq. (47),

$$(H^P(t) - i\hbar\partial_t)|u_\alpha^P(t)\rangle\rangle = \varepsilon_\alpha^P|u_\alpha^P(t)\rangle\rangle . \qquad (50)$$

Since, by construction, this problem lives in $\mathcal{H} \otimes \mathcal{T}$, it is exactly the quasienergy problem formulated in Eq. (27). The remaining reasoning follows the standard route: We fix the phases of the instantaneous eigenstates by requiring

$$\langle\langle u_\alpha^P|\partial_P\, u_\alpha^P\rangle\rangle = 0 , \qquad (51)$$

and start at time $\tau = 0$ with the initial condition

$$|\Psi(\tau=0,t)\rangle\rangle = |u_\alpha^{P(\tau=0)}(t)\rangle\rangle . \qquad (52)$$

Then

$$|\Psi(\tau,t)\rangle\rangle = |u_\alpha^{P(\tau)}(t)\rangle\rangle \exp\left(-\frac{i}{\hbar}\int_0^\tau d\tau'\, \varepsilon_\alpha^{P(\tau')}\right) \qquad (53)$$

is an adiabatic solution to the extended evolution equation (47), provided the propositions of the adiabatic theorem can be met, and returning to $\mathcal{H}$ according to Eq. (48) gives

$$|\psi(t)\rangle = |u_\alpha^{P(t)}(t)\rangle \exp\left(-\frac{i}{\hbar}\int_0^t dt'\, \varepsilon_\alpha^{P(t')}\right) \qquad (54)$$

as an approximate solution to the original Schrödinger equation (46). In short, for adiabatic quantum transport in periodically time-dependent systems with slowly changing parameters the Floquet states adopt a role which is completely analogous to that played by energy eigenstates in conventional situations described by an equation of the type (41). The strategy of "lifting" the Schrödinger equation (41) to the extended space $\mathcal{H} \otimes \mathcal{T}$, applying standard techniques there, and then projecting back to $\mathcal{H}$ is useful not only for understanding the structure of the problem, but also for detailed computations of non-adiabatic corrections [22].

There is, however, a big caveat. As remarked above, the standard adiabatic theorem demands that the adiabatically transported state be separated by an energy gap from all other states. Accordingly, when transferring this theorem to $\mathcal{H} \otimes \mathcal{T}$, one requires that the adiabatically transported Floquet state be separated in *quasienergy* from the other ones. But since one quasienergy-representative from each state falls into each quasienergy Brillouin zone, this condition is almost impossible to satisfy when there is a large number of states. One then expects a multitude of near-degeneracies "modulo $\hbar\omega$", reflecting a dense set of multiphoton

resonances. In such a situation, it appears unlikely that an adiabatic limit exists [23]. However, it appears equally plausibe that, if one does *not* consider the fictitious limit of a parameter variation proceeding "infinitely slowly", but instead specifies that the variation takes place within a finite time interval, most of these resonances are not "seen" long enough by the system to become active. Then effectively adiabatic motion is possible, if major resonances can be avoided. Although it might be hard to formulate this somewhat vague notion in a mathematically precise manner in the general case, the emerging adiabatic principle (not theorem) for Floquet states can provide intuitively clear guidelines for understanding the evolution of periodically driven systems in well-designed particular cases. The following discussion of the driven Bose–Hubbard model exemplifies that the occurrence of effectively adiabatic motion, or its destruction by active resonances, depends on the choice of the frequency.

5. The driven Bose–Hubbard model

The Bose–Hubbard model, as devised by Fisher *et al.* [24], describes Bose particles on a lattice. There exists a tunnelling contact between neigbouring sites, with a strength specified by a hopping matrix element J; each pair of particles occupying the same site increases the energy of the system by an amount U due to repulsion. Thus, for the case of a one-dimensional (1d) lattice with M sites the many-body Hamiltonian reads

$$\hat{H}_0 = -J \sum_{\ell=1}^{M-1} \left(\hat{b}_\ell^\dagger \hat{b}_{\ell+1} + \hat{b}_{\ell+1}^\dagger \hat{b}_\ell \right) + \frac{U}{2} \sum_{\ell=1}^{M} \hat{n}_\ell \left(\hat{n}_\ell - 1 \right) , \tag{55}$$

where $\hat{b}_\ell^\dagger$ ($\hat{b}_\ell$) is the creation (annihilation) operator for a Bose particle at the ℓth lattice site, obeying $[\hat{b}_\ell, \hat{b}_k^\dagger] = \delta_{\ell,k}$, and $\hat{n}_\ell \equiv \hat{b}_\ell^\dagger \hat{b}_\ell$ gives the number of particles on that site. Assuming that there are N particles in total, and that the filling factor $n = N/M$ is integer, the system's ground state undergoes a significant change when the dimensionless control parameter U/J is varied: In the interaction-free limit $U/J \to 0$ it corresponds to a superfluid, given by a Bose–Einstein condensate with all particles occupying the lowest Bloch state,

$$|\text{SF}\rangle = \frac{1}{\sqrt{N!}} \left(\frac{1}{\sqrt{M}} \sum_{\ell=1}^{M} \hat{b}_\ell^\dagger \right)^N |0\rangle , \tag{56}$$

where $|0\rangle$ is the vacuum state. In the opposite limit of vanishing tunnelling contact, $U/J \to \infty$, the individual sites are isolated, so that the systems adopts the Mott-insulating ground state

$$|\text{MI}\rangle = \prod_{\ell=1}^{M} \frac{(\hat{b}_\ell^\dagger)^n}{\sqrt{n!}} |0\rangle . \tag{57}$$

When the lattice is infinitely large, that is, for $M \to \infty$ and $N \to \infty$ while keeping $n = N/M$ constant at an integer value, a sharp transition between the superfluid and the Mott-insulating regime occurs at a critical value $(U/J)_c$, accompanied by the emergence of a finite energy gap. For a 1d lattice with filling factor $n = 1$, one finds $(U/J)_c \approx 3.4$ [25]. The Bose–Hubbard model has received considerable attention recently, since it can be realised with ultracold atoms in d-dimensional optical lattices ($d = 1, 2, 3$) [26], allowing one to investigate the superfluid-to-Mott insulator quantum phase transition in great detail in the laboratory [27, 28, 29, 30].

We view ultracold Bose particles in an optical lattice as prime candidates for exploring the concept of dressed matter waves. Namely, atoms in a 1d lattice can be subjected to a time-periodic lattice modulation, to the effect that a term of the form

$$\hat{H}_1(t) = [K_0 + K_\omega \cos(\omega t)] \sum_{\ell=1}^{M} \ell \hat{n}_\ell \tag{58}$$

is added to the system (55). Here K_ω denotes the amplitude of a drive with angular frequency ω, typically on the order of one to a few kilohertz [4], while K_0 corresponds to a static lattice tilt [5]; the extension to lattices with higher dimension is straightforward. If there were no interaction between the particles, that is, for $U/J = 0$, the total Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t)$ would be identical in form to the systems studied in Secs. 2 and 3: One faces nearest-neighbour coupling combined with homogeneous site-diagonal forcing. Hence, when the resonance condition

$$K_0 = \nu\hbar\omega \tag{59}$$

corresponding to the previous equations (8) and (36) is satisfied, so that the energy of ν "photons" matches the energy shift induced by the static tilt between adjacent sites, one can adapt the results (19) and (40): Under such conditions, the driven system behaves approximately like an undriven one with the modified hopping matrix element

$$J_{\text{eff}} = (-1)^\nu \mathrm{J}_\nu(K_\omega/\hbar\omega)\, J \;. \tag{60}$$

More careful analysis [6, 7] shows that this expression remains valid even for nonzero U at least in the high-frequency regime where $\hbar\omega \gg U$ and $\hbar\omega \gg J$, thus including the strong-coupling case $U/J \gg 1$. The experimental data available so far give clear evidence of this modification (60) both for $\nu = 0$, when there is no static tilt [4], and for $\nu = 1, 2$, when one has "photon"-assisted tunnelling [5]. This finding now directly leads to a further consequence: The ratio U/J governs the superfluid-to-Mott insulator transition in the bare Bose–Hubbard model; this control parameter has to be replaced by U/J_{eff} in the presence of resonant forcing. Since J_{eff} depends significantly on the amplitude K_ω, it should be possible to cross the border between the superfluid and the insulator regime by varying that amplitude [6, 7]. However, the notion of a "superfluid" or a "Mott insulator" refers to the ground state of the Bose–Hubbard model, so that it becomes necessary to guide the ground state of the undriven system $\hat{H}_0$ into the effective ground state of the driven system $\hat{H}_0 + \hat{H}_1(t)$. That "effective ground state", of course, is the Floquet state which originates from the ground state of $\hat{H}_0$ when the drive is turned on, so that the adiabatic principle discussed in Sec. 4 comes into play: In an experiment aiming at a demonstration of a superfluid-to-Mott insulator transition induced by time-periodic forcing, the driving amplitude should be turned on smoothly, such that the system's wave function can follow the Floquet state connected to the unperturbed ground state. But since adiabatic following in a periodically forced many-level system with a "dense" quasienergy spectrum is endangered by a host of multiphoton-like resonances, the precise choice of the protocol is not trivial: The frequency has to be chosen such that major resonances are avoided, while the amplitude has to vary sufficiently slowly in order to allow for a reasonable degree of adiabaticity, but still sufficiently fast in order to pass minor resonances before they become active.

We illustrate these deliberations by numerical calculations for small systems with $N = 7$ particles on $M = 7$ lattice sites. The initial N-body wave function $|\psi(0)\rangle$ at time $t = 0$ is chosen as the ground state of the Bose–Hubbard model (55) with $U/J = 3$, falling into the superfluid regime when the system is sufficiently large. Here we restrict ourselves to $K_0 = 0$, i.e., to $\nu = 0$; related studies for $\nu = 1, 2$ are documented in Ref. [7]. After selecting some frequency ω, and thus specifying the time scale $T = 2\pi/\omega$, the time-dependent force is turned on with an amplitude $K_\omega(t)$ which rises linearly between $t = 0$ and $t_1 = 100\,T$ from $K_\omega/\hbar\omega = 0$ to $K_{\max}/\hbar\omega = 2.4$. The latter value lies close to the first zero of J_0, and thus gives a quite large ratio U/J_{eff}, which should place the system far into the Mott-like regime. Then the amplitude is kept constant at $K_{\max}$ between t_1 and $t_2 = 200\,T$, and finally ramped linearly back to zero between t_2 and $t_3 = 300\,T$. The corresponding N-body wave function $|\psi(t)\rangle$ is computed by plain direct solution of the time-dependent Schrödinger equation, not taking any recourse at all to Floquet theory. From that wave function, the single-particle quasimomentum distribution

$\rho(p, t)$ is obtained according to

$$\rho(p, t) = \frac{1}{M} \sum_{\ell,j} \exp\left[i\frac{(\ell - j)p}{\hbar/a}\right] \langle\psi(t)|\hat{b}_\ell^\dagger \hat{b}_j|\psi(t)\rangle \, , \tag{61}$$

and recorded at integer multiples of T [6, 7]. This momentum distribution is sharply peaked, due to (quasi) long-range phase coherence, in the superfluid phase, but apparently structureless in the Mott regime.
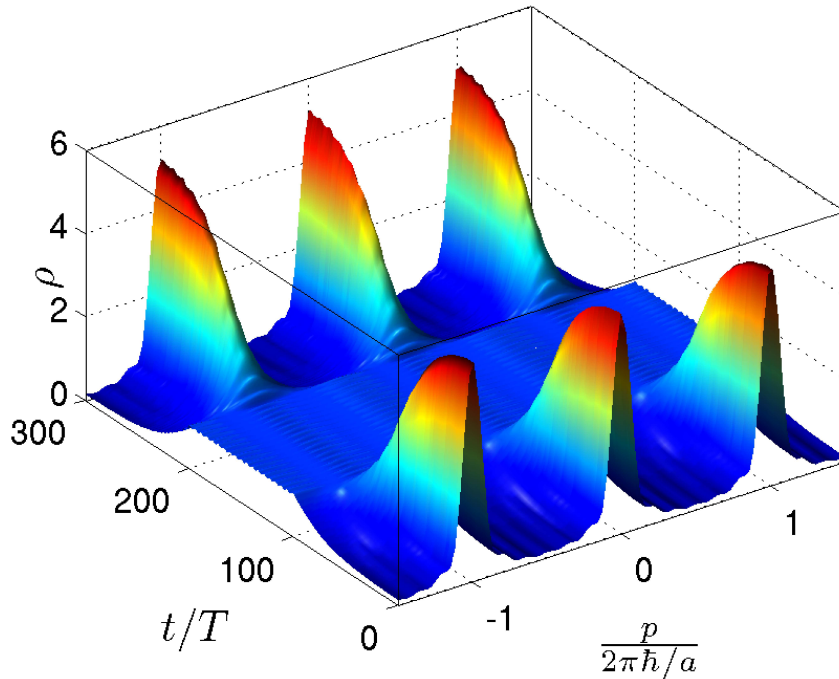

Figure 1. Quasimomentum distribution computed according to Eq. (61) for a driven Bose–Hubbard model with $N = 7$ particles on $M = 7$ sites. The interaction strength is $U/J = 3$, the driving frequency is $\hbar\omega/J = 16$. Between $t = 0$ and $t_1 = 100\,T$ the amplitude $K_\omega(t)$ is increased linearly from zero to $K_{\max} = 2.4\,\hbar\omega$, then kept constant until $t_2 = 200\,T$, and finally ramped linearly back to zero between t_2 and $t_3 = 300\,T$. Here the system is able to follow adiabatically.

Figure 1 shows the results for $\hbar\omega/J = 16$: Initially one finds a strongly peaked distribution, as expected for a superfluid-like state, which then becomes practically flat at $t = t_1$. However, after the driving amplitude $K_\omega(t)$ has been switched off again at $t = t_3$, the sharp pattern reappears: This fact clearly signals that the flat distribution between t_1 and t_2 is not due to loss of coherence resulting from uncontrolled excitations, but rather indicates the Mott-like regime, since otherwise it would not be possible to switch back (almost) adiabatically to the initial state. Thus, this figure provides a glimpse at a quantum phase transition induced by "dressing" a matter wave, although, of course, a truly sharp "transition" cannot be achieved with $N = M = 7$.

But the small system already is sufficently rich to demonstrate that an ideal outcome cannot be taken for granted: Fig. 2 shows a momentum distribution obtained in the same manner for a lower frequency, $\hbar\omega/J = 12$. Whereas a signature like this might not be distinguishable from
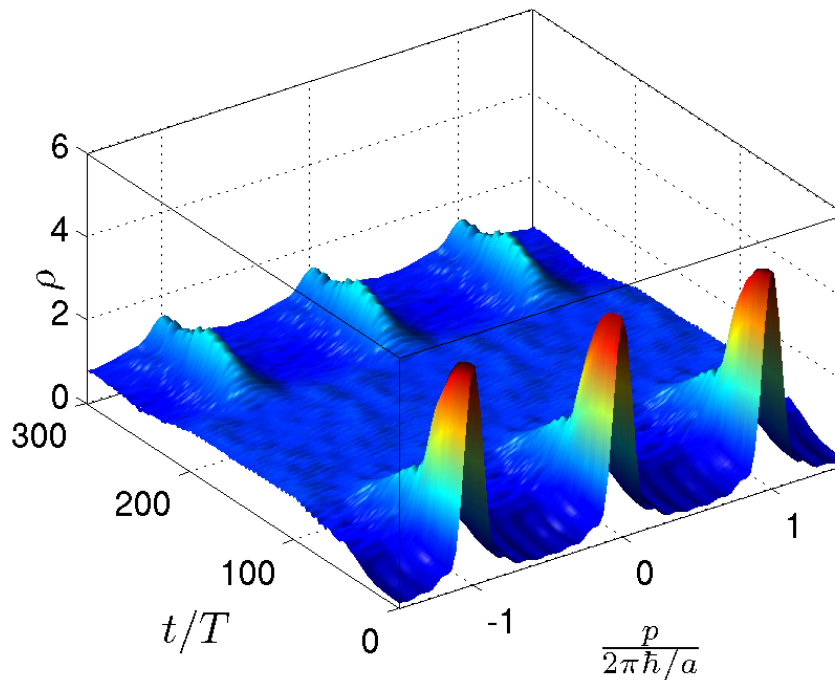
Figure 2. As Fig. 1, but for a lower frequency $\hbar\omega/J = 12$. In this case, adiabatic following is destroyed by Landau–Zener transitions at the avoided quasienergy crossings visible in Fig. 4.

that in Fig. 1 in an actual experiment for times up to t_2, here the initial pattern is not restored at t_3, indicating severe deviations from the desired adiabatic following.

Inspection of the corresponding quasienergy spectra immediately reveals the reason for the different dynamics found in both cases. In Fig. 3 we depict a part of the quasienergy spectrum for the frequency employed in Fig. 1, whereas Fig. 4 shows the spectrum for the lower frequency underlying Fig. 2; both spectra have been computed with $N = M = 5$ by solving the eigenvalue equation (27). In the first case, the quasienergy line emanating from the ground state is not visibly affected by other states, indicating the absence of active resonances and thus enabeling the adiabatic return observed in Fig. 1; essentially; only one single instantaneous Floquet state is populated during the entire process. In contrast, in the second case the quasienergy level originating from the ground state undergoes several large avoided crossings. Incomplete Landau–Zener transitions at these avoided quasienergy crossings then lead to a significant population of the anticrossing Floquet states [22], rendering an adiabatic return to the initial state impossible.

6. Conclusions

Ultracold atoms in time-periodically modulated optical lattices give rise to "dressed matter waves", in analogy to the dressed atoms known from atomic physics [1]. Such dressed systems acquire properties quite different from their "bare" antecedents, the modification of atomic Landé g-factors setting a prominent example [2]. An effect closely related to this g-factor modification is a transition of ultracold bosonic atoms in a modulated optical lattice from a superfluid to a Mott-insulator-like state in response to a variation of the modulation strength; this transition is mediated by a modification of the nearest-neighbour hopping matrix elements relying on precisely the same mechanism as that of the g-factors. An experimental verification of
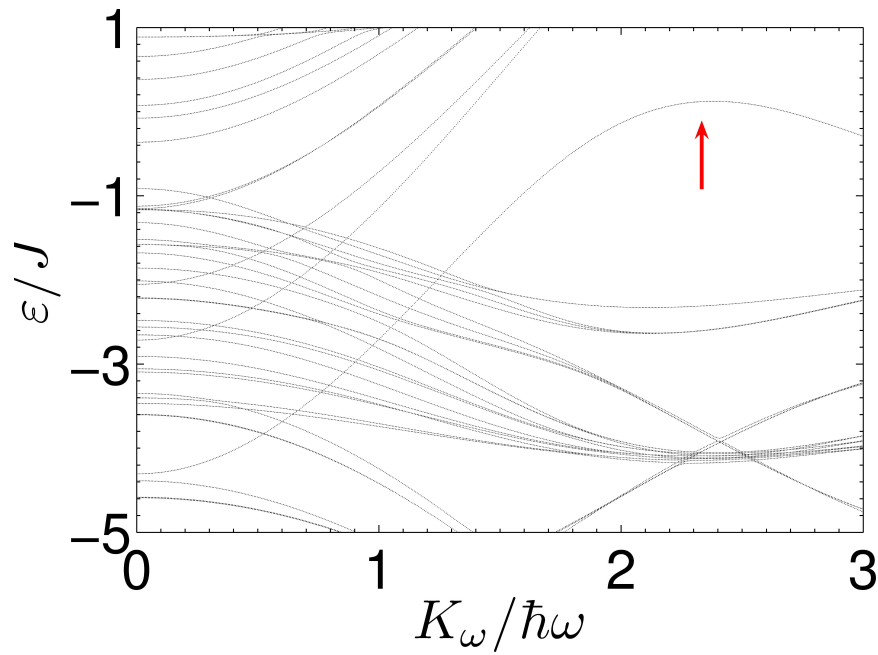
Figure 3. Part of the quasienergy spectrum for the almost ideal case considered in Fig. 1. The arrow marks the quasienergy of the Floquet state evolving adiabatically from the ground state of the undriven Bose–Hubbard system. Computed with $N = M = 5$.
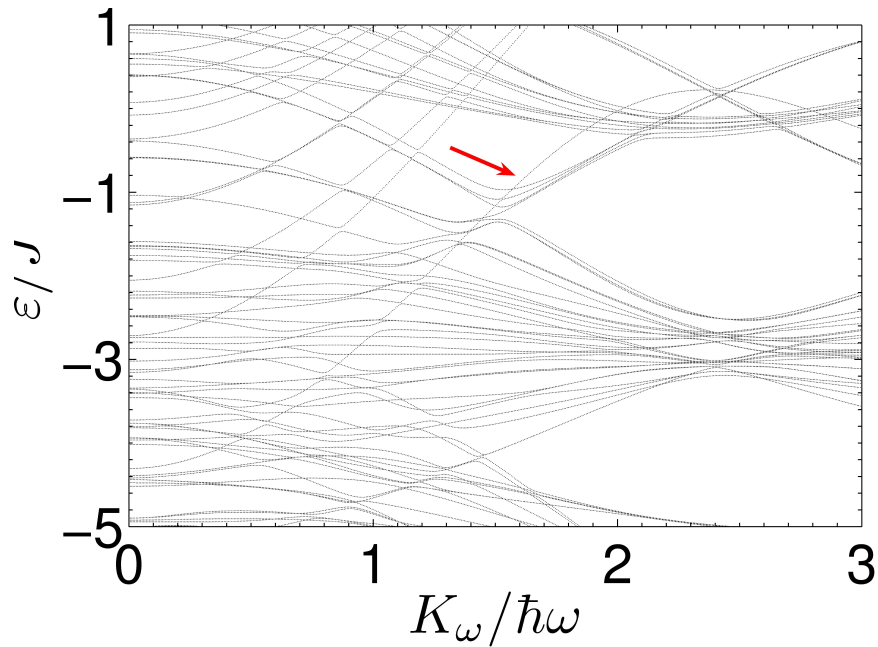

Figure 4. Part of the quasienergy spectrum ($N = M = 5$) for the thwarted case considered in Fig. 2; the arrow indicates the quasienergy associated with the ground state. Observe that here there are several active resonances, corresponding to pronounced avoided level crossings.

this proposal involves adiabatic following of the many-body Floquet state originating from the ground state of the bare system; such adiabatic following is a quite tricky concept in the context of driven matter waves. The question to what extent multiphoton-like resonances can be avoided (or perhaps deliberately be induced and exploited) is open to experimental investigation.

The long-term perspective of these considerations, however, seems to lie elsewhere. Just as the g-factor modification is but *one* facet of the dressed-atom picture, there are further possibilites of controlling the state of a matter wave in an optical lattice by time-periodic forcing. Our present scheme defines a first cornerstone; if achieved, more demanding ones can follow. In particular, it might be interesting to resonantly couple different Wannier states located at the same site, and thus to open up new ways of quantum state engineering. The experimentally established fact that Bose–Einstein condensates in optical lattices can be subjected to strong forcing without destroying their phase coherence [4] is a sound cause for optimism.

Acknowledgments

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