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Discrete-time analogues of Cohen-Grossberg neural networks with time-varying delays

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Abstract. The existing semi-discretisation technique is extended for deriving a discrete-time analogue in the attempt to simulate the global exponential stability characteristics of a unique equilibrium point of a Cohen-Grossberg neural network (CGNN) with time-varying delays. Without assuming the symmetry of connection weight matrices and the boundedness and monotonicity of activation functions, a sufficiency condition given in the form of an $M$-matrix is obtained for the convergence of the analogue towards the unique equilibrium point. The convergence requires no restriction on the size of the time-step and no additional stability conditions.

1. Introduction
Research activities focusing on the global (exponential) stability of Cohen-Grossberg neural network (CGNN) models [2] have grown rapidly over the years and many competitive results are found proposing new development into the design of the models. They include removing the symmetry of connection weight matrices, the boundedness and monotonicity of activation functions as well as the differentiability of stabilising functions so as to enable the networks addressing a large class of real problems [1,5,8,10].

We consider a CGNN model consisting of $m$ elementary processing units (or neurons) whose state variables $x_i(t), i \in \{1, 2, \ldots, m\}$ are governed by the system

$$\frac{dx_i(t)}{dt} = a_i(x_i(t)) \left[ -b_i(x_i(t)) + \sum_{j=1}^{m} c_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} d_{ij} g_j(x_i(t - \tau_j(t))) + I_i \right]$$

(1.1)

for $t > 0$, that is supplemented with an initial condition of the form $x_i(s) = \phi_i(s)$ for $s \in [-\tau, 0]$, where the function $\phi_i(\cdot)$ is piecewise continuous on $[-\tau, 0]$ in which $\tau = \max_{i,j=1}^{m} \{\tau_j\}$ and

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0 \leq \tau_j(t) \leq \tau^\star_j \text{ for } i, j \in \mathbb{I} , t \in \mathbb{R} . \text{ In (1.1), } a_i(\cdot) \text{ denotes an amplification function; } b_i(\cdot) \text{ denotes an appropriate function that supports the stabilising (or negative) feedback term } -a_i(\cdot)b_i(\cdot) \text{ of the network; } c_{ij}, d_{ij} \in \mathbb{R} \text{ denote the weights (or strengths) of the synaptic connections between unit } j \text{ and unit } i ; f_j(\cdot), g_j(\cdot) \text{ denote activation functions; } \tau_j(t) \text{ incorporates the delayed transmission of signal from unit } j \text{ to unit } i \text{ at time } t - \tau_j(t) ; \text{ and } I_i \in \mathbb{R} \text{ denotes an external input introduced from outside the network to the unit } i .

The main assumptions imposed on the CGNN model (1.1) are given as follows:

**Assumption 1**: The amplification function \( a_i : \mathbb{R} \mapsto \mathbb{R}^+ \) is continuous and bounded in the sense of
\[
0 < a_i \leq a_i(u) \leq \overline{a}_i, \quad u \in \mathbb{R} .
\]

**Assumption 2**: The stabilising function \( b_i : \mathbb{R} \mapsto \mathbb{R} \) is continuous and monotonically increasing, particularly,
\[
0 < b_i \leq \frac{b_i(u) - b_i(v)}{u - v}, \quad u, v \in \mathbb{R} .
\]

**Assumption 3**: The activation functions \( f_j, g_j : \mathbb{R} \mapsto \mathbb{R} \) with \( f_j(0) = g_j(0) = 0 \) satisfy the global Lipschitz continuous conditions given by
\[
| f_j(u) - f_j(v) | \leq L_j^f | u - v |, \quad | g_j(u) - g_j(v) | \leq L_j^g | u - v |, \quad u, v \in \mathbb{R} ,
\]
where \( L_j^f, L_j^g \) denote positive constants.

For a given initial vector function \( x(s) = \phi(s) \), where \( \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_m(s))^T \) for \( s \in [-\tau, 0] \), we denote a solution of (1.1) by the vector function \( x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T \) wherein the components \( x_i(t) \) satisfy the system (1.1) for \( t > 0 \) and \( x_i(s) = \phi_i(s) \) for \( s \in [-\tau, 0] \). For a given external input vector \( I = (I_1, I_2, \ldots, I_m)^T \), we denote the corresponding equilibrium point of the CGNN model (1.1) by the constant vector \( x^\star = (x_1^\star, x_2^\star, \ldots, x_m^\star)^T \) whose components \( x_i^\star \) are governed by the algebraic system
\[
b_i(x_i^\star) = \sum_{j=1}^m c_{ij} f_j(x_j^\star) + \sum_{j=1}^m d_{ij} g_j(x_j^\star) + I_i, \quad i \in \mathbb{I} .
\]

**2. Discrete-time analogues**

A discrete-time analogue of the CGNN model (1.1) is derived by extending the semi-discretisation technique that has been successfully implemented for the digital simulations of additive neural networks, cellular neural networks, bidirectional associative memory networks and high-order neural networks with linear stabilising feedback terms [6,7,9,11]. The analogue is given by
\[
x_i(n+1) = e^{-\gamma_i} x_i(n) + \psi_i(\delta) \left\{ \gamma_i x_i(n) + a_i(x_i(n)) \left[ -b_i(x_i(n)) + \sum_{j=1}^m c_{ij} f_j(x_j(n)) \right] + \sum_{j=1}^m d_{ij} g_j(x_j(n - \kappa_i(n))) + I_i \right\}, \quad i \in \mathbb{I} , n \geq 0 ,
\]

(2.1)
where \( h \in (0, \infty) \), \( \gamma_i = \alpha_i h > 0 \) and \( \psi_i(h) = \frac{1}{\tau_i} e^{-\gamma_i h} \). The denominator function \( \psi_i(h) \) is monotonically increasing and bounded in the sense of \( 0 < \psi_i(h) < \frac{1}{\tau_i} \) for \( h \in (0, \infty) \) wherein \( \lim_{h \to 0} \psi_i(h) = 0 \) and \( \lim_{h \to \infty} \psi_i(h) = \frac{1}{\tau_i} \). The initial condition for (2.1) is given by the vector sequence for \( n \geq 0 \) represents another type of analogue of the CGNN model (1.1), whose formulation is based on the first-order explicit Runge-Kutta (or forward Euler) numerical scheme [4]. One observes that the model denotes a special case of the analogue (2.1). For instance, if \( e^{-h} \approx 1 - \gamma_i h \) and \( \psi_i(h) \approx h \), then the analogue (2.1) yields model (2.2). We remark further that model (2.2) with \( \alpha_i = 1 \) and \( \kappa_i(n) = \kappa_i \) has been considered recently by Xiong and Cao [12]. Its application for simulating the convergence dynamics of the CGNN model (1.1) with \( \alpha_i = \kappa_i \) towards a unique equilibrium point \( x^* \) of the analogue (2.1) is governed by the same system (1.2).

The following model given by

\[
\begin{bmatrix}
-\beta_i(x_i(n)) + \sum_{j=1}^{m} c_{ij} f_i(x_j(n)) + \sum_{j=1}^{m} d_{ij} g_i(x_j(n - \kappa_i(n))) + I_j
\end{bmatrix}
\]  

for \( i \in I \), \( n \geq 0 \) represents another type of analogue of the CGNN model (1.1), whose formulation is based on the first-order explicit Runge-Kutta (or forward Euler) numerical scheme [4]. One observes that the model denotes a special case of the analogue (2.1). For instance, if \( e^{-h} \approx 1 - \gamma_i h \) and \( \psi_i(h) \approx h \), then the analogue (2.1) yields model (2.2). We remark further that model (2.2) with \( \alpha_i = 1 \) and \( \kappa_i(n) = \kappa_i \) has been considered recently by Xiong and Cao [12]. Its application for simulating the convergence dynamics of the CGNN model (1.1) with \( \alpha_i = \kappa_i \) towards a unique equilibrium point \( x^* \) of the analogue (2.1) is governed by the same system (1.2).

3. Global exponential stability

The following lemma establishes the existence of a unique equilibrium point \( x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T \) of the analogue (2.1) under the sufficiency condition (3.1). The proof is based on constructing a continuous mapping that represents a homeomorphism on \( \mathbb{R}^m \); see Forti and Tesi [3] and Song and Cao [10] for details.

**Lemma 3.1.** Let the assumptions \( A_1 \) and \( A_2 \) hold. Suppose

\[
\Xi_\alpha = B_0 - C | L'| - D | L|^\mathbb{R}
\]

is a nonsingular \( M \) matrix, where \( B_0 = \text{diag}(l_1, l_2, \ldots, l_m) \), \( | C | = (c_{ij})_{m \times m} \), \( | D | = (d_{ij})_{m \times m} \), \( L' = \text{diag}(L_1', L_2', \ldots, L_m') \) and \( L = \text{diag}(L_1, L_2, \ldots, L_m) \). Then there is a unique solution \( x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T \) for the algebraic system (1.2).

Now, we are ready to prove the global exponential stability of \( x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T \) for a fixed \( h \in (0, \infty) \). Upon introducing the following translations

\[
\begin{align*}
\bar{a}_i(u_i(n)) &= a_i(u_i(n) + x_i^*), \\
\bar{b}_i(u_i(n)) &= b_i(u_i(n) + x_i^*) - h_i(x_i^*), \\
\varphi(l) &= \phi(l) - x_i^*, \\
\bar{f}_i(u_i(n)) &= f_i(u_i(n) + x_i^*) - f_i(x_i^*), \\
\bar{g}_i(u_i(n)) &= g_i(u_i(n) + x_i^*) - g_i(x_i^*)
\end{align*}
\]

we conclude that the analogue (2.1) with the initial conditions (3.2) yields the unique equilibrium point \( x^* \) of the model (2.2) for a fixed \( h \) in a neighborhood of \( \infty \).
into the analogue (2.1), one obtains
\[
\begin{align*}
  u_i(n+1) &= e^{-\gamma_i h} u_i(n) + \psi_i(h) \left\{ \gamma_i u_i(n) + \tilde{a}_i(u_i(n)) \left[ -\tilde{b}_i(u_i(n)) + \sum_{j=1}^m c_{ij} \tilde{f}_j(u_j(n)) \right] + \sum_{j=1}^m d_{ij} \tilde{g}_j(u_j(n-\kappa_j(n))) \right\}, \quad i \in I, \ n \geq 0.
\end{align*}
\] (3.2)

One can verify that the functions \( \tilde{a}_i(\cdot), \tilde{b}_i(\cdot), \tilde{f}_j(\cdot), \tilde{g}_j(\cdot) \) in (3.2) satisfy the assumptions \( A_i - A_3 \).

Due to the uniqueness of the point \( x^* \) (a consequence of Lemma 3.1) and the equivalence between the two analogues (2.1) and (3.2), it is sufficed to investigate the exponential stability of the trivial point \( u^* = 0 \) (i.e., \( u_i^* = 0, \ i \in I \)) of the analogue (3.2). Let us denote

\[ || \varphi || = \max_{i \in I} \left\{ \sup_{t \in [-\kappa,0]} | \varphi_i(I) | \right\} = \max_{i \in I} \left\{ \sup_{t \in [-\kappa,0]} | \phi_i(I) - x^*_i | \right\}. \]

**Theorem 3.1.** Let the value \( h \in (0, \infty) \) be fixed, and let the assumptions \( A_1 - A_3 \) hold. Suppose

\[ \Xi_i = B_i - |C| L^f - |D| L^g \]

is an \( M \)-matrix, where \( |C|, |D|, L^f \) and \( L^g \) are defined in (3.1) and

\[ B_i = \text{diag} \left( \left[ \begin{array}{c} (a_i/\bar{a}_i) b_i, \ (a_i/\bar{a}_i) b_i, \ f_i(\underline{a}_m/\bar{a}_m) h_i \end{array} \right] \right). \]

Then the trivial point \( u^* = 0 \) of the analogue (3.2) is exponentially stable, in the sense that there is a positive real number \( \mu = \mu(h) \) (called the Lyapunov exponent of the analogue (2.1)) for which

\[ |u_i(n)| \leq \rho || \varphi || e^{-\mu h n}, \quad i \in I, \ n \geq 0, \]

where \( \rho \geq 1 \) is a constant.

**Proof.** We have from the property of the \( M \)-matrix \( \Xi_i \) that there is a positive vector \( \xi = (\xi_1, \xi_2, \ldots, \xi_m)^T \) such that

\[ \xi_i (a_i/\bar{a}_i) b_i - \sum_{j=1}^m \xi_j |c_{ij}| L^f_j - \sum_{j=1}^m \xi_j |d_{ij}| L^g_j > 0, \quad i \in I. \]

(3.5)

This implies

\[ b_i > \sum_{j=1}^m \frac{\xi_j}{\xi_i} c_{ij} |L^f_j| + \sum_{j=1}^m \frac{\xi_j}{\xi_i} d_{ij} |L^g_j|, \quad i \in I \]

which means that \( \Xi_i \) is an \( M \)-matrix. The existence of the unique equilibrium point \( x^* \) therefore follows from Lemma 3.1.

Recall that \( \gamma_i = a_i b_i > 0 \) and \( \psi_i(h) = \frac{1 - e^{-\gamma_i h}}{\gamma_i} > 0 \). It will follow from (3.5) that

\[ e^{-\gamma_i h} + \bar{a}_i \psi_i(h) \left[ \sum_{j=1}^m \frac{\xi_j}{\xi_i} c_{ij} |L^f_j| + \sum_{j=1}^m \frac{\xi_j}{\xi_i} d_{ij} |L^g_j| \right] < 1, \quad i \in I. \]
Thus for a given \( h \in (0, \infty) \), one can introduce a perturbation \( \mu = \mu(h) \) such that \( 0 < \mu < \min_{i=1}^{n} \{ \gamma_i \} \) and
\[
e^{(\mu^{-1}h)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| c_{ij} \right| L_{ij} e^{\mu(h)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \left| d_{ij} \right| L_{ij} e^{\mu(T_{i,j}^{-1}h)} \right) \leq 1, \quad i \in \mathbb{I}.
\]
(3.6)

Now let us define nonnegative sequences \( w_i(n) \) by
\[
w_i(n) = \xi_i^{-1} e^{-\mu h} |u_i(n)| \quad \text{for} \quad n \geq -\kappa,
\]
(3.7)

where the sequences \( u_i(n) \) satisfy |\( u_i(n) = |\phi_i(n) | \) for \( i \in \mathbb{I}, n \in [-\kappa, 0] \) and
\[
|u_i(n+1)| \leq e^{-\gamma_i h} |u_i(n)| + \sum_{j=1}^{n} \left| c_{ij} \right| L_{ij} e^{\mu h} w_j(n) + \sum_{j=1}^{n} \left| d_{ij} \right| L_{ij} e^{\mu(T_{i,j}^{-1}h)} w_j(n-\kappa_j(n))
\]
(3.8)

for \( i \in \mathbb{I}, n \geq 0 \). Subsequently, the nonnegative sequences \( w_i(n) \) obey
\[
w_i(n+1) \leq e^{(\mu^{-1}h)} w_i(n) + \sum_{j=1}^{n} \left| c_{ij} \right| L_{ij} e^{\mu h} w_j(n) + \sum_{j=1}^{n} \left| d_{ij} \right| L_{ij} e^{\mu(T_{i,j}^{-1}h)} w_j(n-\kappa_j(n))
\]
(3.9)

for \( i \in \mathbb{I}, n \geq 0 \). It is clear from (3.7) that \( w_i(n) \leq \max_{i=1}^{n} \{ \xi_i^{-1} \} \| \phi \| = \Omega \) for \( i \in \mathbb{I}, n \in [-\kappa, 0] \). This leads us to claim that
\[
w_i(n) \leq \Omega \quad \text{for} \quad i \in \mathbb{I}, n \geq 0.
\]
(3.10)

Suppose (3.10) is not true in the sense that there is an index \( i_0 \in \mathbb{I} \) and an integer \( n_1 > 0 \) for which \( w_{i_0}(n) \leq \Omega \) for \( -\kappa \leq n \leq n_1 - 1 \) and \( w_{i_0}(n_1) > \Omega \). We have from (3.9) that
\[
\Omega < w_{i_0}(n_1) \leq e^{(\mu^{-1}h)} + \sum_{j=1}^{n} \left| c_{ij} \right| L_{ij} e^{\mu h} w_j(n) + \sum_{j=1}^{n} \left| d_{ij} \right| L_{ij} e^{\mu(T_{i,j}^{-1}h)} w_j(n-\kappa_j(n))
\]

which yields the contradiction \( \Omega < w_{i_0}(n_1) \leq \Omega \) by virtue of the condition (3.6). Thus, the claim (3.10) is valid. It follows from (3.7) that
\[
|u_i(n)| \leq \frac{\max_{i=1}^{n} \{ \xi_i^{-1} \} \| \phi \| e^{-\mu h}}{\min_{i=1}^{n} \{ \xi_i^{-1} \} \| \phi \| e^{-\mu h}} \quad \text{for} \quad i \in \mathbb{I}, n \geq 0
\]

and this establishes the estimate (3.4). The proof of the theorem is complete.

We remark that the value \( h \in (0, \infty) \) is fixed but arbitrary. Moreover, the severe restriction \( \bar{L}_i \leq 1 \) is not included in the condition (3.3) for ascertaining the exponential stability of \( x^* \). In this sense, the analogue (2.1) is much superior than the discrete-time model (2.2) which relies on the choice \( h = 1 \) and the severe restriction \( \bar{L}_i \leq 1 \) in order to preserve the convergence dynamics of the continuous-time CGNN (1.1) towards the unique equilibrium point \( x^* \).

The following corollary attempts to capture the corresponding Lyapunov exponent of the CGNN model (1.1) from the sufficiency condition (3.3).
Corollary 3.2. (Case $h \to 0$). Let the assumptions $A_1 - A_3$ hold. If the condition (3.3) is satisfied, then the unique equilibrium point $x'$ of the CGNN model (1.1) is globally exponentially stable with a Lyapunov exponent $0 < \mu_0 < \min_{i \in [1]} \{\gamma_i\}$, that is,
\[
|u_i(t)| \leq \rho \|\varphi\| e^{-\mu_0 t} \quad \text{for } i \in I, \ t > 0,
\] (3.11)
where $\rho \geq 1$ is a constant.

**Proof.** The task is to show the Lyapunov exponent $\mu = \mu(h)$ stated in (3.4) satisfies $0 < \mu < \mu_0 < \min_{i \in [1]} \{\gamma_i\}$ and $\mu \to \mu_0$ (from below) as $h \to 0$.

One can find from the $M$-matrix $\Xi$ in (3.3) that the Lyapunov exponent $0 \leq \mu_0 < \min_{i \in [1]} \{\gamma_i\}$ of the continuous-time CGNN (1.1) is governed by
\[
-a h + \mu_0 + \bar{a} \left( \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} |c_j| |L_j e^{\mu h}| + \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} |d_j| |L_j e^{\mu h}e^{\gamma_i h}| \right) \leq 0 \quad \text{for } i \in I.
\] (3.12)

The reader can refer the relevant details provided by Song and Cao [10] for deriving (3.12).

We have from the inequality (3.6) that
\[
\frac{e^{(\mu-\gamma_i)h} - 1}{\psi_i(h)} + \bar{a} \left( \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} |c_j| |L_j e^{\mu h}| + \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} |d_j| |L_j e^{\mu h}e^{\gamma_i h}| \right) \leq 0, \quad i \in I.
\]

Since the inequality holds for any $h \in (0, \infty)$, one asserts that the positive number $\mu = \mu(h)$ satisfies $\mu \to 0$ as $h$ gets large. With $\overline{\xi}/h = [\overline{\xi}_i/h] h \to \overline{\xi}$ as $h \to 0$, we have
\[
\lim_{h \to 0} \left[ \frac{e^{(\mu-\gamma_i)h} - 1}{\psi_i(h)} + \bar{a} \left( \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} |c_j| |L_j e^{\mu h}| + \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} |d_j| |L_j e^{\mu h}e^{\gamma_i h}| \right) \right]
= \mu - \gamma_i + \overline{\bar{a}} \left( \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} |c_j| |L_j e^{\mu h}| + \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} |d_j| |L_j e^{\mu h}e^{\gamma_i h}| \right) \leq 0, \quad i \in I.
\]

A direct comparison with (3.12) suggests that $\mu \to \mu_0$ from below as $h \to 0$. Thus the function $e^{-\mu h}$ in (3.4) for a fixed $n \geq 0$ satisfies $e^{-\mu h} \to e^{-\mu_0 t}$ for the corresponding $t > 0$ as $h \to 0$. The statement (3.11) therefore follows. The proof is complete.

4. Concluding remarks

We have extended the semi-discretisation technique for obtaining a discrete-time analogue of a Cohen-Grossberg neural network with time-varying delays. The analogue requires no restriction on the size of the time-step and no additional stability condition in order to simulate the global exponential stability characteristics of the unique equilibrium point of the CGNN model. This demonstrates the superiority of the analogue over the discrete-time CGNN model (2.2) studied by Xiong and Cao [12]. We expect however that further development or modification can be introduced into the formulation of the analogue in order to simulate the exponential stability of the CGNN model (1.1) under the $M$-matrix $\Xi$, instead of the $M$-matrix $\Xi$. This type of condition wherein the boundedness of the amplification functions is removed has been implemented for the continuous-time CGNN model (1.1); see Song and Cao [10] for the relevant details.

References