The fractal dimensions of graphs of the Weierstrass function with the Weyl-Marchaud fractional derivative

To cite this article: Y Liang et al 2008 J. Phys.: Conf. Ser. 96 012111

View the article online for updates and enhancements.

Related content

- On the Weierstrass cubic for hyperelliptic functions
  D V Leikin

- $CP(2)$-multiplicative Hirzebruch genera and elliptic cohomology
  V M Buchstaber and E Yu Netay

- Weierstrass Elliptic Function Solutions to Nonlinear Evolution Equations
  Yu Jian-Ping and Sun Yong-Li
The fractal dimensions of graphs of the Weierstrass function with the Weyl-Marchaud fractional derivative

Yongshun Liang, Kui Yao, Wei Xiao

1. School of Science, Nanjing University of Science and Technology, Nanjing 210094, P.R.China
   Email: liangyongshun@gmail.com

2. PLA University of Science and Technology, Nanjing 210093, P.R.China
   Institute of Mathematics, Chuzhou University, Chuzhou 239000, P.R.China
   Email: yaokuinju@gmail.com

3. School of Science, Nanjing University of Science and Technology, Nanjing 210094, P.R.China
   Email: chinaxiao@163.com

Abstract. The fractal dimensions of graphs of the Weierstrass function with the Weyl-Marchaud fractional derivative are calculated with the techniques of K-measure and K-dimension. And the linear relationship between the fractal dimensions of graphs of the Weierstrass function and the orders of their Weyl-Marchaud fractional derivative is obtained. Graphs and numerical results of a given example verify this interesting connection.

1. Introduction

In [13], Zhang gave a very lucid as well as elementary discussion of the fractal dimension of certain random attractors for a Ginzburg-Landau equation with additive noise. She posed a heuristic question of what relationship was between the fractal dimensions of certain fractal function and the Weyl-Marchaud fractional derivative of graphs of this function. It is defined as follows:

$$W(t) = \sum_{j=1}^{\infty} \lambda^{-\alpha j} \sin(\lambda^j t), (0 < \alpha < 1, \lambda > 1).$$  \hspace{1cm} (1.1)

The function which is defined as (1.1) is often called the Weierstrass function. It is a prototype example of a function which is continuous everywhere but differentiable nowhere and has an exponent which is constant everywhere. Its graph is known to have a Box dimension, $2-\alpha$, for sufficiently large $\lambda$ (see, for details, [1]). Incidentally, the Weierstrass function is not just mathematical curiosities but occurs at several places. For example, the graph of this function is known to be a repeller or an attractor of some dynamical systems (cf. [1]). This kind of function can also be recognized as the

1 To whom any correspondence should be addressed.
characteristic function of Lévy flight on a one dimensional lattice, which means that the Lévy flight can be considered as superposition of the Weierstrass-type functions.

It is also very interesting to consider the fractal dimensions of fractal functions for the importance of the application of their fractal dimensions, such as fractal interpolation and nonlinear dynamics system. Other work can also be found in Refs. [3,5,6,7,8,14,15,16].

Let the Riemann-Liouville fractional calculus be defined as:

Definition 1.1 (Cf. [9]). Let \( f \) be piecewisely continuous on \((0, \infty)\), and local integrable on \([0, \infty)\). Then for \( t > 0 \), \( \Re(v) > 0 \), we call

\[
D^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-x)^{-v-1} f(x) dx
\]

the Riemann-Liouville fractional integral of \( f \) of order \( v \). For \( 0 < u < 1 \), we call

\[
D^u f(t) = D[D^{-u} f(t)]
\]

the Riemann-Liouville fractional derivative of \( f \) of order \( u \).

From this definition, we have the following theorem.

Theorem 1.2 (Cf. [10]). Let \( 0 < v, u, \alpha < 1, \alpha + v < 1, u < \alpha \), it holds that

\[
\dim_B \Gamma(g, I) = \dim_B \Gamma(W, I) - v, \dim_B \Gamma(m, I) = \dim_B \Gamma(W, I) + u. \tag{1.2}
\]

Here function \( g \) is a Riemann-Liouville fractional integral of \( W \) of order \( v \) and function \( m \) is the Riemann-Liouville fractional derivative of \( W \) of order \( u \). And \( \Gamma(f, I) \) denotes the graph of \( f \) on interval \( I = [0,1] \).

Now we are mainly interested in considering the Weyl-Marchaud fractional derivative of graphs of the Weierstrass function and the corresponding relationship. First we give the definition of the Weyl-Marchaud fractional derivative.

Definition 1.3 (Cf. [9,11,12]). Let \( f \) be a continuous function, \( 0 < v < 1 \), we call

\[
D^v f(t) = \frac{\Gamma(v)}{\Gamma(1-v)} \int_0^\infty \frac{f(t)-f(t-y)}{y^{1+v}} dy \tag{1.3}
\]

the Weyl-Marchaud fractional derivative of \( f(t) \) of order \( v \).

Let

\[
S_v(a, \lambda) = \frac{\Gamma(v)}{\Gamma(1-v)} \int_0^\infty \frac{\sin(at) - \sin(at-ay)}{y^{1+v}} dy \tag{1.4}
\]

denote the Weyl-Marchaud fractional derivative of \( \sin(at) \).

For \( \lambda > 1, 0 < v < \alpha < 1 \), define

\[
W^\alpha(t) = D^\alpha(W(t)) = \sum_{j=1}^{\infty} \lambda^{-\alpha} S_v(\lambda^j) \tag{1.5}
\]

the Weyl-Marchaud fractional derivative of \( W(t) \) of order \( v \).

2. Fractal dimensions of the Weierstrass function with the Weyl-Marchaud fractional derivative

In this section, we first give several lemmas, then we will give the main results of the present paper. By simply calculation, we have
Lemma 2.1 Let $0 < \nu < 1, \alpha > 1$, then we have

\begin{align}
\text{(1)} \quad |S'_j(v,a)| &\leq C_j(v)a^\nu, \quad (2.1) \\
\text{(2)} \quad |S''_j(v,a)| &\leq C'_j(v)a^{1+\nu}, \quad (2.2) \\
\text{(3)} \quad |S'_{j+k} - S'_j| &\geq C_2(v)a^\nu \quad (2.3)
\end{align}

Here

$$C_j(v) = \frac{v}{\Gamma(1-\nu)} \left( \frac{\pi^{1-\nu}}{1-\nu} - \frac{2\pi^{-\nu}}{v} \right), \quad C'_j(v) = \frac{4v(1+v)}{27\Gamma(1-\nu)} \left( \frac{2}{3} \right)^v.$$

$S'_j(v,a)$ denotes the classical derivative of function $S_j(v,a)$. For $t_j = 3\pi / a(j=1, 3, 5, \cdots)$ and $h = \pi / a$.

Lemma 2.2 (Cf. [1]) Let $f$ be a continuous function on $I = [0,1], C > 0$, and $0 < s < 1$. If we suppose $|f(t) - f(u)| \leq C |t - u|^s$ ($0 \leq t, u \leq 1$), then,

$$\dim\Gamma(f, I) \leq 2 - s.$$

Lemma 2.3 (Cf. [2]) Let $f$ be a continuous function on $I = [0,1]$ and $1 < s < 2$. Let $\Pi = \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}$ be a partition of $I$, $\delta_i = [x_{i-1}, x_i]$ and $|\Pi| = \max_{1 \leq i \leq n} \delta_i$. Then, we have

\begin{align}
\text{(1)} \quad K^s(\Gamma(f, I)) &= \liminf_{\delta \to 0^+} \sum_{\Pi} \text{OSC}(f, \delta) |\delta|^{s-1}. \quad (2.4) \\
\text{(2)} \quad \dim_{K} \Gamma(f, I) < \dim_{\Gamma} \Gamma(f, I). \quad (2.5)
\end{align}

Where $K^s(\Gamma(f, I))$ denotes the $s$-dimension $K$-measure of $\Gamma(f, I)$. And

$$\dim_{K} \Gamma(f, I) < \dim_{\Gamma} \Gamma(f, I).$$

Lemma 2.4 Let $\lambda > 1, 0 < \nu < 1, I = [0,1]$ and $W^\nu(t)$ be the Weyl-Marchaud fractional derivative of the Weierstrass function $W(t)$. Then, $W^\nu(t)$ is continuous on $I$.

Theorem 2.5 Let $\lambda > 1, 0 < \nu < \alpha < 1, I = [0,1]$ and $W^\nu(t)$ be the Weyl-Marchaud fractional derivative of the Weierstrass function $W(t)$. Then,

$$\dim_{K} \Gamma(W^\nu, I) \leq 2 - \alpha + \nu. \quad (2.6)$$

Theorem 2.6 Let $0 < \nu < \alpha < 1, I = [0,1], \lambda > 1$, $C_j(v) = 2C_i(v) / C_2(v), \lambda_i = (2C_j(v) + 1)^{(\nu-\alpha v)^{-1}}, \lambda_2 = (2C_1(v) + 3)^{(\nu-\alpha v)^{-1}}$ and $\lambda_0 = \max(7\pi, \lambda_i, \lambda_2)$. Let $W^\nu$ be the Weyl-Marchaud fractional derivative of the Weierstrass function $W(t)$ of order $\nu$. Then for $\lambda > \lambda_0$, it holds that

$$\dim_{K} \Gamma(W^\nu, I) \geq 2 - \alpha + \nu. \quad (2.8)$$

where $C_i(v), C_j(v)$ are the same as that of Lemma 2.1.
A combination of Theorem 2.6, Theorem 2.7 and Lemma 2.3(2) leads to the following theorem

**Theorem 2.7** Let $0 < v < \alpha < 1, \lambda > \lambda_0$ (where $\lambda_0$ is the same as that of Theorem 2.6). Then

$$\dim_k \Gamma(W^*, I) = \dim_\mu \Gamma(W^*, I) = 2 - \alpha + v.$$ 

Here function $W^*(t)$ is the Weyl-Marchaud fractional derivative of the Weierstrass function $W(t)$ of order $\nu$ which is defined as (1.1).

3. **The connection between the fractal dimensions of $W(t)$ and $W^*(t)$**

In this section, we discuss the connection between the orders of the Weyl-Marchaud fractional derivative and the fractal dimensions of graphs of the Weierstrass function.

For convenience, let $\Gamma(W^*, I)$ denote the common value of the two dimensions of graphs of $W^*(t)$, i.e.,

$$\dim \Gamma(W^*, I) = 2 - \alpha + v. \quad (3.1)$$

On the other hand, we know that for the Weierstrass function $W(t)$, it holds that (More details can be found in Ref. [1])

$$\dim_k \Gamma(W, I) = \dim_\mu \Gamma(W, I) = 2 - \alpha. \quad (3.2)$$

With (3.1) and (3.2), it is obvious that

**Theorem 3.1** Let $0 < \nu < \alpha < 1, I = [0, 1]$ and $\lambda$ be sufficiently large. Then,

$$\dim \Gamma(W^*, I) = \dim \Gamma(W, I) + \nu.$$ 

Here function $W^*(t)$ is the Weyl-Marchaud fractional derivative of the Weierstrass function $W(t)$ of order $\nu$ which is defined as (1.1).

This shows the linear connection between the order of the Weyl-Marchaud fractional derivative and the fractal dimensions of graphs of the Weierstrass function $W(t)$.

4. **Graphs and numerical results**

Like [4], we give graphs and numerical results of certain example.

**Example** Let the Weierstrass function $W(t)$ be defined as follows:

$$W(t) = \sum_{j=1}^{\frac{1}{3}} \sin(3^j t). \quad (4.1)$$

Fig. 4.1 shows the graph of the Weierstrass function $W(t)$, and Fig. 4.2 shows the graph of the function $W^*(t)$ which is the Weyl-Marchaud derivative of $W(t)$ of order 0.1.

Fig. 4.3 shows the graph of the function $W^*(t)$ which is the Weyl-Marchaud derivative of $W(t)$ of order 0.2, and Fig. 4.4 shows the graph of the function $W^*(t)$ which is the Weyl-Marchaud derivative of $W(t)$ of order 0.3.
Fig. 4.1 $W(t)$. Fig. 4.2 $W^*(t), \nu = 0.1$. Fig. 4.3 $W^*(t), \nu = 0.2$. Fig. 4.4 $W^*(t), \nu = 0.3$. Fig. 4.5 Connection between $\nu$ and $\dim \Gamma(W^*, I)$. 

5. Conclusions
To sum up, we have the following conclusions: Let $W^*(t)$ be the Weyl-Marchaud fractional derivative of the Weierstrass function $W(t)$. Then, for sufficiently large $\lambda, I = [0,1]$ and $0 < \nu < \alpha < 1$, it holds that

$$\dim_k \Gamma(W,I) = \dim_\nu \Gamma(W,I) = \dim_k \Gamma(W,I) = \dim_\nu \Gamma(W,I) = 2 - \alpha.$$ 

This is the first time to calculate the fractional derivative of fractal function by the Weyl-Marchaud fractional derivative. The Weyl-Marchaud derivative is important in science, technology and engineering. The graphs and numerical results of the Weierstrass function and their Weyl-Marchaud fractional derivative further show the corresponding connection.

**Acknowledgements**

Research supported by University Natural Science Foundation of Anhui Province(2005kj382，China) and National Natural Science Foundation of China (10671039).

**References**