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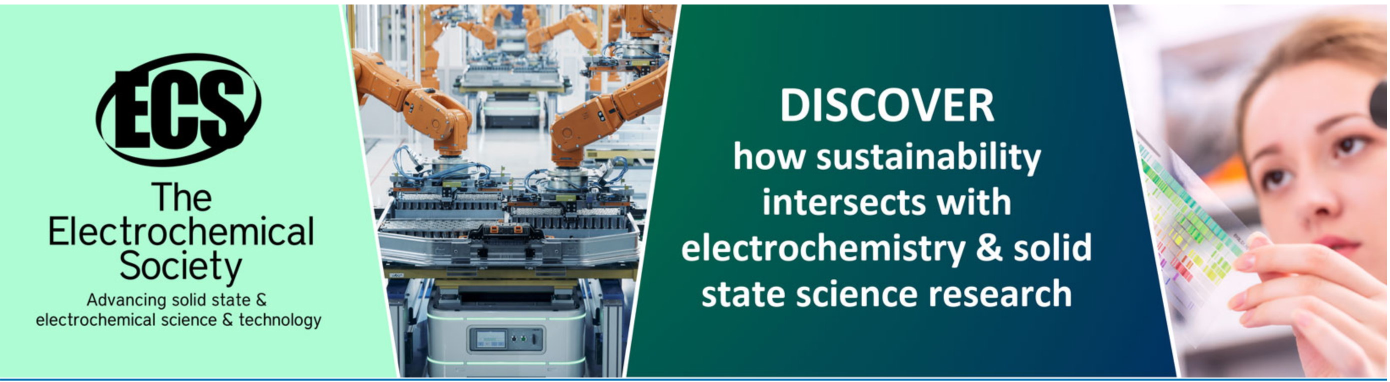

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Spacetimes in Noncommutative Geometry: a definition and some examples

Fabien Besnard

Fabien Besnard, EPF, 3 bis rue Lakanal, Sceaux F-92330

E-mail: fabien.besnard@epf.fr

Abstract. We quickly review the notion of spectral triples - the noncommutative generalization of Riemannian spin manifolds devised by Alain Connes - and propose extensions of it to the Lorentzian and anti-Lorentzian signatures, which we call "spectral spacetimes". We motivate this definition and give several examples.

1. Introduction

There are several meanings one can give to the words "noncommutative spacetime". In this short review we will be concerned with the extension to Lorentzian (mostly plus) or anti-Lorentzian (mostly minus) signature of Alain Connes' Noncommutative Geometry. Several proposals in this direction have already been laid down ([1], [2], [3], [4]), to which we unfortunately cannot do justice here (a more thorough review can be found in [5]). Our approach differs only on a few points from previous ones but these turn out to have important consequences. In particular we can build an elementary and most natural example (section 5.3) which is not covered by existing frameworks. Another important difference is that the C^*-structure is an output rather than an input in our approach.

The paper is organized as follows: in section 2 we recall how the classical theorems of Gelfand and Naimark give a spectral characterization of (locally compact) topological spaces which allows for a noncommutative generalization. This duality is extended to spin geometry in the next section, where we review the definition of spectral triples and give a simple but important commutative and finite example: the canonical spectral triple over a finite weighted graph. In particular we explain how the geodesic distance can be recovered, thanks to a formula of Connes, both in the discrete and the manifold cases. In section 4 we explain the strategy which can be used to generalize Noncommutative Geometry to other, in particular Lorentzian, signatures. The only original proposal in this text is in section 5, where we put forward the definition of "spectral spacetimes". We give some examples of this structure, in particular a discrete example which is canonically constructed over a finite weighted directed graph and is the obvious anti-Lorentzian counterpart of the construction reviewed in section 3. We also define a noncommutative generalization of this example, which we call the "split Dirac structure".

2. The spectral characterization of space

Two classical theorems of Gelfand and Naimark are often viewed as having opened the doorway to the world of Noncommutative Geometry. In order to state them, let us recall some definitions.

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Definition 1 *A* C^*-algebra *is a complete normed* $\mathbb{C}$*-algebra* $\mathcal{A}$ *with an involution* $*$ *such that for all* $a \in \mathcal{A}$, $\|a^*a\| = \|a\|^2$. *An element of* $\mathcal{A}$ *is said to be* positive *iff it is of the form* a^*a *for some* a. *A linear functional* ω *on* $\mathcal{A}$ *is said to be a* state *iff it has norm* 1 *and* $\omega(a) \in \mathbb{R}^+$ *for all positive* $a \in \mathcal{A}$. *A state is said to be* pure *if it cannot be decomposed as a non-trivial convex combination of other states.*

Let us give some examples.

(i) Let X be a locally compact Hausdorff topological space. Then the algebra $\mathcal{C}_0(X)$ of complex continuous functions on X vanishing at infinity, equipped with the supremum norm and complex conjugation of functions, is a commutative C^*-algebra. If X is compact, then $\mathcal{C}_0(X) = \mathcal{C}(X)$ and the algebra is unital. On $\mathcal{C}_0(X)$ the pure states are the evaluation maps $\hat{x} : f \mapsto f(x)$, which send a function f to its value at x.

(ii) Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the algebra of bounded operator on $\mathcal{H}$ equipped with the Hilbert adjoint $*$ and the operator norm. Then any subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ which is stable by $*$ and norm closed is a C^*-algebra. In particular $B(\mathcal{H})$ itself is a C^*-algebra. For any vector ψ of norm 1, one can define $\omega_\psi : a \mapsto \langle \psi, a\psi \rangle$, which can be proven to be a pure state on $B(\mathcal{H})$. (These are not the only pure states on this algebra.)

The theorems proven by Gelfand and Naimark in the 1940's assert that there are essentially no other example of C^*-algebras besides the two above. More precisely, according to the first theorem, any commutative C^*-algebra $\mathcal{A}$ is canonically isomorphic to the algebra of continuous functions vanishing at infinity defined on its space of pure states $\mathcal{P}(\mathcal{A})$, which turns out to be Hausdorff and locally compact. The isomorphism is the map $a \mapsto (\omega \mapsto \omega(a))$, i.e. a is identified by biduality to the map which evaluates pure states on a. The second theorem shows that any C^*-algebra has a unitary faithful representation on some Hilbert space, i.e. is isomorphic to a C^*-algebra of the kind given above as second example.

In view of these theorems, giving a commutative C^*-subalgebra of $B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, or giving a locally compact Hausdorff space amount to the same. The interest of the first point of view is that it can be generalized merely by removing the hypothesis of commutativity, defining the field of "Noncommutative Topology" (for more on this, see [6]).

3. Spectral triples, Connes' distance formula, noncommutative 1-forms

3.1. Definition

In order to go beyond topology, one needs to enrich the structure. Alain Connes found an algebraic equivalent of Riemannian spin manifolds, in the spirit of Gelfand and Naimark.

Definition 2 *A* real even spectral triple *is a multiplet* $S = (\mathcal{A}, \mathcal{H}, \pi, D, C, \chi)$ *with* $\mathcal{A}$ *a* C^**-algebra,* $\mathcal{H}$ *a Hilbert space,* π *a representation of* $\mathcal{A}$ *on* $\mathcal{H}$*,* D *and* χ *linear operators and* C *an antilinear operator, such that:*

(i) $\chi^2 = 1$, $\chi^* = \chi$, $[\chi, \pi(\mathcal{A})] = 0$, $\{\chi, D\} = 0$,

(ii) D *is densely defined, self-adjoint, and satisfy additional analytical conditions,*

(iii) $\mathcal{A}_{\text{Lip}} := \{a \in \mathcal{A} | [D, \pi(a)]$ *is bounded* $\}$ *is dense in* $\mathcal{A}$,

(iv) $C^2 = \pm 1$, $C^*C = 1$, $\{C, D\} = 0$, $C\chi = \pm \chi C$.

In the definition above $\{.,.\}$ stands for the anti-commutator. Some comments are in order.

(i) We have only quoted the real even case, since this is the only one we will deal with. It corresponds to Riemannian spin manifolds of even dimension in the commutative case.

(ii) The operator D is called the Dirac operator. The analytical conditions which we have omitted concern the growth of its eigenvalues.

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(iii) The signs in (iv) define the KO-dimension of S, which is an integer modulo 8. We refer to [7] for details.

Given a Riemannian spin manifold (M, g, σ), with g the metric and σ the spin structure, one can build its so-called *canonical triple* $S_{\text{can}}(M, g, \sigma)$ where the algebra is $\mathcal{C}_0(M)$, $\mathcal{H}$ is the Hilbert space of square integrable spinor fields, D is the usual Dirac operator, given locally by $D = i\sum_k e_k \nabla^S_{e_k}$, where (e_k) is a moving frame and ∇^S is the spin connection, χ is the chirality element of the Clifford bundle, which defines the orientation, and C is the charge conjugation operator, which selects the real form $Cl(M, g)$ of the complex Clifford bundle $\mathbb{C}l(M)$ (it commutes with real vector fields). Under some regularity conditions, $S_{\text{can}}(M, g, \sigma)$ is the only example of spectral triple with a commutative algebra (Connes' reconstruction theorem [8]), up to $D \to D + \rho$, where ρ is a linear operator which acts fibrewise on the spinor fields by $\psi \mapsto (x \mapsto \rho(x)\psi(x))$, with $\rho(x)$ an endomorphism of the spinor module at $x \in M$.

Though the proof of the reconstruction theorem is extremely involved, the metric is quite easily recovered through the geodesic distance d_g it defines. This is done thanks to the following proposition.

Proposition 1 *([9], chap. VI) Let S be a spectral triple with algebra $\mathcal{A}$, and let $\mathcal{P}(\mathcal{A})$ be the space of pure states of $\mathcal{A}$. Then the formula*

$$d_C(\omega, \omega') = \sup_{a \in \mathcal{A}_{\text{Lip}}} \{|\omega(a) - \omega'(a)|, \|[D, \pi(a)]\| \leq 1\} \tag{1}$$

defines a (possibly infinite) distance on $\mathcal{P}(\mathcal{A})$ (Connes' distance). If $S = S_{\text{can}}(M, g, \sigma)$, then $d_C = d_g$.

The crucial point in the proof of the second part of the proposition, is the observation that if a is a smooth complex function, $[D, \pi(a)] = i\gamma(da)$, where $\gamma : \Omega^1(M) \to \mathbb{C}l(M)$ is the map which identifies (complex) 1-forms on M with sections of the Clifford bundle, through the musical isomorphism and the canonical inclusion of TM into $Cl(M, g)$. As an aside, we immediately get the equality

$$\{\sum_i \pi(a_i)[D, \pi(b_i)], a_i, b_i \in \mathcal{C}^\infty(M)\} = \gamma(\Omega^1(M)) \tag{2}$$

The left-hand side of (2) does not make use of commutativity. Hence, for any spectral triple $S = (\mathcal{A}, \pi, \ldots)$, we can define noncommutative 1-forms to be the elements of $B(\mathcal{H})$ of the form $\omega = \sum_i \pi(a_i)[D, \pi(b_i)]$, where a_i, b_i are "smooth" elements in $\mathcal{A}$, that is, elements on which the derivation $[|D|, .]$ can act an arbitrary number of times.

3.2. A discrete example

One of the virtue of Noncommutative Geometry is its ability to handle the discrete and the continuous objects within the same formalism. As a very simple example, consider two points separated by a distance $\delta > 0$. Let us consider the spectral triple with algebra $\mathcal{A} = \mathbb{C}^2$, representation $\pi(a_1, a_2) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ acting on $\mathcal{H} = \mathbb{C}^2$ equipped with the canonical scalar product, and Dirac operator $D = \frac{1}{\delta}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (the chirality and charge conjugation do not matter for the moment). Since $[D, \pi(a_1, a_2)] = \frac{a_2 - a_1}{\delta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, it is immediate to check that Connes' distance between the two points (seen as pure states) is exactly δ.

It turns out that the generalization of this example to any finite metric space is not as obvious as one could expect. Since this will be important in the sequel, let us consider in some detail the case of a finite weighted graph $G = (V, E, \delta)$, with V the set of vertices, E the set of edges, and $\delta : E \to]0, +\infty[$ a weight function. This can be seen as a finite model of a manifold. The weight function and the adjacency structure of the graph give rise to a geodesic distance d on V. We now want to build a spectral triple such that its Connes' distance is equal to d. The solution, known for a long time ([10]), is to use a representation of the algebra with non-trivial multiplicities. We will present it here in a systematic way using the notion of *split graph* ([5]). The split graph of G is $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{\delta})$, where $\tilde{V} = E \times \{-; +\}$, $\tilde{E} = \{((e, -); (e, +))|e \in E\}$, and $\tilde{\delta}((e, -); (e, +)) = \delta(e)$. Since $\tilde{E} \simeq E$, this can be seen as a splitting of G into the disconnected union of its edges. Let us consider an orientation of the graph G, seen as a couple (s, t) of maps (source, target) from E to V. We will use the shorthand notation $e_- := s(e)$ and $e_+ := t(e)$. We can now define the so-called canonical spectral triple $S_{\text{can}}(G, s, t)$. The algebra is $\mathcal{A} = \mathcal{C}(V) \simeq \mathbb{C}^{|V|}$, the Hilbert space is $\mathcal{H} = \mathcal{C}(\tilde{V}) \simeq \mathbb{C}^{2|E|}$, equipped with the L^2 scalar product, and the representation is defined by

$$(\pi(a)F)(e, \pm) = a(e_\pm)F(e, \pm) \tag{3}$$

for every $a \in \mathcal{A}$, $F \in \mathcal{H}$ and $e \in E$. Finally the Dirac operator, chirality and real structures are given by:

$$DF(e, \pm) = \frac{1}{\delta(e)}F(e, \mp); \quad \chi F(e, \pm) = \pm F(e, \pm), \quad CF(e, \pm) = \pm\overline{F(e, \pm)} \tag{4}$$

One can check that with these definitions, Connes' distance exactly matches the geodesic distance defined by the weight δ on the graph G ([5]).

4. NCG in the anti-Lorentzian signature

4.1. Features common to all approaches

As pointed out in the introduction, many authors have investigated the question of extending Connes' Noncommutative Geometry to semi-Riemannian signature. In all approaches, the main difference with the Riemannian case is that the Dirac operator is no longer self-adjoint with respect to a scalar product, but with respect to a non-degenerate non-positive product, that in the sequel we will call *the Krein product*. The reason for this is the following: given a space and time oriented spin manifold (M, g, σ) with g of signature (p, q), and its spinor bundle S, there is a canonical family of non-degenerate products defined on each fibre S_x by the requirement that each vector $v \in T_xM$ must act in a self-adjoint way ([11]). That is, $(.,.)_x$ is a product in this family if, and only if:

$$(v \cdot \psi, \psi')_x = (\psi, v \cdot \psi')_x \tag{5}$$

for each $v \in T_xM$, $\psi, \psi' \in S_x$. It is easy to see that the different products in the family are non-zero real multiple of one another. The existence of a time orientation allows one to glue these products together in order to form a smooth non-degenerate spinor metric on S ([12]), and finally to define a non-degenerate product on $\Gamma_c(S)$, that is, on compactly supported spinor fields, through

$$(\Psi, \Psi') = \int_M (\Psi(x), \Psi'(x))_x \text{vol}_g \tag{6}$$

where the integral is with respect to the canonical semi-Riemannian volume form. It is then possible to find a differential form ω such that $\eta := \gamma(\omega)$ is a fundamental symmetry, i.e. it satisfies the following requirements: $\eta^2 = \text{Id}$, $\eta^\times = \eta$, where $\times$ is the adjunction of operators for the product $(.,.)$, and $(\Psi, \eta \cdot \Psi) > 0$ for all non-zero spinor field Ψ. Hence η turns the indefinite product $(.,.)$ into a scalar product $\langle .,. \rangle_\eta = (.,\eta.)$. The completion $\mathcal{H}$ of $\Gamma_c(S)$ with respect to $\langle .,. \rangle_\eta$ is thus a Hilbert space, on which a non-degerate product $(.,.)$ is defined, related to the scalar product by the fundamental symmetry η. This is called a Krein space ([13]). The

Dirac operator $D = i\sum_k g(e_k, e_k)e_k\nabla^S_{e_k}$ is formally Krein-self adjoint with respect to $(.,.)$ and is even essentially Krein self-adjoint in the case where the metric can be rotated to a geodesically complete Riemannian one (see [14], Satz 3.19).

As a consequence, in the existing literature on semi-Riemannian Noncommutative Geometry, one replaces the Hilbert space with a Krein space and D is required to be Krein self-adjoint in some sense ([1], [2], [3], [4]). The fundamental symmetry is often seen to be part of the structure ([2], [3]).

4.2. The signature problem

The previous approaches, however, do not distinguish between the different signatures. It is hence impossible to speak of Lorentzian noncommutative manifold, for instance. We here put forward a proposal in this direction. The idea comes from the following theorem (here anti-Lorentzian means signature $(+,-,\ldots,-)$, Lorentzian means signature $(-,+,\ldots,+)$):

Theorem 1 *([12]) Let (M, g) be a semi-Riemannian space and time orientable spin manifold of even dimension. Then (M, g) is*

(i) anti-Lorentzian iff there exists a never vanishing 1-form β such that $C\beta = \beta C$ and $(.,\gamma(\beta)^{-1}.)$ is positive definite.

(ii) Lorentzian iff there exists a never vanishing 1-form β such that $C\beta = \beta C$ and $(.,\gamma(\beta)^{-1}\chi.)$ is positive definite.

Since a good theory of noncommutative 1-forms is available, this motivates the definition of spectral spacetimes, to come in the next section. Moreover, the form β above is exact iff it is the differential of a temporal function, and by a theorem of Sánchez ([15], [16]), this shows that a manifold is anti-Lorentzian and stably causal iff there exists an exact 1-form with the above property, thus providing a way to generalize stable causality in the noncommutative setting (in a way similar to [3]).

5. Spectral spacetimes

5.1. Definition

Motivated by the above considerations, we put forward the following definition.

Definition 3 *An* even anti-Lorentzian spectral spacetime *is a multiplet $S = (\mathcal{A}, K, (.,.), \pi, D, C, \chi)$, where:*

(i) $(K, (.,.))$ is a Krein space,

(ii) $\mathcal{A}$ is an algebra and π is a representation of it on K,

(iii) D is an operator on K such that $D^\times = D$,

(iv) $\chi \in B(K)$ is such that $\chi^2 = 1$, $[\pi(a), \chi] = 0$ for all $a \in \mathcal{A}$, $\chi D = -D\chi$ and $\chi^\times = -\chi$,

(v) C is an antilinear operator on K such that $C^2 = \pm 1$, $CD = -DC$, $C\chi = \pm\chi C$, $C^\times C = 1$.

(vi) there exists a noncommutative 1-form β, called a time-orientation form*, such that $[C, \beta] = 0$ and $\langle .,. \rangle_\beta := (., \beta^{-1}.)$ is positive definite.*

Such a spectral spacetime is said to be stably causal *if β can be chosen to be of the form $\beta = [D, \pi(a)]$ for some $a \in \mathcal{A}$.*

The Lorentzian case is obtained by inserting a chirality operator in the definition of $\langle .,. \rangle_\beta$, just as in theorem 1. Let us observe that it is not straightforward to generalize this definition to other signatures, such as $(2, n-2)$, since the theory of noncommutative p-forms, for $p > 1$ involves the notion of junk, so that these forms cannot be made to act on K ([17]). In the sequel "spectral spacetime" without any other qualification will always mean even anti-Lorentzian spectral spacetime.

An example of this structure is of course given by the *canonical spectral spacetime* $S_{\text{can}}(M, g, \sigma)$ defined out of an even dimensional space and time orientable spin anti-Lorentzian manifold (M, g, σ) in the way described in section 4.1. However, it is important to note that definition 3 is incomplete since we do not take care of the analytical conditions. In particular we cannot know whether $\mathcal{A}$ stands for an algebra of smooth or continuous functions, or some other degree of regularity. Of course, this is sufficient to treat the finite-dimensional case. Since we already know what a spacetime is in the commutative case, we can apply our definition in the almost-commutative case, which is, up to now, the only one needed for particle physics (there are two possibilities: tensorizing a Lorentzian manifold with a finite spectral triple or an anti-Lorentzian manifold with an "antieuclidean spectral triple", details are given in [5]). Moreover, in order to apply the stable causality condition as stated in the definition to the classical case, we must add a unit to the algebra, or equivalently consider a compactification of the manifold. Alternatively we can restrict to a compact subset of the original manifold.

5.2. How to recover a C^*-algebra

At first sight, definition 3 seems to be in sharp contrast with the definition of spectral triple on an essential point: no C^*-structure is postulated on $\mathcal{A}$. This may look strange, since this is the way a topological space, independent of any notion of metric, can be recovered out of a commutative algebra, as explained in section 2. In truth, C^*-structures have not disappeared. They are defined through the scalar products $\langle ., . \rangle_\beta$, where β runs over the space of time-orientation forms. Each such scalar product determines an involution $*_\beta$ on operators, hence a C^*-structure on $B(K)$. In the case $\mathcal{A} = \mathcal{C}(M)$, we have $f^{*_\beta} = \bar{f} \in \mathcal{A}$ for every β. We thus see that for all β, $*_\beta$ defines a C^*-involution on $\mathcal{C}(M)$ which is unique. In the other cases there is no guarantee that $*_\beta$ stabilizes the algebra for at least a single β, or if it does, that the different involutions coincide. We think we have to live with this, and take it as an important insight. This is where our approach differs the most from the existing ones where a particular fundamental symmetry commuting with the algebra is part of the structure. Still, there are two ways that we can see which permit to recover C^*-algebras out of spectral spacetimes. Note that in contrast with other approaches, these C^*-structures will be "end products": they are not fixed at the beginning.

Let us start with some notations. Given a spectral spacetime S, we will write $\mathrm{T}-\mathrm{or}(S)$ for the set of time-orientation forms. Let us define $\mathrm{Obs}(S) := \{\beta \in \mathrm{T}-\mathrm{or}(S) | \forall a \in \mathcal{A}, \pi(a)^{*_\beta} \in \pi(\mathcal{A})\}$. We call *observers* the elements of $\mathrm{Obs}(S)$. The name has two origins: the first is that in the case $S = S_{\text{can}}$, the elements of $\mathrm{Obs}(S) = \mathrm{T}-\mathrm{or}(S)$ can be seen as partitions of spacetimes into observers' trajectories (first by dualizing the form to a timelike vector field, then by integration). The second is that in the general case, each element of $\mathrm{Obs}(S)$ can be used to define its own Jordan algebra of observables. Indeed, each $\beta \in \mathrm{Obs}(S)$ defines an involution of $\pi(\mathcal{A})$, which is consequently a C^*-subalgebra of $(B(\mathcal{K}), *_\beta)$, the C^*-algebra of bounded operators on the Hilbert space $(\mathcal{K}, \langle ., . \rangle_\beta)$. The elements of $\pi(\mathcal{A})$ which are $*_\beta$-self-adjoint thus form the Jordan algebra of observables corresponding to the "observer" β. If π is injective, i.e. if the representation is faithful, we can carry the C^*-structure defined by $*_\beta$ on $\mathcal{A}$ itself. If $\mathrm{Obs}(S) \neq \emptyset$ and π is injective, in which case we say that S *is reconstructible*, there is thus at least one C^*-structure on $\mathcal{A}$. In case there are several, we can easily see that they are isomorphic. More precisely, $(\pi(\mathcal{A}), *_\beta) \simeq (\pi(\mathcal{A}), *_{\beta'})$ by $x \mapsto \beta'\beta^{-1}x(\beta'\beta^{-1})^{-1}$.

Unfortunately the set $\mathrm{Obs}(S)$ may very well be empty (examples are given in [5]). In such cases we can fall back on a kind of dual construction, which we now briefly describe[1]. We define $U(S) := \{a \in \mathcal{A} | \forall \beta \in \mathrm{T}-\mathrm{or}(S), \pi(a)^{*_\beta} = \pi(a)\}$. The elements of $U(S)$ are called *universal*

[1] More details will be given in future work. Our reference on Jordan algebras is [18].

observables. We can also define the subset $U_C(S)$ which contains the *real* universal observables, that is, those which commute with C. It is easy to see that $U(S)$ and $U_C(S)$ are both real Jordan-Banach algebras (see [18] for the definition). Clearly in the case $S = S_{\text{can}}(M, g, \sigma)$, one has $U(S) = U_C(S) = \mathcal{C}(M, \mathbb{R})$ and they are associative Jordan algebras. It can be proven that $U_C(S)$ is always associative. Every Jordan-Banach algebra gives rise to a so-called universal C^*-algebra, which in our case is just the closed associative algebra generated by $U(S)$ over the complex numbers. This algebra is commutative iff $U(S)$ is associative, in which case we can apply the first Gelfand-Naimark theorem in order to recover a topological space.

5.3. The split graph again

Let us now describe an example of spectral spacetime which generalizes to the anti-Lorentzian signature the canonical spectral triple over a finite weighted graph. We use the notations of section 3.2. We take the same algebra $\mathcal{A}$, acting by the same representation π on the same space $\mathcal{C}(\tilde{V})$, which we now denote by $\mathcal{K}$ instead of $\mathcal{H}$ in order to remind ourselves that it is now endowed with a Krein product, defined by

$$(F, G)_\omega = i \sum_{e \in E} \left(F(e, -)\overline{G(e, +)} - F(e, +)\overline{G(e, -)} \right) \tag{7}$$

The Dirac operator is:

$$D = \frac{-i}{\delta(e)} F(e, \mp) \tag{8}$$

It only differs from the euclidean Dirac operator by a factor of $-i$. The chirality is the same, but the real structure now acts off-diagonally by:

$$CF(e, \pm) = \pm \overline{F(e, \mp)} \tag{9}$$

These formula are obtained by a systematic "Wick rotation" procedure (see [5]), but we can check directly that they satisfy the axioms of an anti-Lorentzian spectral spacetime, which is moreover reconstructible. The time-orientation forms are the operators of the form

$$\beta F(e, \pm) = \pm i b(e) F(e, \mp) \tag{10}$$

where $e \mapsto b(e)$ is a positive-valued function defined on the set of edges E. This form is exact iff $b(e) = \frac{f(e^+) - f(e^-)}{\delta(e)}$ for some function $f : V \to \mathbb{R}$. The existence of this function prevents the existence of directed cycle, and it is a well-known result in graph theory that directed acyclic graphs has a so-called topological ordering ([19]). This means that the spectral spacetime just constructed is stably causal iff the transitive closure of the oriented graph (G, s, t) is a partial ordering. Note that the orientation (s, t) which played a marginal role in the euclidean case (it only served to define the representation) is now a crucial part of the structure. More precisely, a change of orientation on G induces a canonical transformation on $\mathcal{H} = \mathcal{K}$, which exchanges the endpoints of some of the edges of the split graph. This is unitary for the scalar product, but not for the Krein product (7). Hence changing the orientation of G just amounts to a change of unitary representation in the spectral triple case, but can lead to a different isomorphism class of anti-Lorentzian spectral spacetimes.

5.4. A noncommutative generalization: the split Dirac structure

We will now present a noncommutative finite-dimensional example of spectral spacetime. It is a generalization of the construction of section 5.3 obtained in the following way:

IOP Conf. Series: Journal of Physics: Conf. Series **968** (2018) 012002 doi:10.1088/1742-6596/968/1/012002

- At each vertex $v \in V$ we attach a $n = 2p$-dimensional real vector space X_v equipped with an anti-Lorentzian bilinear form g_v.
- We replace the algebra $\mathcal{A} = \mathcal{C}(V)$ by the algebra of *even* sections $a : v \mapsto a_v \in \mathbb{C}l(X_v, g_v)^0$ of the discrete Clifford bundle $\bigsqcup_{v\in V} \mathbb{C}l(X_v, g_v)$.
- We put an irreducible representation space S_v of $\mathbb{C}l(X_v, g_v)$, that is, a module of Dirac spinors, at each vertex v. The representation map is $\rho_v : \mathbb{C}l(X_v, g_v) \to \mathrm{End}(S_v)$. We fix a spinor product $(.,.)_v$ on S_v. The future cone X_v^+ in X_v is defined to be the set of vectors $\xi \in X_v$ such that $(\psi, \rho_v(\xi)\psi)_v \geq 0$ for all $\psi \in S_v$.
- We replace $\mathcal{K} = \mathcal{C}(\tilde{V})$ by the space of sections $F : (e, \pm) \mapsto F(e, \pm) \in S_{e^\pm}$ of the discrete spinor bundle $\bigsqcup_{(e,\pm)\in\tilde{V}} S_{e^\pm}$ over the split graph $\tilde{G}$.
- The representation of the algebra on K is defined by:

$$(\pi(a)F)(e, \pm) = a(e^\pm) \cdot F(e, \pm) \tag{11}$$

  where we have written $\cdot$ to denote the action of Clifford algebra elements on spinors by way of the representation $\rho_{e^\pm}$.
- We fix parallel transport operators $h_e^+ : S_{e^-} \to S_{e^+}$, and write $h_e^- = (h_e^+)^{-1}$.
- The Krein product on K is defined to be:

$$(F, G) = \sum_{e\in E} ((F(e, +), h_e^+ G(e, -)) + (F(e, -), h_e^- G(e, +)) \tag{12}$$

- The Dirac operator D is:

$$(DF)(e, \pm) = \frac{\mp 1}{\delta(e)} \gamma_e^\pm h_e^\pm F(e, \mp) \tag{13}$$

  where $\gamma_e^\pm \in X_{e^\pm}$ is a "gamma matrix" and $\delta_e \in]0, +\infty[$.
- The chirality χ is defined by:

$$(\chi F)(e, \pm) = \chi_{e^\pm} F(e, \pm) \tag{14}$$

  where $\chi_{e^\pm}$ is the chirality operator of $\mathbb{C}l(X_{e^\pm})$.
- Finally we define $(CF)(e, \pm) = h_e^\pm C_{e^\mp} F(e, \mp)$.

The structure defined above is called *the split Dirac structure* over the weighted directed graph $G = (V, E, s, t, \delta)$. The family $(h_e^\pm)_{e\in E}$ of parallel transport operators is called the *discrete connection.* We say that the discrete connection is

- *metric* iff the h_e are Krein-unitary,
- *preserves spin* iff the diagrams

$$\begin{tikzcd} S_{e^-} \arrow[r, "C_{e^-}"] \arrow[d, "h_e^+"'] & S_{e^-} \arrow[d, "h_e^+"] \\ S_{e^+} \arrow[r, "C_{e^+}"] & S_{e^+} \end{tikzcd}$$

  commute for all $e \in E$,

- *is Clifford* iff for all e there exists a linear map $\Lambda_e : X_{e^-} \to X_{e^+}$ such that the following diagram commutes for all $v \in X_{e^-}$:

$$\begin{tikzcd} S_{e^-} \arrow[r, "v"] \arrow[d, "h_e^+"'] & S_{e^-} \arrow[d, "h_e^+"] \\ S_{e^+} \arrow[r, "\Lambda_e(v)"] & S_{e^+} \end{tikzcd}$$

- *preserves orientation* if $\chi_{e^\pm} h_e^\pm = h_e^\pm \chi_{e^\mp}$.

It can be shown that in the continuous case these definitions match the usual ones ([12]). In particular, if the parallel transport operators of a connection on the spinor bundle have all these properties, then they come from the spin connection. One can also prove the following theorems:

Theorem 2 *([5]) The split Dirac structure is an anti-Lorentzian spectral spacetime if and only if*

(i) The elements $\gamma_e^\pm$ satisfy

$$\gamma_e^+ = h_e^+ \gamma_e^- h_e^-$$

for all e.

(ii) The discrete connection is metric, spin and orientation preserving.

Moreover if the $\gamma_{e^\pm}$ span $X_{e^\pm}$ then the discrete connection is Clifford.

Theorem 3 *([5]) Under the conditions of theorem 2, and if in addition $n > 2$ and $(h_e)_{e\in E}$ is Clifford, then the following are equivalent:*

- *the split Dirac structure is recontructible,*
- *there exists a future-directed "covariantly constant vector field", i.e. $v \mapsto u_v \in X_v^+$ such that for all vertices v, w, u_w is the image of u_v by the product of the parallel transport operators encountered along any path leading from v to w,*
- *the discrete holonomy group fixes a timelike direction.*

This theorem allows in particular to build examples of non-reconstructible anti-Lorentzian spacetimes. These cannot be Wick rotations of spectral triples.

6. Conclusion

To end this short review, we quote a few directions for future research that we think are relevant. There are three different kinds of problems that we think are worth investigating. First, there are technical issues concerning the general theory of spectral spacetimes: finding the correct analytical conditions which would allow for a reconstruction theorem (results in [4] would probably be very helpful), generalizing the definition to the odd-dimensional case, and maybe other signatures (this last problem being probably more difficult). The second direction is to develop some aspects of the theory, namely the study of stable causality (making a bridge to Franco's approach to causality in Noncommutative Geometry, and maybe isocone theory should be possible), and the classification of the finite-dimensional case (along the lines of [20], [21]). Finally, the split Dirac structure seems interesting to study for its own sake. In particular it is possible to relate it to the discretization of the Dirac operator defined in Marcolli and van Suijlekom at least in the flat case. Is it possible to go beyond ? One could also look forward for applications in particle physics (noncommutative lattice models, also in the spirit of [22]).

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