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Distances between Interval-valued Intuitionistic Fuzzy Sets

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Abstract. We give a geometrical interpretation of the interval-valued fuzzy set (IVFS). Based on the geometrical background, we propose new distance measures between IVFSs and compare these measures with distance measures proposed by Burillo and Bustince (1996) and Grzegorzewski (2004), respectively. Furthermore, we extend three methods for measuring distances between IVFSs to interval-valued intuitionistic fuzzy sets.

1. Introduction

Most of problems in real life situation such as economics, engineering, environment, social sciences and medical sciences not always involve crisp data.

So we cannot successfully use the traditional methods because of various types of uncertainties presented in those problems. Since Zadeh [9] introduced fuzzy sets in 1965, many approaches and theories treating imprecision and uncertainty have been proposed. Some of these theories, like as intuitionistic fuzzy set theory and interval-valued fuzzy set theory and interval-valued intuitionistic fuzzy set theory, are extensions of fuzzy set theory and the others try to handle imprecision and uncertainty in different ways. Some authors [2,4] pointed out that there is strong connection between intuitionistic fuzzy sets (IFSs) and IVFSs, i.e., intuitionistic fuzzy set theory and interval-valued fuzzy set theory are equipollent generalizations of fuzzy set theory. Some authors have investigated IVFSs and its relevant topics, for example, Burillo and Bustince [3] researched entropy and distance for IVFSs, Grzegorzewski [5] studied distance between IVFSs based on the Hausdorff metric, Zeng and Li [12] studied the relationship between entropy and similarity measure of IVFSs.

In this paper, we give a geometrical interpretation of the IVFS and take into account all three parameters describing the IVFS. So, based on the geometrical background, we propose new distance measures between IVFSs and compare these measures with above-mentioned distance measures proposed by Burillo and Bustince [3] and Grzegorzewski [5], respectively. Furthermore, we extend

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three methods for measuring distances between IVFSs to interval-valued intuitionistic fuzzy sets (IVIFSs).

2. Geometrical interpretation of an IVFS

We recall the notion of interval-valued fuzzy set or Φ -fuzzy set introduced by Zadeh [10,11] and Sambuc [6] and so begin by presenting the notation we are going to use:

Let $[I]$ be the set of all closed subintervals of the interval $[0,1]$ and $M = [M_L, M_U] \in [I]$, where M_L and M_U are the lower extreme and the upper extreme, respectively. Given $M \in [I]$, \bar{M} is the interval $[1 - M_U, 1 - M_L]$ and $W = M_U - M_L$ is the width.

An interval-valued fuzzy set (IVFS) A in a universe of discourse X is an expression given by

$$A = \{(x, M_A(x) : x \in X)\}, \tag{1}$$

where the function $M_A : X \rightarrow [I]$ defines the degree of membership of an element x to A .

Let $IVF(X)$ denote all IVFSs on X . For each $A \in IVF(X)$, let $A(x) = [M_{AL}(x), M_{AU}(x)]$, where $M_{AL}(x) \leq M_{AU}(x)$ for any $x \in X$. Then fuzzy set $M_{AL} : X \rightarrow [I]$ and $M_{AU} : X \rightarrow [I]$ are called a lower fuzzy set of A and an upper fuzzy set of A , respectively.

For each $A \in IVF(X)$, we will call the amplitude of membership of the element x in the set A the following expression

$$W_A(x) = M_{AU}(x) - M_{AL}(x) \tag{2}$$

evidently $0 \leq W_A(x) \leq 1$ for all $x \in X$.

If A is a fuzzy set, $M_{AL}(x) = M_{AU}(x)$ for each $x \in X$, i.e. $W_A(x) = 0$. So, we should present the amplitude of membership to handling an IVFS but not a fuzzy set.

To present the geometrical interpretation of the IVFS, we consider a universe X and subset Y in the Euclidean plane with Cartesian coordinates. For a fixed IVFS A , a function f_A from X to Y can be constructed, such that if $x \in X$, then $p = f_A(x) \in Y$, and the point $p \in Y$ has the coordinates $(M_{AL}(x), M_{AU}(x))$ for which $0 \leq M_{AL}(x) \leq M_{AU}(x) \leq 1$.

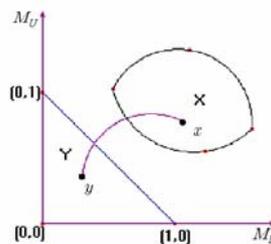


Fig. 1 A geometrical interpretation of an IVFS

The above geometrical interpretation can be used as an example when considering a situation at beginning of negotiations - cf. Fig. 2 (applications of interval-valued fuzzy sets for group decision making, negotiations and other real situations are presented in real life). Each expert i is represented as a point having coordinates $(M_L(i), M_U(i), W(i))$. Expert A: $(1,0,0)$ - fully accepts a discussed idea. Expert B: $(0,0,0)$ - fully rejects it. The experts placed on the segment AB fixed their points of view (their amplitude margins equal zero for segment AB, so each expert is convinced that the extent $M_L(i)$ is equal to the extent $M_U(i)$; segment AB represents a fuzzy set). Expert C: $(0,1,1)$ is absolutely hesitant, i.e. undecided - he or she is the most open to the influence of the arguments presented.

A line parallel to a segment AB describes a set of experts with the same level of amplitude. For example, in Fig. 2, two sets presented with amplitudes equal to $W(m)$ and $W(n)$, where $0 < W(m) < W(n) < 1$. In other words, Fig. 2 (the triangle ABD) is an orthogonal projection of the real situation (the triangle ABC) presented in Fig. 3. An element of an IVFS has three coordinates $(M_L(i), M_U(i), W(i))$ (cf. (2)), hence the most natural representation of an IVFS is to draw a cuboid (with edge length equal to 1), and because of (2), the triangle ABC (Fig. 3) represents an IVFS. As before (Fig. 2), the triangle ABD is the orthogonal projection of the triangle ABC

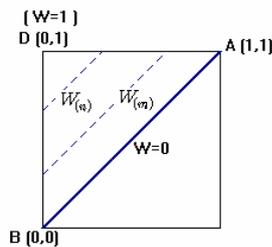


Fig. 2 An orthogonal projection of (three-dimension) representation (triangle ABD in Fig. 3) of an IVFS

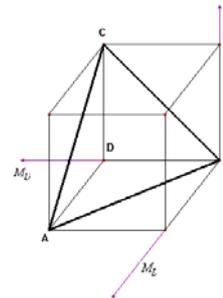


Fig. 3 A three-dimension the real representation of an IVFS

This representation of an IVFS will be another point of departure for considering the distances and entropy for IVFSs.

3. Distances between IVFSs

Burillo and Bustince [3] suggested some methods for measuring distances between sets that are generalizations of the well known Hamming distance, Euclidean distance and their normalized forms as follows: For any two IVFSs $A = \{(x_i, M_A(x_i)) : x \in X\}$ and $B = \{(x_i, M_B(x_i)) : x \in X\}$ of the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$,

- the Hamming distance $d'(A, B)$:

$$d'(A, B) = \frac{1}{2} \sum_{i=1}^n [|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)|], \quad (3)$$

- the normalized Hamming distance $l'(A, B)$:

$$l'(A, B) = \frac{1}{2n} \sum_{i=1}^n [|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)|], \quad (4)$$

- the Euclidean distance $e'(A, B)$:

$$e'(A, B) = \left\{ \frac{1}{2} \sum_{i=1}^n [(M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2] \right\}^{1/2}, \quad (5)$$

- the normalized Euclidean distance $q'(A, B)$:

$$q'(A, B) = \left\{ \frac{1}{2n} \sum_{i=1}^n [(M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2] \right\}^{1/2}. \quad (6)$$

Now we modify these distances. So, we propose to take into account the three parameters characterization of IVFSs: the lower degree of membership $M_{AL}(x)$, the upper degree of membership $M_{AU}(x)$ and the amplitude margin $W_A(x)$.

- the Hamming distance $d''(A, B)$:

$$d''(A, B) = \frac{1}{2} \sum_{i=1}^n [|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)| + |W_A(x_i) - W_B(x_i)|], \quad (7)$$

- the normalized Hamming distance $l''(A, B)$:

$$l''(A, B) = \frac{1}{2n} \sum_{i=1}^n [|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)| + |W_A(x_i) - W_B(x_i)|], \quad (8)$$

- the Euclidean distance $e''(A, B)$:

$$e''(A, B) = \left\{ \frac{1}{2} \sum_{i=1}^n [(M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2 + (W_A(x_i) - W_B(x_i))^2] \right\}^{1/2}, \quad (9)$$

- the normalized Euclidean distance $q''(A, B)$:

$$q''(A, B) = \left\{ \frac{1}{2n} \sum_{i=1}^n [(M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2 + (W_A(x_i) - W_B(x_i))^2] \right\}^{1/2}. \quad (10)$$

We claim that our approach ensures that the distances for fuzzy sets and IVFSs can be easily compared since it reflects distances in three dimensional space, while distances due to Burillo and Bustince [3] are orthogonal projections of the real distances. Obviously, these distances satisfy the conditions of the metric.

Example 1. Let us consider following IVFSs A, B, C, G and E of $X = \{x\}$:

$$A = \{(x, [1,1])\}, \quad B = \{(x, [0,0])\}, \quad C = \{(x, [0,1])\}, \quad G = \{(x, [\frac{1}{2}, \frac{1}{2}])\}, \quad E = \{(x, [\frac{1}{4}, \frac{3}{4}])\}$$

and their geometrical interpretation is presented in Fig. 4. We calculate the Euclidean distances between the above IVFSs using the formula (5) (i.e. omitting the third parameter):

$$e'(A, C) = e'(B, C) = \sqrt{\frac{1}{2}}, \quad e'(A, B) = 1, \quad e'(A, G) = e'(B, G) = e'(C, G) = \frac{1}{2}, \quad e'(E, G) = \frac{1}{4}.$$

These results are not of the sort that one can agree with. As Fig. 3, the triangle ABC (Fig. 4) has all edges equal to $\sqrt{2}$. So we should obtain $e'(A, C) = e'(B, C) = e'(A, B)$. But Burillo and Bustince's results show only that $e'(A, C) = e'(B, C)$, but $e'(A, C) \neq e'(A, B)$ and $e'(B, C) \neq e'(A, B)$. Also $e'(E, G)$, i.e., it is half of the height of triangle ABC multiplied by $\frac{1}{2}$, is not the value we want. Let us calculate the same Euclidean distances using (9). Then we obtain

$$e''(A, C) = e''(B, C) = e''(A, B) = 1, \quad e''(A, G) = e''(B, G) = \frac{1}{2}, \quad e''(E, G) = \frac{3}{4}, \quad e''(C, G) = \frac{3}{2}.$$

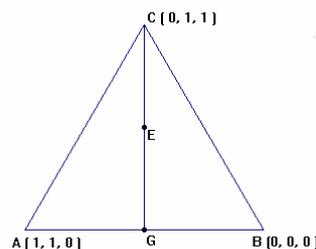


Fig. 4 A geometrical interpretation of the IVFS in Example 1.

Formula (9) gives the results we expect, i.e. $e''(A, C) = e''(B, C) = e''(A, B) = 2e''(A, G) = 2e''(B, G)$ and $e''(E, G)$ is equal to half of the height of the triangle with all edges equal to $\sqrt{2}$ multiplied by $\frac{1}{2}$, i.e. $\frac{\sqrt{3}}{4}$.

Besides Hamming distance and Euclidean distance, some distances based on the Hausdorff metric are also used in the fuzzy sets theory. For any two subsets U and V of a Banach space X the Hausdorff metric is defined by $d_H(U, V) = \max\{\sup_{u \in U} \inf_{v \in V} |u - v|, \sup_{v \in V} \inf_{u \in U} |u - v|\}$. If $X = \mathbf{R}$ and $U = [u_1, u_2]$ and $V = [v_1, v_2]$ are intervals, then above equation reduces to

$$d_H(U, V) = \max\{|u_1 - v_1|, |u_2 - v_2|\}.$$

Grzegorzewski [5] suggested how to measure the distance between IVFSs on arbitrary finite universe of discourse utilizing the Hausdorff metric. For any two IVFSs $A = \{(x_i, M_A(x_i)) : x \in X\}$ and $B = \{(x_i, M_B(x_i)) : x \in X\}$ of $X = \{x_1, x_2, \dots, x_n\}$,

- the Hamming distance $d_h(A, B)$:

$$d_h(A, B) = \sum_{i=1}^n \max\{|M_{AL}(x_i) - M_{BL}(x_i)|, |M_{AU}(x_i) - M_{BU}(x_i)|\}, \quad (11)$$

- the normalized Hamming distance $l_h(A, B)$:

$$l_h(A, B) = \frac{1}{n} \sum_{i=1}^n \max\{|M_{AL}(x_i) - M_{BL}(x_i)|, |M_{AU}(x_i) - M_{BU}(x_i)|\}, \quad (12)$$

- the Euclidean distance $e_h(A, B)$:

$$e_h(A, B) = \left\{ \frac{1}{2} \sum_{i=1}^n \max\{(M_{AL}(x_i) - M_{BL}(x_i))^2, (M_{AU}(x_i) - M_{BU}(x_i))^2\} \right\}^{1/2}, \quad (13)$$

- the normalized Euclidean distance $q_h(A, B)$:

$$q_h(A, B) = \left\{ \frac{1}{n} \sum_{i=1}^n \max\{(M_{AL}(x_i) - M_{BL}(x_i))^2, (M_{AU}(x_i) - M_{BU}(x_i))^2\} \right\}^{1/2}. \quad (14)$$

Now, we give some results on elementary properties of these concepts.

Proposition 1. Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite universe of discourse. Then function $d_h, l_h, e_h, q_h : IVF(X) \rightarrow \mathbf{R}^+ \cup \{0\}$ given by (11)-(14), respectively, are metrics.

Proposition 2. For any two IVFSs $A = \{(x_i, M_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i)) : x_i \in X\}$ of the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, the following inequalities hold:

$$d_h(A, B) \leq n, \quad (15)$$

$$l_h(A, B) \leq 1, \quad (16)$$

$$e_h(A, B) \leq \sqrt{n}, \quad (17)$$

$$q_h(A, B) \leq 1. \quad (18)$$

Proposition 3. For any two IVFSs $A = \{(x_i, M_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i)) : x_i \in X\}$ of the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, the following inequalities hold:

$$d'(A, B) \leq d_h(A, B) \leq d''(A, B), \quad (19)$$

$$l'(A, B) \leq l_h(A, B) \leq l''(A, B), \quad (20)$$

$$e'(A, B) \leq e_h(A, B) \leq e''(A, B), \quad (21)$$

$$q'(A, B) \leq q_h(A, B) \leq q''(A, B). \quad (22)$$

4. Distances between Interval-valued fuzzy sets

As a generalization of the notion of IFSSs, Atanassov and Gargov [2] introduced the notion of interval-valued intuitionistic fuzzy sets in the spirit of IVFSs.

An interval-valued intuitionistic fuzzy set (IVIFS) A on a universe X is an object having the form

$$A = \{(x, M_A(x), N_A(x)) : x \in X\}, \quad (23)$$

where $M_A : X \rightarrow [I]$ and $N_A : X \rightarrow [I]$ denote membership function and non-membership function of A , respectively, and satisfy $0 \leq M_{AU}(x) + N_{AU}(x) \leq 1$ for any $x \in X$.

Let $IVIF(X)$ denote all interval-valued fuzzy sets on X . Even though we can represent a fuzzy set in an intuitionistic-type representation, we can not always represent any IVFS in interval-valued intuitionistic-type representation. For example, let A be an IVFS in $X = \{x\}$ such that $M_A = [\frac{1}{4}, \frac{1}{2}]$. Then $(M_A, \overline{M}_A) = ([\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}])$ is not an IVIFS because $M_{AU} + \overline{M}_{AU} = \frac{1}{2} + \frac{3}{4} > 1$. However, if an IVFS A satisfy the condition $M_{AU} + \overline{M}_{AU} \leq 1$, i.e. $M_{AU} + 1 - M_{AL} \leq 1$, then the IVFS A can represent interval-valued intuitionistic-type representation (M_A, \overline{M}_A) .

We extend the Burillo and Bustince's distances to IVIFSs as (3)-(6). For any two IVIFSs $A = \{(x_i, M_A(x_i), N_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i), N_B(x_i)) : x_i \in X\}$ of the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$,

- the Hamming distance $d_1'(A, B)$:

$$d_1'(A, B) = \frac{1}{4} \sum_{i=1}^n [|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)| + |N_{AL}(x_i) - N_{BL}(x_i)| + |N_{AU}(x_i) - N_{BU}(x_i)|] \quad (24)$$

- the normalized Hamming distance $l_1'(A, B)$:

$$l_1'(A, B) = \frac{1}{4n} \sum_{i=1}^n [|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)| + |N_{AL}(x_i) - N_{BL}(x_i)| + |N_{AU}(x_i) - N_{BU}(x_i)|] \quad (25)$$

- the Euclidean distance $e_1'(A, B)$:

$$e_1'(A, B) = \{ \frac{1}{4} \sum_{i=1}^n [(M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2 + (N_{AL}(x_i) - N_{BL}(x_i))^2 + (N_{AU}(x_i) - N_{BU}(x_i))^2] \}^{\frac{1}{2}}, \quad (26)$$

- the normalized Euclidean distance $q_1'(A, B)$:

$$q_1'(A, B) = \{ \frac{1}{4n} \sum_{i=1}^n [(M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2 + (N_{AL}(x_i) - N_{BL}(x_i))^2 + (N_{AU}(x_i) - N_{BU}(x_i))^2] \}^{\frac{1}{2}}. \quad (27)$$

Now, we consider the amplitude margin to modify these distances as (7)-(10).

- the Hamming distance $d_1''(A, B)$:

$$d_1''(A, B) = \frac{1}{4} \sum_{i=1}^n [|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)| + |N_{AL}(x_i) - N_{BL}(x_i)| + |N_{AU}(x_i) - N_{BU}(x_i)| + |W_{M_A}(x_i) - W_{M_B}(x_i)| + |W_{N_A}(x_i) - W_{N_B}(x_i)|] \quad (28)$$

- the normalized Hamming distance $l_1''(A, B)$:

$$l_1''(A, B) = \frac{1}{4n} \sum_{i=1}^n [|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)| + |N_{AL}(x_i) - N_{BL}(x_i)| + |N_{AU}(x_i) - N_{BU}(x_i)| + |W_{M_A}(x_i) - W_{M_B}(x_i)| + |W_{N_A}(x_i) - W_{N_B}(x_i)|] \quad (29)$$

- the Euclidean distance $e_1''(A, B)$:

$$e_1''(A, B) = \{ \frac{1}{4} \sum_{i=1}^n [(M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2 + (N_{AL}(x_i) - N_{BL}(x_i))^2 + (N_{AU}(x_i) - N_{BU}(x_i))^2 + (W_{M_A}(x_i) - W_{M_B}(x_i))^2 + (W_{N_A}(x_i) - W_{N_B}(x_i))^2] \}^{\frac{1}{2}}, \quad (30)$$

- the normalized Euclidean distance $q_1''(A, B)$:

$$q_1''(A, B) = \{ \frac{1}{4n} \sum_{i=1}^n [(M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2 + (N_{AL}(x_i) - N_{BL}(x_i))^2 + (N_{AU}(x_i) - N_{BU}(x_i))^2 + (W_{M_A}(x_i) - W_{M_B}(x_i))^2 + (W_{N_A}(x_i) - W_{N_B}(x_i))^2] \}^{\frac{1}{2}}. \quad (31)$$

Clearly these distances satisfy the conditions of the metric (cf. [7]). Finally, we extend the Grzegorzewski's distances to IVIFSs as (11)-(14). For any two IVIFSs $A = \{(x_i, M_A(x_i), N_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i), N_B(x_i)) : x_i \in X\}$ of the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$,

- the Hamming distance $d_H(A, B)$:

$$d_H(A, B) = \frac{1}{2} \sum_{i=1}^n [\max\{|M_{AL}(x_i) - M_{BL}(x_i)|, |M_{AU}(x_i) - M_{BU}(x_i)|\} + \max\{|N_{AL}(x_i) - N_{BL}(x_i)|, |N_{AU}(x_i) - N_{BU}(x_i)|\}] \quad (32)$$

- the normalized Hamming distance $l_H(A, B)$:

$$l_H(A, B) = \frac{1}{2n} \sum_{i=1}^n [\max\{|M_{AL}(x_i) - M_{BL}(x_i)|, |M_{AU}(x_i) - M_{BU}(x_i)|\} + \max\{|N_{AL}(x_i) - N_{BL}(x_i)|, |N_{AU}(x_i) - N_{BU}(x_i)|\}] \quad (33)$$

- the Euclidean distance $e_H(A, B)$:

$$e_H(A, B) = \left\{ \frac{1}{2} \sum_{i=1}^n [(\max\{|M_{AL}(x_i) - M_{BL}(x_i)|, |M_{AU}(x_i) - M_{BU}(x_i)|\})^2 + (\max\{|N_{AL}(x_i) - N_{BL}(x_i)|, |N_{AU}(x_i) - N_{BU}(x_i)|\})^2] \right\}^{\frac{1}{2}} \quad (34)$$

- the normalized Euclidean distance $q_H(A, B)$:

$$q_H(A, B) = \left\{ \frac{1}{2n} \sum_{i=1}^n [(\max\{|M_{AL}(x_i) - M_{BL}(x_i)|, |M_{AU}(x_i) - M_{BU}(x_i)|\})^2 + (\max\{|N_{AL}(x_i) - N_{BL}(x_i)|, |N_{AU}(x_i) - N_{BU}(x_i)|\})^2] \right\}^{\frac{1}{2}} \quad (35)$$

Proposition 4. Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite universe of discourse. Then function $d_H, l_H, e_H, q_H : IVF(X) \rightarrow \mathbf{R}^+ \cup \{0\}$ given by (32)-(35), respectively, are metrics.

Proposition 5. For any two IVIFSs $A = \{(x_i, M_A(x_i), N_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i), N_B(x_i)) : x_i \in X\}$ of the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, the following inequalities hold:

$$d_H(A, B) \leq n, \quad (36)$$

$$l_H(A, B) \leq 1, \quad (37)$$

$$e_H(A, B) \leq \sqrt{n}, \quad (38)$$

$$q_H(A, B) \leq 1. \quad (39)$$

Proposition 6. For any two IVIFSs $A = \{(x_i, M_A(x_i), N_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i), N_B(x_i)) : x_i \in X\}$ of the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, the following inequalities hold:

$$d_1^{\cdot}(A, B) \leq d_H(A, B) \leq d_1^{\cdot\cdot}(A, B), \quad (40)$$

$$l_1^{\cdot}(A, B) \leq l_H(A, B) \leq l_1^{\cdot\cdot}(A, B), \quad (41)$$

$$e_1^{\cdot}(A, B) \leq e_H(A, B) \leq e_1^{\cdot\cdot}(A, B), \quad (42)$$

$$q_1^{\cdot}(A, B) \leq q_H(A, B) \leq q_1^{\cdot\cdot}(A, B). \quad (43)$$

When generalizing any notion it is desirable that the new object should be consistent with the primary one and it should reduce to that primary one in some particular cases. As it was mentioned above each IVFS can be IVIFS under some conditions. Thus it would be desirable that our definitions (24)-(35) should reduce to the Burillo and Bustince's distances (3)-(6), our distances (7)-(10) and Grzegorzewski's distances (11)-(14), respectively, for ordinary IVFSs. One can check easily that

Proposition 7. For any IVIFSs $A, B \in X = \{x_1, x_2, \dots, x_n\}$ such that $A = \{(x_i, M_A(x_i), \overline{M}_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i), \overline{M}_B(x_i)) : x_i \in X\}$, the following equalities hold:

$$d'(A, B) = d_1'(A, B), l'(A, B) = l_1'(A, B), e'(A, B) = e_1'(A, B), q'(A, B) = q_1'(A, B), \quad (44)$$

$$d''(A, B) = d_1''(A, B), l''(A, B) = l_1''(A, B), e''(A, B) = e_1''(A, B), q''(A, B) = q_1''(A, B), \quad (45)$$

$$d_h(A, B) = d_H(A, B), l_h(A, B) = l_H(A, B), e_h(A, B) = e_H(A, B), q_h(A, B) = q_H(A, B). \quad (46)$$

Remark. Since IFSSs and IVFSs are equipollent generalizations of fuzzy sets, our definitions (24)-(35) should also reduce to the Szmidt and Kacprzyk's distances [8] and Grzegorzewski's distances [5], respectively, for ordinary IFSSs.

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