Variational principles for Lax fifth-order equation and the (2+1) dimensional potential KdV equation

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Variational Principles for Lax Fifth-order Equation and the (2+1) Dimensional Potential KdV Equation

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Abstract. Using the semi-inverse method proposed by Ji-Huan He, variational principles are established for two nonlinear equations arising in physics, i.e., the Lax fifth-order equation and the (2+1) dimensional potential KdV equation.

1. Introduction

Recently a large amount of work has been done on the variational theory and its applications [1-15]. Here, we use the Lax fifth-order equation [16] and the (2+1) dimensional potential KdV equation [17] as examples to show how to establish variational formulations using the semi-inverse method proposed by Ji-Huan He [2].

2. Variational Formulations

2.1 The Lax fifth-order equation

Consider the Lax fifth-order equation [16]

\[ u_t + 30u^2u_x + 20uu_{x2} + 10uu_{x3} + u_{x5} = 0, \]  

(1)

We introduce a special function \( \Phi \) defined as

\[ \Phi_x = -u, \]  

(2)

\[ \Phi_t = 10u^3 + 10uu_{x2} + 5(u_x)^2 + u_{x4}, \]  

(3)

so that Eq. (1) is automatically satisfied. We will apply the semi-inverse method [2-6] to search for the needed variational formulation:

\[ J(u, \Phi) = \int \int L \, dx \, dt. \]  

(4)

Here \( L \) is a trial Lagrangian defined by

\[ L = u\Phi_x + \left(10u^3 + 10uu_{x2} + 5(u_x)^2 + u_{x4}\right)\Phi_x + F(u), \]  

(5)

where \( F \) is an unknown function of \( u \) and/or its derivatives. The merit of the trial Lagrangian is that the stationary condition with respect to \( \Phi \) leads to (1).

Now the stationary condition with respect to \( u \) is
\[ \Phi_x + 30u^2\Phi_x + 10u_2\Phi_x - 10(u_x\Phi_x)_x + (10u\Phi_x)_x + \Phi_{5x} + \frac{\delta F}{\delta u} = 0. \]  \hfill (6)

Here, \( \delta F / \delta u \) is called He’s variational derivative with respect to \( u \).

Concerning (2) and (3), we set
\[ \frac{\delta F}{\delta u} = -(\Phi_i + 30u^2\Phi_x + 10u_2\Phi_x - 10(u_x\Phi_x)_x + (10u\Phi_x)_x + \Phi_{5x}) \]
\[ = 20u^3 + 5(u_x)^2 + 10uu_{xx}. \]  \hfill (7)

So, the unknown \( F \) can be determined as
\[ F_1 = 5u^4 - 5(u_x)^2u, \]  \hfill (8)

or
\[ F_2 = 5u^4 + \frac{5}{2}u^2u_{xx}. \]  \hfill (9)

Therefore, we obtain the following variational formulation
\[ J_1(u, \Phi) = \int \left\{ u\Phi_i + (10u^3 + 10uu_{xx} + 5(u_x)^2 + u_{4x}) \Phi_x + 5u^4 - 5(u_x)^2u \right\} dx dt, \]  \hfill (10)

or
\[ J_2(u, \Phi) = \int \left\{ u\Phi_i + (10u^3 + 10uu_{xx} + 5(u_x)^2 + u_{4x}) \Phi_x + 5u^4 + \frac{5}{2}u^2u_{xx} \right\} dx dt. \]  \hfill (11)

2.2 KdV equation

The (2+1) dimensional potential KdV equation \cite{17} is given by
\[ \{u_t + au(u_x)^2 + bu_{3x}\}_x + ku_{yy} = 0, \]  \hfill (12)

where \( a, b, \) and \( k \) are constants.

Similarly we obtain the following variational formulations
\[ J_1 = \int \left\{ \frac{1}{2}uu_{xx} - \frac{a}{3}u_x^2u_{3x} + \frac{b}{2}(u_{2x})^2 + \frac{k}{2}uu_{yy} \right\} dx dy dt, \]  \hfill (13)

\[ J_2 = \int \left\{ \frac{1}{2}uu_{xx} - \frac{a}{3}u_x^2u_{3x} - \frac{b}{2}u_{x3}u_{xx} + \frac{k}{2}uu_{yy} \right\} dx dy dt, \]  \hfill (14)

\[ J_3 = \int \left\{ -\frac{1}{2}u_xu_t - \frac{a}{3}u_x^2u_{3x} + \frac{b}{2}(u_{2x})^2 + \frac{k}{2}uu_{yy} \right\} dx dy dt, \]  \hfill (15)

and
\[ J_4 = \int \left\{ -\frac{1}{2}u_xu_t - \frac{a}{3}u_x^2u_{3x} - \frac{b}{2}u_{x3}u_{xx} + \frac{k}{2}uu_{yy} \right\} dx dy dt. \]  \hfill (16)

3. Conclusion

We obtain the variational formulations for the discussed equations, which might find some potential applications.

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References