Application of variational iteration method to non-homogeneous non-linear dissipative wave equations

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Application of variational iteration method to non-homogeneous non-linear dissipative wave equations

M Rostamian\textsuperscript{a}, A Barari\textsuperscript{b}, D D Ganji\textsuperscript{a\textdagger}
\textsuperscript{a} Department of Mechanical Engineering, University of Mazandaran, Babol, Iran
\textsuperscript{b} Department of Civil Engineering, University of Mazandaran, Babol, Iran
Email: ddg_davood@yahoo.com

Abstract. In this paper, the solution of the non-linear non-homogeneous dissipative wave equation is obtained by means of the variational iteration method. This equation describes the propagation of a wave, and it arises in a wide variety of physical problems. The results reveal that the variational iteration method is very effective, convenient and quite accurate to systems of nonlinear partial differential equations.

1. Introduction
In this paper we will apply the variational iteration method (VIM) \cite{1-9} to the non-linear non-homogeneous dissipative wave equation.

The non-linear non-homogeneous dissipative wave equation is \cite{10}:

\begin{equation}
 u_{tt} = u_{xx} - (\frac{\partial}{\partial t})\left(uu_x\right) + 2e^{-t}\sin x - 2e^{-t}\sin x\cos x,
\end{equation}

(1)

This equation describes the propagation of a wave, and it arises in a wide variety of physical problems. This problem includes a vibrating string, vibrating membrane, longitudinal vibrations of an elastic rod or beam, shallow water waves, acoustic problems for the velocity potential for a fluid flow through which sound can be transmitted, transmission of electric signals along a cable, shock waves, chemical exchange processes in chromatography, sediment transport in rivers and waves in plasmas, and both electric and magnetic fields in the absence of charge and dielectric \cite{11}.

Variational iteration method (VIM) \cite{1-9} is the most effective and convenient for both linear and nonlinear equations. The VIM is to construct correction functional using general Lagrange multipliers identified optimally via the variational theory.

2. Basic idea of Variational iteration method
To clarify the basic ideas of VIM, we consider the following differential equation:

\begin{equation}
 Lu + Nu = g(t),
\end{equation}

(2)

\textsuperscript{1} To whom any correspondence should be addressed.
where $L$ is a linear operator, $N$ is a nonlinear operator and $g(t)$ is a homogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left( Lu_n(\tau) + Nu_n(\tau) - g(\tau) \right) d\tau,$$

where $\lambda$ is a general lagrangian multiplier which can be identified optimally via the variational theory. The subscript $n$ indicates the $n$th approximation and $u_n$ is considered as a restricted variation, i.e., $\delta \tilde{u}_n = 0$.

3. Example

We consider Equation (1), The initial and boundary conditions posed are:

$$u(x, 0) = \sin x, \quad u_t(x, 0) = -\sin x, \quad u(0, t) = u(\pi, t) = 0,$$

(4)

Exact solution of this equation is:

$$u(x,t) = e^{-t} \sin x,$$

(5)

3.1. Application of variational iteration method

To solve Equations (1), (4) using VIM, we have the correction functional as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left( \frac{\partial^2 u_n(x,\tau)}{\partial \tau^2} + \frac{\partial^2 u_n(x,\tau)}{\partial x^2} \right) + \left( \frac{\partial u_n(x,\tau)}{\partial t} \right) u_n(x,\tau) - \lambda \left( \frac{\partial^2 u_n(x,\tau)}{\partial x \partial \tau} \right) d\tau,$$

(6)

Its stationary conditions can be obtained as follows:

$$1 - \lambda^* \bigg|_{\tau=t} = 0,$$

$$\lambda \bigg|_{\tau=t} = 0,$$

$$\lambda^* \bigg|_{\tau=t} = 0,$$

(7)

The lagrangian multiplier can there be identified as:

$$\lambda = \tau - t,$$

(8)

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t) \left( \frac{\partial^2 u_n(x,\tau)}{\partial x^2} + \left( \frac{\partial u_n(x,\tau)}{\partial \tau} \right) \right) + \left( \frac{\partial^2 u_n(x,\tau)}{\partial x \partial \tau} \right) u_n(x,\tau) - 2e^{-t} \sin x + 2e^{-t} \sin x \cos x$$

(9)
Now we start with an arbitrary initial approximation:

\[ u_0(x, t) = x(1 - t), \quad (10) \]

Using the above variational formula (9), we have:

\[
u_t = \frac{\partial^2 u_0(x, \tau)}{\partial \tau^2} - \frac{\partial^2 u_0(x, \tau)}{\partial x^2} + \left( \frac{\partial u_0(x, \tau)}{\partial \tau} \right)
+ \left( \frac{\partial^2 u_0(x, \tau)}{\partial x \partial \tau} \right) u_0(x, \tau) - \int_0^t (\tau - t) \left( \frac{\partial u_0(x, \tau)}{\partial x} + \frac{\partial^2 u_0(x, \tau)}{\partial x \partial \tau} \right) u_0(x, \tau) - d\tau,
\]

(11)

Substituting Equation (10) into Equation (11) and after simplification, we have:

\[
u_t = x - xt - \frac{1}{3} xt^3 + t^2 x + t^2 e^{-t} \sin x - t^2 e^{-t} \sin x \cos x,
\]

(12)

In the same way, we obtain \( u_2(x, t) \) as follows

\[
u_2 = \frac{2}{3} t^2 e^{-t} \sin x - 2t \sin x - \frac{1}{6} t \sin x - \frac{1}{3} t^2 x - \frac{11}{3} e^{-t} x - \frac{2}{3} e^{-t} t^3 x + \frac{3}{4} e^{-2t} \sin x - \frac{3}{2} \sin x \cos x + 4e^{-t} x \cos x + 13t \sin x \cos x - t + \frac{2}{3} t^2 x + 8t \sin x \cos x - 20e^{-t} x - \frac{1}{3} x^5 - 10e^{-t} x + 21x - 40x \cos^2 x - 20e^{-tx} \cos x - 20e^{-t} x \cos x - \frac{1}{3} e^{-t} t^3 \sin x + 20x \cos x - 24t \sin x \cos x + \frac{3}{2} e^{-t} \sin x + \frac{2}{3} e^{-t}
\]

(13)

\[
\cos x - \frac{11}{3} t^2 e^{-t} \sin x \cos x + \frac{11}{3} e^{-t} t^3 \sin x \cos x + \frac{2}{3} e^{-t} t^4 \cos x + \frac{3}{4} e^{-t} t^4 \sin x \cos x + \frac{1}{3} e^{-t} t^5 \sin x \cos x
\]

\[
+ \frac{4}{3} e^{-t} t^4 \sin x \cos x - 44 \sin x \cos x - 10e^{-t} x \cos x + 20e^{-t} x \sin x + 44 \sin x \cos x
\]

\[
\cos x - 26e^{-t} \sin x + \frac{1}{3} x^6 - x^5 - 10r^2 e^{-i} \sin x - \frac{11}{3} t^3 \sin x e^{-t} \sin x + \frac{101}{4} \sin x + \frac{22}{3} \sin x \cos x + \frac{20e^{-t} t^4 \sin x} {\cos^3 x} + 3e^{-2t} \sin x \cos x - \frac{9}{2} t^2 e^{-t^2} \sin x
\]

\[
\cos x + \frac{3}{4} x^6 e^{-2t} \sin x + e^{-t} t^4 \sin x \cos x + \frac{2}{3} e^{-t} t^3 \sin x \cos x + \frac{2}{3} e^{-t} \cos x - \frac{9}{2} e^{-t} \sin x
\]

\[
\sin x + \frac{9}{4} e^{-2t} \sin x \cos^2 x + \frac{9}{2} e^{-t} \sin x \cos^2 x + \frac{1}{2} t^2 e^{-2t} \sin x + \frac{3}{2} e^{-2t} \sin x + \frac{3}{2} \sin x
\]

and so on; in the same way the rest of the components of the iteration formula can be obtained.

Figures (1), (2) and Table (1) show comparison between results of VIM and exact solution.
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Figure 1: Results of exact solution
4. Conclusions
The variational iteration method has been successfully used to study non-linear non-homogeneous dissipative wave equation. This equation describes the propagation of a wave, and it arises in a wide variety of physical problems.

The results obtained here were compared with the exact solutions. The results revealed that variational iteration method is powerful mathematical tool for solutions of nonlinear partial differential equations in terms of accuracy and efficiency.

References