Application of homotopy perturbation method to the Zakharov-Kuznetsov equation

To cite this article: A Barari et al 2008 J. Phys.: Conf. Ser. 96 012082

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Application of Homotopy perturbation method to the Zakharov-Kuznetsov equation

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Abstract. This paper investigates a nonlinear dispersive Zakharov-Kuznetsov ZK (3, 3, 3) equation. This equation governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. The homotopy perturbation method is implemented for solving the equation. The obtained results are compared with those obtained by the Adomian method, revealing that the homotopy perturbation method is very effective, convenient and quite accurate to systems of nonlinear partial differential equations.

1. Introduction

The investigation of the traveling wave solution play an important role in nonlinear science. These solutions may well describe various phenomena in nature, such as vibrations, solitons and propagation with a finite speed. The wave phenomena observed in fluid dynamics, plasma and elastic media. Rosenau and Hyman [1] introduced a class of PDEs:

\[ K(m,n): u_t + a(u^m)_x + (u^n)_{xxx} = 0, \quad m > 0, \quad 1 < n \leq 3, \quad (1) \]

which is a generalization of the korteweg-de vries (KDV) equation. For values of \( m \) and \( n \), the \( K(m,n) \) equation has solitary waves which are compactly supported. For \( m = n \) these are solitary waves or so-called compactons.

Recently, Wazwaz [2] gave the new solitary patterns for the nonlinear dispersive \( K(m,n) \) equations:

\[ u_t - a(u^m)_x + (u^n)_{xxx} = 0, \quad m, n > 1. \quad (2) \]

The new solitary wave special solutions with compact support for the nonlinear dispersive \( K(m,n) \) equations:
are presented by Wazwaz [3]. Ismail and Taha [4] used a finite difference method and a finite element method to investigate the approximate solutions of \( k (2,2) \) and \( k (3,3) \) in Equation (1). The main goal of this paper is to investigate the ZK equation of the form (shortly called ZK \((m,n,k)\)):

\[

u_t + a (u^n)_x + (u^n)_{xxx} = 0, \quad m,n > 1,
\]

(3)

where \( a, b, c \) are arbitrary constants and \( m, n, k \) are integers. This equation governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [5].

In this letter, we apply the homotopy-perturbation method (HPM) to the ZK equation. The HPM deforms a difficult problem into a simple problem which can be easily solved.

2. Basic idea of Homotopy-perturbation method

Linear and Nonlinear phenomena are of fundamental importance in various fields of science and engineering. Most models of real-life problems are still very difficult to solve. Therefore, approximate analytical solutions such as Homotopy-perturbation method (HPM) [6-13] were introduced. This method is the most effective and convenient ones for both linear and nonlinear equations.

Perturbation method is based on assuming a small parameter. The majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all and the approximate solutions obtained by the perturbation methods, in most cases, are valid only for small values of the small parameter.

Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exists. Thus, it is essential to check the validity of the approximations numerically and/or experimentally. To overcome these difficulties, HPM have been proposed recently.

To explain this method, let us consider the following function:

\[

A(u) - f(r) = 0, \quad r \in \Omega
\]

(5)

with the boundary conditions of:

\[

B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma,
\]

(6)

where \( A, B, f(r) \) and \( \Gamma \) are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain \( \Omega \), respectively.

Generally speaking the operator \( A \) can be divided into a linear part \( L \) and a nonlinear part \( N (u) \). Equation (5) can therefore, be written as:

\[

L(u) + N(u) - f(r) = 0
\]

(7)

By the homotopy technique, we construct a homotopy

\[ v(r, p) : \Omega \times [0,1] \rightarrow R \]

Which satisfies
\[ H(v, p) = (1 - p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \]
\[ p \in [0, 1], r \in \Omega, \] (8)

Or
\[ H(v, p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0, \] (9)

where \( p \in [0, 1] \) is an embedding parameter, while \( u_0 \) is an initial approximation of Equation (5), which satisfies the boundary conditions. Obviously, from Equations (8) and (9) we will have:
\[ H(v, 0) = L(v) - L(u_0) = 0, \] (10)
\[ H(v, 1) = A(v) - f(r) = 0, \] (11)

The changing process of \( p \) from zero to unity is just that of \( v(r, p) \) from \( u_0 \) to \( u(r) \) . In topology, this is called deformation, while \( L(v) - L(u_0) \) and \( A(v) - f(r) \) are called homotopy.

According to the HPM, we can first use the embedding parameter \( p \) as a “small parameter”, and assume that the solutions of Equations (8) and (9) can be written as a power series in \( p \):
\[ v = v_0 + pv_1 + p^2v_2 + ..., \] (12)

Setting \( p = 1 \) yields in the approximate solution of Equation (5) to:
\[ u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ..., \] (13)

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantage. The series (13) is convergent for most cases. However, the convergent rate depends on the nonlinear operator \( A(v) \). Moreover, He made the following suggestions [13]:

- The second derivative of \( N(v) \) with respect to \( v \) must be small because the parameter may be relatively large, i.e. \( p \to 1 \).
- The norm of \( L^{-1}\frac{\partial N}{\partial v} \) must be smaller than one so that the series converges.

3. Example
We consider the ZK (3, 3, 3) equation with initial condition of:
\[ u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyy} = 0, \]
\[ u(x, y, 0) = \frac{3}{2} \lambda \sinh \left[ \frac{1}{6} (x + y) \right], \quad (14) \]

where \( \lambda \) is an arbitrary constant. We assume \( \lambda = 1 \).

### 3.1. Application of Homotopy-perturbation method

We consider the following process after separating the linear and nonlinear parts of the equation. A homotopy-perturbation method can be constructed as follows:

\[
H(v, p) = (1 - p)\frac{\partial}{\partial t}v(x, y, t) - \frac{\partial}{\partial t}u_0(x, y, t) + \\
p\left( \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(v(x, y, t)^3) + 2 \frac{\partial^3}{\partial x^3}v(x, y, t)^3 + 2 \frac{\partial^3}{\partial y \partial x}v(x, y, t)^3 \right) = 0, 
\]

(15)

Substituting \( v = v_0 + pv_1 + \ldots \) in to Equation (15) and rearranging the resultant equation based on powers of p-terms, one has:

\[
p^0 : \frac{\partial}{\partial t}v_0(x, y, t) = 0, 
\]

(16)

\[
p^1 : 12\left( \frac{\partial}{\partial x}v_0(x, y, t)(\frac{\partial}{\partial y}v_0(x, y, t))^2 + 6(v_0(x, y, t))^2(\frac{\partial^3}{\partial x^3}v_0(x, y, t)) + 6(v_0(x, y, t))^2 \right) \\
(\frac{\partial^3}{\partial y^2 \partial x}v_0(x, y, t)) + 12(\frac{\partial}{\partial x}v_0(x, y, t))^3 + 24v_0(x, y, t)(\frac{\partial}{\partial y}v_0(x, y, t))(\frac{\partial^2}{\partial y \partial x}v_0(x, y, t)) + \\
12v_0(x, y, t)(\frac{\partial^2}{\partial y^2}v_0(x, y, t))(\frac{\partial}{\partial x}v_0(x, y, t)) + (\frac{\partial}{\partial t}v_1(x, y, t)) + 36v_0(x, y, t)(\frac{\partial}{\partial x}v_0(x, y, t)) + \\
(\frac{\partial^2}{\partial x^2}v_0(x, y, t)) + 3(v_0(x, y, t))^2(\frac{\partial^2}{\partial x}v_0(x, y, t)) = 0, 
\]

(17)
\[ p^2 : 24v_0(x,y,t)\left(\frac{\partial}{\partial y}v_0(x,y,t)\right)\left(\frac{\partial^2}{\partial y^2}v_0(x,y,t)\right) + 36v_0(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t) \]

\[ + 3v_0(x,y,t)\frac{\partial}{\partial t}v_0(x,y,t) + 36v_0(x,y,t)v_1(x,y,t) \]

\[ + 36v_0(x,y,t)v_1(x,y,t) + 12v_0(x,y,t)v_1(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t) + \]

\[ 12v_0(x,y,t)v_1(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t) + \]

\[ 36v_0(x,y,t)v_1(x,y,t) + 24v_0(x,y,t)v_1(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t) + \]

\[ 12v_0(x,y,t)v_1(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t) + 24v_0(x,y,t)v_1(x,y,t)\frac{\partial}{\partial x}v_0(x,y,t) + \]

\[ 12v_0(x,y,t)v_1(x,y,t)\frac{\partial}{\partial y}v_0(x,y,t) = 0, \quad (18) \]

with the following conditions:

\[ v_0(x,y,0) = \frac{3}{2} \lambda \sinh \left[ \frac{1}{6} (x + y) \right], \]

\[ v_i(x,y,0) = 0, \quad i = 1, 2, \ldots \quad (19) \]

With the effective initial approximation for \( v_0 \) from the conditions (19) and solutions of Equations (16, 17 and 18) may be written as follows:

\[ v_0(x,y,t) = \frac{3}{2} \sinh \left( 1.6666 \times 10^{-1} (x + y) \right), \]

\[ v_1(x,y,t) = 4.6875 \times 10^{-1} t \cosh (1.6666 \times 10^{-1} x + 1.6666 \times 10^{-1} y) \]

\[ -8.4375 \times 10^{-1} t \cosh (5 \times 10^{-1} x + 5 \times 10^{-1} y), \]

\[ v_2(x,y,t) = 1.25 \times 10^{-11} t^2 \begin{cases} -145546875063 \sinh (0.5 x + 0.5 y) \\ +16406250015 \sinh (1.6666 \times 10^{-1} x + 1.66 \times 10^{-1} y) \\ +179296875050 \sinh (8.333 \times 10^{-1} x + 8.33 \times 10^{-1} y) \end{cases} \]
In the same manner, the rest of components were obtained using the maple package. According to the HPM, we can conclude that:

\[
\begin{align*}
    u(x, t) = \lim_{p \to 1} v(x, t) &= v_0(x, t) + v_1(x, t) + \ldots,
\end{align*}
\]

(23)

Therefore, substituting the values of \( v_0(x, t), v_1(x, t) \) and \( v_2(x, t) \) from Equations (20, 21, 22) in to Equation (23) yields:

\[
\begin{align*}
    u(x, y, t) &= 1.5\sinh\left(1.6666 \times 10^{-1}(x + y)\right) + 4.6875\times 10^{-1}t\cosh(1.6666 \times 10^{-1}x + 1.6666 \times 10^{-1}y) - 8.4375\times 10^{-1}t\cosh(5\times 10^{-1}x + 5\times 10^{-1}y) + 1.25 \times 10^{-11}t^2 \\
    &\quad - 145546875063\sinh(0.5x + 0.5y) \\
    &\quad + 16406250015\sinh(1.6666 \times 10^{-1}x + 1.66 \times 10^{-1}y) \\
    &\quad + 179296875050\sinh(8.3333 \times 10^{-1}x + 8.33 \times 10^{-1}y),
\end{align*}
\]

(24)

It can be seen good agreement between results of HPM and ADM as shown in Figures (1), (2) and Tables (1), (2) [14].

**Table 1.** Comparison between results of different solutions at \( y = 0.1, t = 0.001 \)

<table>
<thead>
<tr>
<th>x</th>
<th>ADM</th>
<th>HPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0247511230</td>
<td>0.0246252670</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0372538287</td>
<td>0.0371268281</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0497591215</td>
<td>0.0496304978</td>
</tr>
<tr>
<td>0.15</td>
<td>0.0622678698</td>
<td>0.0621371434</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0747809423</td>
<td>0.0746476322</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0872992080</td>
<td>0.0871628320</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0998235361</td>
<td>0.0996836096</td>
</tr>
<tr>
<td>0.35</td>
<td>0.1123547965</td>
<td>0.1122108326</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1248938594</td>
<td>0.1247453685</td>
</tr>
<tr>
<td>0.45</td>
<td>0.1374415954</td>
<td>0.1372880853</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1499988761</td>
<td>0.1498398507</td>
</tr>
</tbody>
</table>
Table 2. Comparison between results of different solutions at $y = 0.1$, $t = 0.01$

<table>
<thead>
<tr>
<th>$x$</th>
<th>ADM</th>
<th>HPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.02250084376</td>
<td>0.02125119915</td>
</tr>
<tr>
<td>0.05</td>
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<td>0.03374657278</td>
</tr>
<tr>
<td>0.1</td>
<td>0.04750793906</td>
<td>0.04623962824</td>
</tr>
<tr>
<td>0.15</td>
<td>0.06001600128</td>
<td>0.05873123598</td>
</tr>
<tr>
<td>0.2</td>
<td>0.07252823132</td>
<td>0.07122226361</td>
</tr>
<tr>
<td>0.25</td>
<td>0.08504549806</td>
<td>0.08371357580</td>
</tr>
<tr>
<td>0.3</td>
<td>0.09758687076</td>
<td>0.09620603449</td>
</tr>
<tr>
<td>0.35</td>
<td>0.1100986191</td>
<td>0.1087004988</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1226362133</td>
<td>0.1211978254</td>
</tr>
<tr>
<td>0.45</td>
<td>0.1351823239</td>
<td>0.1336988681</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1477378221</td>
<td>0.1462044784</td>
</tr>
</tbody>
</table>

Figure 1. results of the Adomian decomposition method at $t=0.01$

Figure 2. results of the Homotopy perturbation method at $t=0.01$

4. Conclusions
The homotopy perturbation method is employed successfully to study problem of ZK (3,3,3). The results obtained here were compared with the ADM. The results revealed that the homotopy perturbation method is powerful mathematical tool for solutions of nonlinear differential equations in terms of accuracy and efficiency.

References