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Xian-Feng Zhou\textsuperscript{1,2} and Bao-Ming Qiao\textsuperscript{3}

\textsuperscript{1} Department of Mathematics, University of Science and Technology of China, Hefei 230026, P.R.China
\textsuperscript{2} School of Mathematics and Computational Science of Anhui University, Hefei 230039, P.R.China
\textsuperscript{3} Department of Mathematics, Shangqiu Normal College, Shangqiu 476000, P.R.China

Email: xianfengzhou@sina.com; bmqiao@126.com.

Abstract. In this paper, we consider all-delay stability of non-degenerate differential systems and degenerate differential systems. We obtain several criteria and generalize some results established before.

1. Introduction
Delay phenomenon is common in economic systems, engineering systems and so on. More and more scholars devote their energy to studying delay phenomenon and lots of important results were obtained, for example [1-14]. Stability is an important character of a system and all-delay stability is a special case of stability. But there are few results about all-delay stability, see [2, 13]. In this paper, we’ll discuss it. We give some preliminaries first.

Definition 1.1 We say that matrix couple \((A, E)\) is regular if there exists \(\lambda\) such that 
\[
\det(\lambda E - A) \neq 0.
\]

Definition 1.2 For equation 
\[
\dot{x}(t) = f(t, x(t), x(t-\tau)),
\]
where \(x(t) \in \mathbb{R}^n, \tau \in \mathbb{R}^+ = [0, \infty)\), we say the solution of (1.1) is all-delay stable if for \(\forall \tau \in \mathbb{R}^+ = [0, \infty)\), the solution of equation (1.1) is asymptotically stable.

Lemma 1.1 Suppose \(f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0\) is complex coefficient polynomial. Then for any \(k \in \mathbb{R}^+\), when \(|x|\) is large enough, we have 
\[
|a_n x^n| > k |a_{n-1} x^{n-1} + \cdots + a_1 x + a_0|,
\]

\(1\) To whom any correspondence should be addressed.
where $|.|$ denote module, $a_{n} \neq 0$.

**Remark** If $f(\lambda) = a_{n}(\lambda)\lambda^{n} + a_{n-1}(\lambda)\lambda^{n-1} + \cdots + a_{1}(\lambda)\lambda + a_{0}$ satisfies $\lim_{|\lambda| \to \infty} a_{n}(\lambda)$ exists, $\lim_{|\lambda| \to \infty} a_{n}(\lambda) \neq 0$ and $\lim_{|\lambda| \to \infty} a_{j}(\lambda)$ exists ($0 \leq j \leq n-1$), then the conclusion of Lemma 1.1 holds.

**Lemma 1.2** ([13]) For system

$$\dot{x}(t) = A x(t) + \sum_{i=1}^{m} B_{i} x(t - \tau_{i}),$$

(1.2)

where $x(t) \in \mathbb{R}^{n}$, $A, B_{i} \in \mathbb{R}^{n \times n}$, $\tau_{i} > 0$ ($i = 1, 2, \cdots, m$), its characteristic equation is

$$h(\lambda, \tau_{1}, \tau_{2}, \cdots, \tau_{m}) = \det(\lambda I - A - \sum_{i=1}^{m} B_{i} e^{-\lambda \tau_{i}}) = 0.$$  

(1.3)

The system (1.2) is asymptotically stable if and only if the real parts of all roots of equation (1.3) are negative.

**Lemma 1.3** ([14]) For singular differential system

$$\dot{x}(t) = Ax(t),$$

(1.4)

where $x(t) \in \mathbb{R}^{n}$, $E, A \in \mathbb{R}^{n \times n}$. If $\det E = 0$, then system (1.4) is asymptotically stable if and only if the real parts of all roots of characteristic equation $h(\lambda, \tau) = \lambda E - A$ are negative.

In this paper, we always assume that matrix couple $(E, A)$ is regular. According to [1], the solutions of system (1.2), system (2.2) and system (2.4) are existent and unique.

2. Main results

**Theorem 2.1** System (1.2) is all-delay stable if and only if

(i) the real parts of all roots of equation

$$h(\lambda, 0, \cdots, 0) = \det |\lambda I - A - \sum_{i=1}^{m} B_{i} e^{-\lambda \tau_{i}}| = 0$$

are negative;

(ii) for $\forall \tau_{i} \in \mathbb{R}^{+}$, there exists some $T > 0$ such that when $T$ is large enough we have

$$h(iy, \tau_{1}, \cdots, \tau_{m}) = \det(iy I - A - \sum_{i=1}^{m} B_{i} e^{-i \tau_{i}}) \neq 0, \quad y \in [-T, T].$$

**Proof. Necessity.** If (i) does not hold, the solution of system

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_{i} x(t) = (A + \sum_{i=1}^{m} B_{i}) x(t)$$

is not asymptotically stable. This contradicts with the fact that System (1.2) is all-delay stable. If (ii) does not hold, we know from Lemma (1.2) that the solution of system (1.2) is not asymptotically stable. This contradicts to the all-delay stability of system (1.2).

**Sufficiency.** We only need to prove: $\forall \tau_{i} \in \mathbb{R}^{+}$, the real parts of all roots of equation (1.3) are negative. On the one hand, characteristic equation (1.3) can be reduced to

2
\[ \lambda^n + \lambda^{n-1} R_1(e^{-\lambda \tau_1}, e^{-\lambda \tau_2}, \ldots, e^{-\lambda \tau_m}) + \cdots + \lambda R_{n-1}(e^{-\lambda \tau_1}, e^{-\lambda \tau_2}, \ldots, e^{-\lambda \tau_m}) + R_n(e^{-\lambda \tau_1}, e^{-\lambda \tau_2}, \ldots, e^{-\lambda \tau_m}) = 0, \]

where \( R_i(Z_1, Z_2, \ldots, Z_m) \) is polynomial about \( Z_1, Z_2, \ldots, Z_m \), \( i = 1, 2, \ldots, m \). For \( \forall \tau_i \in R^+ \) and \( \text{Re } \lambda \geq 0 \), we have \( |e^{-\lambda \tau}| \leq 1 \). Therefore there exists \( M > 0 \) such that

\[ |R_i(e^{-\lambda \tau_1}, e^{-\lambda \tau_2}, \ldots, e^{-\lambda \tau_m})| \leq M \quad i = 1, 2, \ldots, m. \]

and

\[ |\lambda^n + \lambda^{n-1} R_1 + \cdots + \lambda R_{n-1} + R_n| \geq |\lambda^n| |[1 - \frac{|R_1|}{|\lambda|} - \cdots - \frac{|R_n|}{|\lambda|^n}]| > 0 \]

when \( \text{Re } \lambda \geq 0 \) and \( |\lambda| > M \). Therefore for \( \forall \tau_i \in R^+ \), equation (1.3) has no root when \( |\lambda| > M \) and \( \text{Re } \lambda \geq 0 \). On the other hand, we know from (i) that all roots of equation (2.1) are distributed in left complex plane. When \( \tau_i \) increases from zero, roots of characteristic equation (1.3) are distributed in right complex plane if and only if there exists some \( \tilde{\lambda} \) such that \( \tilde{\lambda} \) passes imaginary axis on \([-M, M]\). This is impossible by condition (ii). Therefore all roots of (1.3) are in left complex plane.

**Theorem 2.2** For degenerate system

\[ E \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t - \tau_i), \quad (2.2) \]

where \( E \in R^{mxn} \), \( \text{det } E = 0 \), \( \tau_i > 0 \) and \( A, B_i \in R^{mxn} \), \( i = 1, 2, \ldots, m \), characteristic equation of (2.2) is

\[ h(\lambda, \tau_1, \tau_2, \cdots, \tau_m) = |\lambda E - A - \sum_{i=1}^{m} B_i e^{-\lambda \tau_i}| = 0. \quad (2.3) \]

System (2.2) is all-delay if and only if

(i) the real parts of all roots of \( h(\lambda, 0, \cdots, 0) = 0 \) are negative;

(ii) for \( \forall \tau_i \in R^+ \), there exists some \( T > 0 \) such that when \( T \) is large enough

\[ h(iy, \tau_1, \tau_2, \cdots, \tau_m) = \text{det}(iyE - A - \sum_{i=1}^{m} B_i e^{-iy \tau_i}) \neq 0, \quad y \in [-T, T]. \]

**Proof. Necessity.** Assume that (i) does not hold. By Lemma (1.3), the solution of system (2.2) isn’t asymptotically stable when \( \tau_i = 0 \). This contradicts with the fact that system (2.2) is all-delay stable. If (ii) doesn’t hold, then there exist \( y \in R \) and \( \tau_i \in R^+ \) \( (i = 1, 2, \cdots, m) \) such that

\[ h(iy, \tau_1, \tau_2, \cdots, \tau_m) = \text{det}(iyE - A - \sum_{i=1}^{m} B_i e^{-iy \tau_i}) = 0. \]

Then the solution \( x(t) = e^{iyt} x(0) \) of system (2.2) is not asymptotically stable. This is a contradiction.
Sufficiency. ∀ \( \tau_i \in R^+ \), we only need to prove the fact that the real parts of all roots of equation (2.3) are negative. Characteristic equation (2.3) can be reduced to

\[
\lambda^i R_0(e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) + \lambda^{i-1} R_1(e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) + \cdots + \\
\lambda R_{i-1}(e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) + R_i(e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) = 0,
\]

where \( i \in N \). When \( \Re(\lambda) \geq 0 \) and \( |\lambda| \to \infty \), polynomial \( R_0, R_1, R_2, \cdots, R_i \) are limited. When \( \Re \lambda \geq 0 \) and \( |\lambda| \) is large enough, we have by Lemma (2.1) that

\[
| \lambda^i R_0(e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) + \lambda^{i-1} R_1(e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) + \cdots + \\
\lambda R_{i-1}(e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) + R_i(e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) | \geq |
\]

\[
| \lambda^i R_0 | - | \lambda^{i-1} R_1 + \cdots + \lambda R_{i-1} + R_i | \not\to 0.
\]

Therefore for ∀ \( \tau_i \in R^+ \), there exists \( M > 0 \) such that equation (2.3) has no root when \( |\lambda| > M \) and \( \Re \lambda \geq 0 \). From (i), we know that when \( \tau_i = 0 \) the roots of equation (2.3) are distributed in left complex plane. When \( \tau_i \) increases from zero, the roots of characteristic equation (2.3) are distributed in right complex plane if and only if there exists some \( \lambda \) such that \( \lambda \) passes imaginary axis on \([-M, M] \). This is impossible by condition (ii). Therefore all roots of equation (2.3) are in left complex plane.

**Theorem 2.3** For neutral degenerate differential system

\[
E x(t) + Ax(t) + \sum_{i=1}^{m} B_i x(t - \tau_i) + \sum_{i=1}^{m} C_i x(t - \zeta_i) = 0,
\]

where \( E, A, B_i, C_i \in R^{n \times n} \), \( \det E = 0 \), \( x(t) \in R^n \), \( \tau_i > 0 \), \((i = 1, \cdots, m)\), characteristic equation of system (2.4) is

\[
h(\lambda, \tau_1, \tau_2, \cdots, \zeta_m) = | \lambda E + A + \sum_{i=1}^{m} B_i e^{-\lambda \tau_i} + \sum_{i=1}^{m} C_i \lambda e^{-\lambda \zeta_i} | = 0.
\]

System (2.4) is all-delay stable if and only if

(i) the real parts of all roots of \( h(\lambda, 0, \cdots, 0) = 0 \) are negative;

(ii) for ∀ \( \tau_i \in R^+ \), there exists some \( T > 0 \) such that when \( T \) is large enough

\[
h(iy, \tau_1, \cdots, \tau_m) = \det(iy E + A + \sum_{i=1}^{m} B_i e^{-iy \tau_i} + \sum_{i=1}^{m} C_i \lambda e^{-iy \zeta_i}) \neq 0, \quad y \in [-T, T].
\]

**Proof** Take \( x(t) = z(t) \), then the system (2.4) can be reduced to

\[
\begin{pmatrix}
-I & 0 \\
0 & I
\end{pmatrix}
\begin{pmatrix}
x(t) \\
z(t)
\end{pmatrix}
+ 
\begin{pmatrix}
0 & I \\
A & E
\end{pmatrix}
\begin{pmatrix}
x(t) \\
z(t)
\end{pmatrix}
+ 
\begin{pmatrix}
0 \\
\sum_{i=1}^{m} B_i 
\end{pmatrix}
\begin{pmatrix}
x(t - \tau_1) \\
z(t - \tau_1)
\end{pmatrix}
= 0.
\]

Note
\[ \bar{x}(t) = \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \quad \bar{E} = \begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 0 & I \\ A & E \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 & 0 \\ \sum_{i=1}^{m} B_i & \sum_{i=1}^{m} C_i \end{pmatrix}, \]

then we have

\[ \bar{E} \bar{x}(t) + \bar{A} \bar{x}(t) + \bar{B} x(t - \tau_i) = 0. \quad \text{(2.6)} \]

It is true that system (2.4) has the same stability as system (2.6). Characteristic function of equation (2.6) can be reduced to

\[ \left| \lambda \bar{E} + \bar{A} + \bar{B} e^{-\lambda \tau_i} \right| = \left| \begin{array}{cc} 0 & I \\ \sum_{i=1}^{m} B_i e^{-\lambda \tau_i} & \sum_{i=1}^{m} C_i e^{-\lambda \tau_i} \end{array} \right| e^{-\lambda \tau_i} \]

\[ = \left| \begin{array}{cc} -\lambda I & I \\ A + \sum_{i=1}^{m} B_i e^{-\lambda \tau_i} & E + \sum_{i=1}^{m} C_i e^{-\lambda \tau_i} \end{array} \right| e^{-\lambda \tau_i} \]

\[ = \left| \lambda E + A + \sum_{i=1}^{m} B_i e^{-\lambda \tau_i} + \sum_{i=1}^{m} C_i e^{-\lambda \tau_i} \right|. \]

It is obvious that system (2.4) has the same characteristic roots as system (2.6). We obtain

\[ \left| iy \bar{E} + \bar{A} + \bar{B} y e^{-iy \tau_i} \right| = \det(iy E + A + \sum_{i=1}^{m} B_i e^{-iy \tau_i} + \sum_{i=1}^{m} C_i \lambda e^{-iy \tau_i}). \]

Therefore system (2.6) is all-delay stable if and only if characteristic equation satisfies (i) and (ii).

References