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A Stage – Structured Predator-Prey Model with Disease in the Prey

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ABSTRACT A non-linear mathematical model for a prey-predator community is proposed and analyzed. In the model, prey gets infected and predator population is structured into two stages of life, immature and mature with a time lag between two stages. Boundedness and non-negativity of the solutions of the system have been proved. Criterion for the stability of the system in the absence of delay is derived and bifurcation is found. The critical value of delay parameter for which stability change may occur is obtained.

1 Introduction
Epidemiological models have become important tools in analyzing the spread and control of infectious diseases. Andersen and May [2] formulated epidemic models to describe only the spread of the disease and obtained some threshold results. Few investigators [3, 4, 6] have paid attention to study prey-predator model with infection. Also, there are species, which behave differently in their different stages of life. We have in mind mammalian population, which exhibit two stages immature and mature, with a time lag ‘τ’ that is the time from birth to maturity.

Aiello and Freedman [1] in their paper worked on such a population where stage-structured was modeled by the introduction of a constant time delay and studied local and global behavior of equilibrium and showed that time delays are not necessarily destabilizing. Zhang et al [7] discussed optimal harvesting policy for a predator model and derived necessary and sufficient condition for the co-existence or extinction of species. Sun et al [8] obtained the conditions for the global stability and the existence of Hopf bifurcation at the positive equilibrium of predator-prey system with a finite delay. Recently, Song and Chen [5] discussed the optimal harvesting policy and stability for a two species competitive system with stage-structured and obtained conditions for the existence of a globally asymptotically stable positive equilibrium point.

In view of all above, we propose here a prey-predator community in which prey gets infected and predator population is structured into two stages-immature and mature. We consider the simple case when only mature predator consumes the infected prey.

2 MATHEMATICAL MODEL
We consider a prey-predator model with infected prey and stage-structured predator population, governed by the following set of differential equations:

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\[ \frac{dS}{dt} = rS \left( 1 - \frac{S + I}{k} \right) - \theta IS , \]
\[ \frac{dI}{dt} = \theta IS - \mu I - mIy_m , \]
\[ \frac{dy_i}{dt} = \alpha y_m - \alpha y_m (t - \tau)e^{-\gamma t} - \gamma y_i , \]
\[ \frac{dy_m}{dt} = \alpha y_m (t - \tau)e^{-\gamma t} - \beta y_m^2 + zmIy_m \]

With initial conditions for (1)
\[ \phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t) \in C_+ = C[-\tau, 0), R^4], \phi_i(t) > 0, \quad i = 1, 2, 3, 4 \]

Here, S(t) and I(t) denote the susceptible prey and infected prey population density respectively; \( y_i(t) \) is immature predator population and \( y_m(t) \) mature predator population density In modeling the system (1), we make following assumptions; 

H1: we assume that only susceptible prey S is capable or reproducing logistically with growth rate constant \( r > 0 \) and carrying capacity \( k > 0 \) The infected prey I is removed by death or by predation with death rate constant \( \mu > 0 \) and predation coefficient \( m > 0 \), before having the possibility of reproducing However the infective prey contributes with S to population growth toward the carrying capacity  

H2: It has been assumed that disease is not genetically inherited and spread among the prey population only The susceptible prey gets transmitted to infected prey in direct proportion to their densities with a proportionality constant \( \theta > 0 \), that is called transmission coefficient, the infected prey do not recover or become immune  

H3: Predator population is structured into two stages with a constant time lag \( \tau \) between them It has been assumed that birth rate of immature predator is proportional to the existing mature predator population and its death rate is proportional to the existing immature predator with proportionality constant \( \alpha > 0 \) and \( \gamma > 0 \) respectively For mature predator population, death rate is of logistic nature and is proportional to the square of the population with proportionality constant \( \beta > 0 \)  

For the continuity of initial conditions, we require
\[ y_i(0) = \int_{-\tau}^{0} \alpha y_m(t)e^{\gamma t} dt \]

the total surviving immature predator population from the observed birth on \(-\tau \leq t \leq 0\)  

H4: Further, it is assumed that mature predators are capable of capturing infected prey This is in accordance with the fact that the infected individuals are less active and can be caught more easily The coefficient in conversing prey into predator is \( \pi (0 < \pi \leq 1) \)  

In order to ensure that the model (1) is biologically feasible and all standard results of existence, uniqueness and continuous dependence on initial conditions are satisfied; we verify that the solution of (1) is non-negative and bounded  

**Theorem 21:** With the given initial conditions, solutions of system (1) are non-negative for all \( t > 0 \)

**Proof:** First we prove that \( y_m(t) > 0 \forall t > 0 \) If not, let there exists a \( \left( \hat{r} \right) > 0 \) such that \( y_m(\hat{r}) = 0 \)  

Let \( t_0 = \inf(t > 0 : y_m(t) = 0) \), then
\[ y'_m(t_0) = \alpha e^{-\gamma t_0} \Phi_4(t_0 - \tau) > 0, \quad 0 \leq t_0 \leq \tau \]
\[ = \alpha e^{-\gamma t_0} y_m(t_0 - \tau) > 0, \quad t_0 > \tau \]
so $y'_m(t_0) > 0$ Hence for sufficiently small $\eta > 0$, $y'_m(t_0-\eta) > 0$. But by the definition of $t_0$, $y'_m(t_0-\eta) \leq 0$, this contradiction proves that $y'_m(t) > 0 \forall t > 0$.

Now, consider the equation

$$\xi'' = -\rho \xi'(t) - a e^{-\rho \tau} y_m(t - \tau), \quad \xi(0) = y_i(0)$$

or

$$\xi''(t) + \gamma \xi'(t) = - a e^{-\rho \tau} y_m(t - \tau),$$

on integrating we get

$$\xi(t) = e^{-\rho t \tau} \left[ y_i(0) - \int_0^\tau a e^{-\rho (\sigma - \tau)} y_m(\sigma - \tau) d\sigma \right]$$

From the continuity of initial conditions, we have

$$\xi(\tau) = e^{-\rho \tau^2} \int_0^\tau a e^{-\rho (\sigma - \tau)} d\sigma - \int_0^\tau a e^{-\rho (\sigma - \tau)} y_m(\sigma - \tau) d\sigma$$

or

$$\xi(\tau) = 0 \quad \therefore \xi(t) > 0 \text{ for } t \in [0, \tau]$$

Also $y'_i(t) > \xi'(t)$, so $y_i(t) > \xi(t)$ on $0 < t \leq \tau$, which in term implies that $y_i(t) > 0$ for $0 \leq t \leq \tau$. Now, as in [1] we can prove by induction that $y_m(t) > 0 \forall t > 0$.

Also, since

$$\frac{dS}{dt} \bigg|_{S=0} = 0, \quad \frac{dl}{dt} \bigg|_{l=0} = 0,$$

continuity of solutions proves non-negativity of solutions.

**Theorem 22:** All solutions of (1) which initiate in $R_4^+$ are confined in a region defined as:

$$\Omega = \left\{ (S, I, y_i, y_m) \in R_4^+: 0 \leq S(t) + I(t) + y_i(t) + y_m(t) \leq \frac{(r + \epsilon)^2}{4r\epsilon} + \frac{(\alpha + \epsilon)^2}{4\beta\epsilon}, 0 < \epsilon \min(\mu, r) \right\}.$$ 

**Proof:** let us consider $z = S + I + y_i + y_m$, then

$$\dot{z} + \epsilon z \leq rS \left( 1 - \frac{S + I}{k} \right) - \mu d + \alpha y_m - y'_i - \beta y^2_m + \epsilon (S + I + y_i + y_m)$$

Now choosing $0 < \epsilon \min(\mu, \gamma)$ we get

$$\dot{z} + \epsilon z \leq - \left[ \frac{rS^2}{k} - (r + \epsilon)s \right] - \left[ \beta y^2_m - (\alpha + \epsilon)y'_m \right],$$

$$= - \left[ \frac{rS^2}{k} - (r + \epsilon) \frac{k}{2} \sqrt{r} \right] - \left[ \sqrt{\beta} y_m - \frac{(\alpha + \epsilon)}{2\sqrt{\beta}} \right] \right]$$

$$+ \frac{(r + \epsilon)^2 k}{4r} + \frac{(\alpha + \epsilon)^2}{4\beta}$$

$$\dot{z} + \epsilon z \leq \frac{(r + \epsilon)^2 k}{4r} + \frac{(\alpha + \epsilon)^2}{4\beta}$$

On integrating and taking limit as $t \to \infty$ we get
The model equation (1) has interior equilibrium point \( E^* (S^*, I^*, y_i^*, y_m^*) \) where

\[
S^* = k \left( r - \left( \frac{\theta + r}{k} \right) I \right), \quad I^* = \frac{K \theta - \mu - m \alpha e^{-\tau \gamma}}{k \theta \left( \frac{\theta + r}{k} \right) + \gamma \alpha e^{-\tau \gamma}},
\]

\[
y_i^* = \frac{\alpha (1 - e^{-\tau \gamma}) y_i^*}{\gamma}, \quad y_m^* = \frac{\alpha e^{-\tau \gamma} + \gamma \alpha e^{-\tau \gamma}}{\gamma}
\]

The characteristic equation corresponding to interior equilibrium \( E^* \) is given by

\[
\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0
\]

where

\[
a_1 = \frac{rs^*}{k} + r + 2 \beta y_m^* - \gamma m I^*,
\]

\[
a_2 = \frac{rs^*}{k} \left( r + 2 \beta y_m^* - \gamma m I^* \right) + r (2 \beta y_m^* - \gamma m I^*) + \gamma m^2 I^* y_m^* + \left( \frac{r}{k} + \theta \right) \partial_{S^*} I^*,
\]

\[
a_3 = \frac{rs^*}{k} \left( r (2 \beta y_m^* - \gamma m I^*) + \gamma m^2 I^* y_m^* \right) + \left( \frac{r}{k} + \theta \right) \partial_{S^*} I^* \left( \gamma + 2 \beta y_m^* - \gamma m I^* \right),
\]

\[
a_4 = \frac{rs^*}{k} \gamma m^3 I^* y_m^* + \left( \frac{r}{k} + \theta \right) \partial_{S^*} I^* \left( 2 \beta y_m^* - \gamma m I^* \right),
\]

\[
b_1 = \beta y_m^* - \gamma m I^*, \quad b_2 = (\beta y_m^* - \gamma m I^*) \left( \frac{rs^*}{k} + \gamma \right)
\]

\[
b_3 = (\beta y_m^* - \gamma m I^*) \left( \frac{rs^*}{k} + \gamma \right), \quad b_4 = (\beta y_m^* - \gamma m I^*) \left( \frac{rs^*}{k} + \gamma \right) \partial_{S^*} I^*
\]

Here we have used \( \alpha e^{-\tau \gamma} = \beta y_m^* - \gamma m I^* \)

In the absence of delay \( \tau = 0 \), characteristic equation (2) becomes

\[
\lambda^4 + (a_1 - b_1) \lambda^3 + (a_2 - b_2) \lambda^2 + (a_3 - b_3) \lambda + (a_4 - b_4) = 0
\]

Since \( a_j > b_j \) for \( j = 1, 2, 3, 4 \) and when \( (a_1 - b_1) (a_2 - b_2) (a_3 - b_3) > (a_4 - b_4) (a_4 - b_4) \), from Routh-Hurwitz criterion, all roots of equation (3) have negative real parts. So the system (1) is locally asymptotically stable in the absence of delay. When \( \tau \neq 0 \) (since \( a_4 \neq b_4 \)), stability of system can change only when there exist one root of equation (2) having its real part zero i.e. \( \text{Re} \lambda = 0 \). Let \( \lambda = i \omega \) be one of such root. Keeping \( \lambda = i \omega \) in equation (2), on equating real and imaginary parts, we get

\[
\omega^4 - a_2 \omega^2 + a_4 = (b_4 - b_2 \omega^2) \cos \omega + (b_2 \omega - b_4 \omega^3) \sin \omega
\]

\[
- a_1 \omega^3 + a_3 \omega = (b_3 \omega - b_1 \omega^3) \cos \omega - (b_1 - b_3 \omega^2) \sin \omega
\]

On squaring and adding above equations, we have

\[
\mu \leq \frac{(r + \epsilon)^2 k}{4r} + \frac{(\alpha + \epsilon)^2}{4 \beta}
\]
\[
\begin{align*}
\omega^2 + \omega^4 (a_1^2 - 2a_2 - b_1^2) + \omega^4 (a_2^2 + 2a_2 - a_3 - b_2^2 + 2b_1 b_3) \\
+ \omega^2 (a_3^2 + 2b_2 - b_3^2 - 2a_2 a_4) + a_4^2 - b_4^2
\end{align*}
\]  
(6)

On substituting \( \omega^2 = v \), in equation (6) becomes

\[
\begin{align*}
v^3 + v^4 (a_1^2 + 2a_2 - b_1^2) + v^4 (a_2^2 + 2a_2 - a_3 - b_2^2) + \\
v(a_3^2 + 2b_2 - b_3^2 - 2a_2 a_4) + a_4^2 - b_4^2 = 0
\end{align*}
\]  
(7)

In order to exist real solution of equation (6), there must be a positive solution of equation (7)

Let \( E^* \) be locally asymptotically stable in the absence of delay and following inequalities hold

\[
c_1 = a_1^2 - 2a_2 - b_1^2 > 0, \quad c_2 = a_2^2 + 2a_2 - 2a_3 - b_2^2 + 2b_1 b_3 > 0,
\]

\[
c_3 = a_3^2 + 2b_2 - b_3^2 - 2a_2 a_4 > 0 \quad \text{and} \quad c_1 c_3 > c_1^2 (a_4^2 - b_4^2) + c_2^2
\]

Then all conditions of Routh-Hurwitz criterion hold for fourth order equation (7) are satisfied and hence all the roots of equation (7) are negative So there exist no real solution in equation (6) and hence interior equilibrium \( E^* \) remains stable for all \( \tau > 0 \), if it is stable for \( \tau = 0 \)

Again on solving equation (4) and (5), we get a critical value of delay for which system becomes unstable

\[
\tau_c = \frac{1}{\omega} \sin^{-1} \left[ \frac{\left( \omega^4 - a_4 \omega^3 + a_4 \omega^2 + b_4 \omega \right)}{\left( b_4 \omega - b_4 \omega^3 \right)} \right]
\]

Here \( \tau_c \) is the least positive value of delay for which stability change or bifurcation occurs

4 NUMERICAL EXAMPLES

To explain the applicability of the results obtained, we consider the following set of values of parameters in our computation

\[
r = 1, \quad k = 12, \quad \theta = 0.05, \quad \mu = 0.02, \quad m = 1, \quad \pi = 0.05, \quad \alpha = 0.04, \quad \gamma = 0.02, \quad \beta = 0.01
\]

First we see the effect of increasing values of time delay by choosing various values of \( \tau \) as 0, 5 and 10 as in Table 1

\[
\begin{array}{cccc}
\tau & S^* & I^* & y_i^* \\
0 & 10517646 & 0211765 & 0 \\
5 & 8435371 & 0509233 & 5079323 \\
10 & 7669340 & 0618666 & 6285544
\end{array}
\]

It can be checked that when \( \tau = 0 \), all conditions for stability are satisfied and \( E^* \) is locally asymptotically stable point in the absence of delay

\[
\begin{array}{cccc}
\tau & S^* & I^* & y_i^* \\
0 & 8435371 & 0509233 & 5079323 \\
05 & 7430085 & 0557307 & 5383209 \\
06 & 6538198 & 0581043 & 5533254
\end{array}
\]

From table 1, we can note that increase in time delay increases the equilibrium level of immature predator and decreases the equilibrium level of mature predator. Decrease in equilibrium level of mature predator causes increases in equilibrium level of infected prey and decrease in equilibrium
level of susceptible prey Again taking $\tau = 5$ and for three different value of $\theta$, viz $\theta = 0.5, 0.75, 1$, we get table 2, (for same values of other parameters)

It is noted from the above table, that as the disease transmission coefficient $\theta$ increases, the equilibrium level of infected prey and predators (both immature and mature) increases and decreases the equilibrium level of susceptible prey

5 DISCUSSION

In this paper we have proposed and analyzed a non-linear mathematical model to study a prey-predator model in which prey population is divided into two classes –susceptible prey and infected prey. Predator population is structured into two stages, immature and mature with a time lag $\tau$ between these two stages that is time from birth to maturity. It has been assumed that only infected prey and mature predator behave like prey-predator ie only mature predators are able to capture (or eat) less active infected prey The existence of nontrivial equilibrium point has been shown and its stability behavior has been studied. It has been proved that system remains stable for all values of time delay if it is stable in the absence of delay under certain conditions. We have also obtained the critical value of delay parameter below which the system is stable and above which stability change may occur.

REFERENCES