Journal of Physics: Conference Series

PAPER • OPEN ACCESS

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IOP Conf. Series: Journal of Physics: Conf. Series **974** (2018) 012072 doi:10.1088/1742-6596/974/1/012072

THE CONVERGENCE PROBLEMS OF EIGENFUNCTION EXPANSIONS OF ELLIPTIC DIFFERENTIAL OPERATORS

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Abstract. In the present research we investigate the problems concerning the almost everywhere convergence of multiple Fourier series summed over the elliptic levels in the classes of Liouville. The sufficient conditions for the almost everywhere convergence problems, which are most difficult problems in Harmonic analysis, are obtained. The methods of approximation by multiple Fourier series summed over elliptic curves are applied to obtain suitable estimations for the maximal operator of the spectral decompositions. Obtaining of such estimations involves very complicated calculations which depends on the functional structure of the classes of functions. The main idea on the proving the almost everywhere convergence of the eigenfunction expansions in the interpolation spaces is estimation of the maximal operator of the partial sums in the boundary classes and application of the interpolation Theorem of the family of linear operators. In the present work the maximal operator of the elliptic partial sums are estimated in the interpolation classes of Liouville and the almost everywhere convergence of the multiple Fourier series by elliptic summation methods are established. The considering multiple Fourier series as an eigenfunction expansions of the differential operators helps to translate the functional properties (for example smoothness) of the Liouville classes into Fourier coefficients of the functions which being expanded into such expansions. The sufficient conditions for convergence of the multiple Fourier series of functions from Liouville classes are obtained in terms of the smoothness and dimensions. Such results are highly effective in solving the boundary problems with periodic boundary conditions occurring in the spectral theory of differential operators. The investigations of multiple Fourier series in modern methods of harmonic analysis incorporates the wide use of methods from functional analysis, mathematical physics, modern operator theory and spectral decomposition. New method for the best approximation of the square-integrable function by multiple Fourier series summed over the elliptic levels are established. Using the best approximation, the Lebesgue constant corresponding to the elliptic partial sums is estimated. The latter is applied to obtain an estimation for the maximal operator in the classes of Liouville.

1. Introduction

The mathematical models of the various vibrating systems are partial differential equations elliptic type and such equations, in the case of self adjoint elliptic operators, can be represented as a limit of eigenfunction expansions of the elliptic operators. The problem of funding the solution of partial differential equations will be reduced to the determining the conditions under which eigenfunction expansions of the functions from certain classes, which are used as a boundary values for the solution of the model equations, are convergent. Most complete solution of the expansion problems can be obtained by proper development of the spectral theory of elliptic differential operators. It is well known that the spectral theory of differential operators also used to describe slow flows of viscous incompressible fluids. Many physical process taking place in real space can be described using the spectral theory of elliptic differential operators. Many of the equations of physical sciences and engineering involve operators of elliptic type. Most important among these is non-relativistic quantum theory, which is based upon the spectral analysis of second order elliptic differential operators. Applications of second order elliptic operators to geometry and stochastic analysis are also now of great importance. Spectral theory of the elliptic differential operators is an extremely rich field which has been studied by many qualitative and quantitative techniques like Sturm-Liouville theory, separation of variables, Fourier and Laplace transforms, perturbation theory, eigenfunction expansions, variational methods, microlocal analysis, stochastic analysis and numerical methods including finite elements.

2. Eigenfunction expansions of the elliptic operators

We define Torus as a cube $[-\pi, \pi]^N$:

$$T^N = \{x = (x_1, x_2, \cdots, x_N) \in R^N : -\pi < x_i \leq \pi, i = 1, \cdots, N\},$$

naturally isomorphic to R^N/Z^N. Precisely, for $x, y \in R^N$ we say that

$$x \equiv y,$$

if $x - y \in 2\pi Z^N$. Here $\equiv$ is an equivalence relation that partitions R^N into equivalence classes, where $2\pi Z^N$ is the additive subgroup of R^N and Z^N is integer coordinates.

Let consider an arbitrary differential operator with constant coefficients:

$$A(D) = \sum_{|\alpha| \leq m} a_\alpha \, D^\alpha,$$

where $D^\alpha = D_1^{\alpha_1} \ldots D_N^{\alpha_N}$, $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$, α is a multi-index.

The polynomial $A(\xi)$ is associated with differential operator $A(D)$ by replace D with $\xi \in R^N$ and it is called a symbol of operator $A(D)$, and the homogeneous polynomial

$$A_h(\xi) = \sum_{|\alpha| = m} a_\alpha \, \xi^\alpha$$

is called its principle symbol. The operator $A(D)$ is said to be elliptic of order m if its principle symbol is positive, i. e. $A_h(\xi) > 0$ for all $\xi \in R^N, \ \xi \neq 0$.

The operator $A(D)$ is considered in the Hilbert space $L_2(T^N)$ as an unbounded operator (we refer for the details [2]) with domain of definition $C^\infty(T^N)$ the class of infinitely differentiable functions on T^N, this class is dense in $L_p(T^N)$.

In case the coefficients are real, the $A(D)$ will satisfy the symmetric condition:

$$(Au, v) = (u, Av), \ \forall \, u, v \in C^\infty(T^N).$$

In addition, since the operator $A(D)$ is elliptic, then by Gårding's inequality the operator $A(D)$ is semi-bounded

$$(Au, u) \geq \lambda_A(u, u), \quad \forall \, u \in C^\infty(T^N),$$

where nonnegative constant λ_A is called lower bound of A. Hence, thanks for Friedrichs's theorem, asserts that for every symmetric semi-bounded operator there are at least one self-adjoint extension with the same lower bound, then there is a self-adjoint extension $\bar{A}$ in $L_2(T^N)$

IOP Conf. Series: Journal of Physics: Conf. Series **974** (2018) 012072 doi:10.1088/1742-6596/974/1/012072

of operator $A(D)$ which, indeed, its closure, and they are coincided on the domain of definition i. e. $\bar{A}u = Au,\ u \in C^\infty(T^N)$.

By von Neumann's spectral theorem, the operator $\bar{A}$ has a spectral decomposition of unity $\{E_\lambda\}$, and then it can be represented in the following form

$$\bar{A} = \int_{\lambda_A}^{\infty} \lambda\, dE_\lambda,$$

the projections E_λ increase monotonically, and continuous on the left, moreover

$$\lim_{\lambda\to\infty} \|E_\lambda u - u\|_{L_2(T^N)} = 0, \quad u \in L_2(T^N).$$

The operator $\bar{A}$ has a complete orthonormal system of eigenfunctions $\{(2\pi)^{-N/2}e^{inx}\}$ in $L_2(T^N)$ corresponding to the eigenvalues $A(n),\ n \in Z^N$. Thus, the spectral decomposition of $f \in L_2(T^N)$ coincides with partial sums of the multiple Fourier series of function f related to $A(n)$. A interesting fact that the lower order coefficients $a_\alpha, |\alpha| < m$ of $A(D)$ do not influence the convergence of the spectral decomposition $E_\lambda f$, provided the function f is sufficiently smooth. Then one can reduce the study of convergence for partial sum of eigenfunction expansions to the study of simpler case, that is, applying the summation over expanding its principle symbol $A_h(n)$.

The spectral decomposition E_λ can be written as an integral operator:

$$E_\lambda f(x) = \int_{T^N} \Theta_\lambda(x,y) f(y)\, dy,$$

where the kernel

$$\Theta_\lambda(x,y) = (2\pi)^{-N} \sum_{A(n)<\lambda} e^{i(n,x-y)}$$

is called the spectral function of operator $\bar{A}$. Indeed, the operator E_λ is a convolution operation $\Theta_\lambda * f$, such that

$$E_\lambda f(x) = \int_{T^N} \Theta_\lambda(x-y) f(y)\, dy.$$

The following is proved in the paper [12]:

Theorem 1.1 Let $\alpha > \frac{N-1}{2}$. Then the maximal operator E_* is bounded from $L_1^\alpha(T^N)$ to $L_1(T^N)$:

$$\|E_* f(x)\|_{L_1(T^N)} \le C\, \|f\|_{L_1^\alpha(T^N)}, \quad \forall\, f \in L_1^\alpha(T^N), \quad \alpha > \frac{N-1}{2},$$

where the constant C depends only on N and α. Furthermore

$$\lim_{\lambda\to\infty} E_\lambda f(x) = \lim_{\lambda\to\infty} \sum_{A(n)<\lambda} \hat{f}(n)\, e^{i(n,x)} = f(x)$$

almost everywhere in T^N.

In this paper we extend this result for the case of general liuoville classes $L_p^\alpha(T^N)$.

Theorem 1.2 Let numbers s and p be related by the condition

$$s > (N-1)(\frac{1}{p} - \frac{1}{2}), \ \ 1 \le p \le 2.$$

Then the maximal operator E_* is bounded from $L_p^s(T^N)$ to $L_p(T^N)$:

$$\|E_*f\|_{L_p(T^N)} \leq C_p(N,s)\ \|f\|_{L_p^s(T^N)},\ \ \forall f \in L_p^s(T^N),$$

where the constant $C_p(N,s)$ depends on the degree p of integrability of f, dimension N and smoothness s.

The necessary and sufficient conditions for the almost everywhere convergence of the multiple series and Fourier integrals by spherical Riesz means in the classes $L_p(T^N)$ are obtained in [17],[18]. The papers [5], [6] are devoted to the problems on the almost everywhere convergence of multiple Fourier series and integrals by Riesz method corresponding to the elliptic differential operators. The methods in these papers based on the application of the Poisson formula to estimate the maximal operator for the Fourier series, where the conditions for the order of the Riesz means s: $s > (N-1)/2$ is required. The almost everywhere convergence of the partial sums of the Fourier integrals in the classes of Liouville $L_p^\alpha(R^N)$ are investigated in the paper [8], where the condition $\alpha > (N-1)/2$ guarantees the almost everywhere convergence of the spectral expansions. So far no results on the almost everywhere convergence of multiple Fourier series in the classes of $L_p^\alpha(T^N)$. In such case the difficulty arises because of not applicability of Poisson formula. In the current work the almost everywhere convergence problems of multiple Fourier series in the Liouville classes are investigated by using the estimation of Lebesgue constant and applying Jackson theorems. The almost everywhere convergence of the Fourier-Laplace series on the unit sphere is obtained in [4]. The almost everywhere convergence of spectral expansions of the general elliptic differential operators in the Liouville classes are investigated in [10], and the summability is established for the expansions of the function from L_p^α under the conditions $\alpha > N(\frac{1}{p}-\frac{1}{2}), 1 \leq p \leq 2$. The extension of the latter result to the case of pseudo-differential operators is carried out in [7].

3. Liouville Space $L_p^s(T^N)$

We deal with the functions from a class which are termed as a Liouville class L_p^s where $s > 0$ (coincide for s integers with Sobolev class). We study the differentiability and smoothness of functions . There are several ways to interpret smoothness and numerous ways to describe it and quantify it. A fundamental fact is that smoothness can be measured and fine-tuned the Fourier transform. Indeed, the investigation of the subject is based on this point. Certain spaces of functions are introduced to serve the purpose of measuring smoothness, for instance, Lipschitz, Sobolev and Hardy spaces. However, the main function spaces we introduce here are Sobolev space and its generalization Liouville space.

The Liouville space is considered as generalization of Sobolev Spaces, where the classes $W_p^k(T^N)$ can also be described in terms of Fourier transform. By that one can extend the definition of Sobolev space to the case in which the index k is real. We explain the idea by putting $p=2$. By using the Plancherels Theorem we obtain that the norm for $f \in W_2^k$ can be introduced in the form

$$\|f\|^2_{W_2^k} = \sum_{n\in Z^N}\sum_{|\alpha|\leq k}|n^\alpha|^2|\hat{f}(n)|^2.$$

The following inequality

$$c_k(1+|\xi|^2)^{\frac{k}{2}} \leq \sum_{|\alpha|\leq k}|\xi^\alpha|^2 \leq C_k(1+|\xi|^2)^{\frac{k}{2}},$$

which holds for all $\xi \in R^N$ with some positive constants c_k, C_k, gives a motivation for the following definition:

Let s be a real number and let $1 < p < \infty$. The Liouville space $L_p^s(T^N)$ is defined as the space of all functions u in $L_p(T^N)$, such that

$$\sum_{n \in Z^N} (1 + |n|^2)^{\frac{s}{2}} \hat{f}(n) e^{inx} \in L_p(T^N). \tag{1}$$

Thus the norm of f in $L_p^s(T^N)$ has a form

$$\|f\|_{L_p^s(T^N)} = \Big\| \sum_{n \in Z^N} (1 + |n|^2)^{\frac{s}{2}} \hat{f}(n) e^{inx} \Big\|_{L_p(T^N)}.$$

We observe that when $s = 0$ then $L_p^s = L_p$. The space L_p^s coincides with the space W_p^k when $s = k$ a positive integer number. We also note that when $s \geq 0$ the elements of L_p^s are always L_p, i.e. $L_p^s \subseteq L_p$.

If $0 < s_1 < s$ then $L_p^s \subset L_p^{s_1}$. Moreover we have the embedding

$$L_p^s \to L_q^{s_1}, \quad L_p^s \subset L_q^{s_1}$$

provided

$$s - \frac{N}{p} = s_1 - \frac{N}{q}.$$

4. Fractional Powers of Elliptic Operators

Let A be an elliptic differential operator of order m on the torus T^N. Let Λ be a subset of the complex plane (in the applications this will, as a rule, be an angle with the vertex at the origin). In spectral theory it is useful to consider operators depending on a parameter $\lambda \in \Lambda$. An example of such an operator is the resolvent $(A - \lambda I)^{-1}$.

Let proceed with the definition of the elliptic differential operators. For this reason we consider a polynomial on $\xi \in R^N$ of even order $m = 2m'$ with coefficients from $C^\infty(T^N)$

$$A(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha.$$

The formal differential operator

$$A(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

is said to be an elliptic of order m, if $\forall x \in T^N$ and $\xi \in R^N$ for $\xi \neq 0$ the principal symbol $A(x, \xi)$ is positive:

$$A(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha > 0.$$

The polynomial $A(x, \xi)$ is called a symbol of the operator $A(x, D)$. We denote by A an operator acting in the Hilbert space $L_2(T^N)$ with the domain of definition $D(A) = C^\infty(T^N)$:

$$Au = A(x, D)u(x), \quad u \in C^\infty(T^N)$$

.

We assume that the operator A is symmetric

$$(Au, v) = (u, Av), \quad \forall u, v \in C^\infty(T^N).$$

Furthermore the operator A is called semi-bounded if there exists positive number κ such that the following inequality holds

$$(Au, u) \geq \kappa(u, u), \quad \forall u \in C^{\infty}(T^N).$$

It is proven that there exists $\lambda_0 > 0$ such that the resolvent $R_\lambda = (A - \lambda I)^{-1}$ is defined for $|\lambda| \geq \lambda_0$. The spectrum $P(A)$ of A is a discrete subset of the complex plane. We choose a number ρ satisfying the condition that the disk $|\lambda| < 2\rho$ does not intersect with the spectrum $P(A)$. Now we select a contour of the form $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$:

$$\Gamma_1 = \{\lambda \in C : \lambda = re^{i\pi}, +\infty > r > \rho\}$$

$$\Gamma_2 = \{\lambda \in C : \lambda = \rho e^{i\phi}, \pi > \phi > -\pi\}$$

$$\Gamma_1 = \{\lambda \in C : \lambda = re^{i\pi}, \rho < r < +\infty\}.$$

Consider the integral

$$A_z = \frac{i}{2\pi} \int_\Gamma \lambda^z (A - \lambda I)^{-1} d\lambda,$$

where λ^z is defined as a holomorphic function of $\lambda \in C \backslash (-\infty, 0]$, equal to $e^{z \ln \lambda}$ for $\lambda > 0$.

Note that the integral on the $\int_\Gamma \lambda^z (A - \lambda I)^{-1} d\lambda$ converges in the operator on $L_2(T^N)$ for $Re(z) < 0$ and also A_z is a bounded operator on $L_2(T^N)$.

With the help this operator we can define fractional power of the elliptic operator as follows: Let $z \in C$ and $k \in Z$ be such that $Re(z) < k$. Put, on $C^\infty(T^N)$

$$A^z = A^k A_{z-k}.$$

Such defined operator is independent of the choice of integer k and for arbitrary $k \in Z$ and $s \in R$, the function A^z is holomorphic operator function of z in the half-plane $Re(z) < k$ with values in the Banach space $L(H^s(T^N), H^{s-mk}(T^N))$ of bounded linear operators from $H^s(T^N)$ to $H^{s-mk}(T^N)$.

Let the distribution $f \in D'(T^N)$ and $f = \sum_{n \in Z^N} \hat{f}(n) e^{i(n,x)}$ be the Fourier series of f. Let denote eigenvalues of A by λ_n. Then

$$A^z f(x) = \sum_{n \in Z^N} \lambda_n^z \hat{f}(n) e^{i(n,x)},$$

which follows from the following identity

$$A^z\{e^{i(n,x)}\} = \frac{i}{2\pi} \int_\Gamma \lambda^z (A - \lambda I)^{-1} e^{i(n,x)} d\lambda =$$

$$\frac{ie^{i(n,x)}}{2\pi} \int_\Gamma \lambda^z (\lambda_n - \lambda)^{-1} d\lambda = \lambda_n^z e^{i(n,x)},$$

which is obtained by using the Cauchy formula.

5. The Lebesgue constant for the multiple Fourier series summed over elliptic levels

Let B denotes a closed, balanced set in R^N. We denote by B_ϵ the union of the balls of radius ϵ with centers in the boundary of B, and by $\mu(B_\epsilon)$ the measure of B_ϵ. We assume that for the set B the upper surface measure in the sense of Minkowski is bounded:

$$\lim_{\epsilon\to 0}\frac{\mu(B_\epsilon)}{2\epsilon}<\infty.$$

For convex, centrally symmetric bodies with bounded surface satisfy above condition, in the paper [20] the following estimation is obtained:

$$\int_{T^N}\Big|\sum_{n\in\lambda B\cap Z^N} e^{i(x,n)}\Big|\,dx = O(1)\lambda^{\frac{N-1}{2}},\quad \lambda\to\infty.$$

The partial sum $E_\lambda f(x)$ of the Fourier series of f summed over elliptic levels is the convolution $f*\Theta_\lambda$ with the corresponding Dirichlet kernel (spectral function)

$$\Theta_\lambda(x)=\sum_{A(n)\le\lambda} e^{i(x,n)}.$$

The Lebesgue constant corresponding to the self adjoint operator A can be defined as follows

$$L_\lambda=\int_{T^N}|\Theta_\lambda(x)|\,dx=\int_{T^N}\Big|\sum_{A(n)\le\lambda} e^{i(x,n)}\Big|\,dx.$$

Lemma. For the Lebesgue constant one has

$$L_\lambda=\int_{T^N}|\Theta_\lambda(x)|\,dx=O(1)\lambda^{\frac{N-1}{2m}},\quad \lambda\to\infty.$$

In the estimation in [19] as a set B we take $B=\{x\in R^N : A(x)\le 1\}$. The homothetic transform of the S is $\lambda S=\{x\in R^N : A(\frac{x}{\lambda})\le 1\},\ \lambda>0$. Using the condition that the $A(x)$ is homogeneous operator of order m: $A(tx)=t^m\lambda^{-m}A(x)$, we obtain:

$$\int_{T^N}|\Theta_\lambda(x)|\,dx=\int_{T^N}\Big|\sum_{A(n)\le\lambda} e^{i(x,n)}\Big|\,dx=\int_{T^N}\Big|\sum_{A(n/\sqrt[m]{\lambda})\le 1} e^{i(x,n)}\Big|\,dx=\int_{T^N}\Big|\sum_{\sqrt[m]{\lambda}B\cap Z^N} e^{i(x,n)}\Big|\,dx.$$

by taking into account the definition B and From the estimation [] we derive that

$$\int_{T^N}\Big|\sum_{n\in\sqrt[m]{\lambda}B\cap Z^N} e^{i(x,n)}\Big|\,dx=O(1)\lambda^{\frac{N-1}{2m}}.$$

This estimation will be used to establish the almost everywhere convergence of the Fourier series in the classes of $L_p^s(T^N)$.

6. The almost everywhere convergence of the multiple Fourier series

In order to apply interpolation theorem of Stein we need to have appropriate estimation for the maximal operator in the classes of $L_1^s(T^N)$ and $L_2^s(T^N)$. We conclude that if $f\in L_2^s(T^N),\ s>0$, then

$$\sum_{n\in Z^N}|\hat f(n)|^2\log^2(1+|n|)\le\|f\|^2_{L_2^s(T^N)},$$

and the elliptic partial sums of f

$$E_\lambda f(x) = \sum_{A(n)<\lambda} \hat{f}(n) e^{i(n,x)}$$

converges almost everywhere on the torus T^N. Moreover, the maximal operator $E^* f(x)$ is estimated as follows

$$\|E_* f\|_{L_2(T^N)} \leq C \Big(\sum_{n\in Z^N} |\hat{f}(n)|^2 \log^2(1+|n|) \Big)^{\frac{1}{2}} \leq C\|f\|^2_{L_2^s(T^N)}.$$

Now we are ready to apply the interpolation theorem of Stein. But the interpolation theorem is valid for linear operator. We need to linearize the operators T_λ^s. Let $\mu(x)$ be a measurable function on T^N that take at most finitely many distinct values and satisfy the condition $0 \leq \mu(x) \leq \mu_0 < \infty$ and $s(z) = \frac{N-1}{2} z + \delta,\ 0 \leq \Re(z) \leq 1,\ \delta > 0$. We define an analytic family of linear operators:

$$T_{\mu(x)}^{s(z)} = \sum_{A(n)\leq\mu(x)} (1+|A(n)|^2)^{s(z)} \hat{g}(n) e^{inx}, \quad g \in C^\infty(T^N).$$

In the case $p = 2$, which corresponds to the value of z when $\Re(z) = 0$:

$$\|T_{\mu(x)}^{s(i\,\Im(z))} g(x)\|_{L_2(T^N)} \leq C_1 e^{\pi|\Im(z)|/2} \|g\|_{L_2(T^N)}, \quad \Re(z) = 0.$$

Secondly on the line $\Re(z) = 1$, we have

$$\|T_{\mu(x)}^{s(1+i\,\Im(z))} g(x)\|_{L_1(T^N)} \leq C_2 e^{\pi|\Im(z)|/2} \|g\|_{L_1(T^N)}, \quad \Re(z) = 1.$$

Thus, by the interpolation we have for the values of $t : 0 < t < 1$:

$$\|T_{\mu(x)}^{s(t)} g(x)\|_{L_p(T^N)} \leq K_t \|g\|_{L_p(T^N)},$$

where $s(t) = \frac{N-1}{2} t + \delta,\ \frac{1}{p} = \frac{1-t}{2} + \frac{t}{1}$. The definition of $T_{\mu(x)}^{s(t)}$ and $L_p^s(T^N)$ gives

$$\|E_*^{s(t)} f(x)\|_{L_p(T^N)} \leq K_t \|f\|_{L_p^{s(t)}(T^N)},$$

where K_t satisfies $\log K_t \leq C$. And if note that

$$\frac{1}{p} = \frac{1-t}{2} + \frac{t}{1}$$

then

$$s(t) = \frac{N-1}{2} t + \delta = \frac{N-1}{2} \left(\frac{2}{p} - 1 \right) + \delta > (N-1) \left(\frac{1}{p} - \frac{1}{2} \right),\ \delta > 0.$$

This means for $s > (N-1)\left(\frac{1}{p} - \frac{1}{2}\right),\ 1 < p < 2$ we have

$$\|E_*^s f(x)\|_{L_p(T^N)} \leq C\|f\|_{L_p^s(T^N)}.$$

The statement of the Theorem 1.2 is established. As a consequence of the estimation for maximal operator we can prove the almost everywhere convergence of the multiple Fourier series summed over elliptic levels.

Acknowledgments
The current research is supported by Universiti Malaysia Pahang under Universiti Research Grant Scheme RDU170364.

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