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Application of the homotopy perturbation method for solving second-order non-linear wave equations

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Abstract. In this paper, the solution of a non-linear wave equation is obtained by means of the homotopy perturbation method. This equation describes the propagation of a wave and it arises in a wide variety of physical problems. The results reveal that the homotopy perturbation method is very effective, convenient and quite accurate to systems of nonlinear partial differential equations.

1. Introduction

The second-order non-linear wave equation \cite{1}:

\[
\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} - u^2(x,t) + u(x,t) \frac{\partial u(x,t)}{\partial x},
\]

(1)

This equation describes the propagation of a wave (or disturbance), and it arises in a wide variety of physical problems. Some of these problems include a vibrating string, vibrating membrane, longitudinal vibrations of an elastic rod or beam, acoustic problems for the velocity potential for a fluid flow through which sound can be transmitted, transmission of electric signals along a cable, shock waves, chemical exchange processes in chromatography, sediment transport in rivers and waves in plasmas, and both electric and magnetic fields in the absence of charge and dielectric \cite{2}. One of the most important applications of wave equation is shallow water waves.

In the last few decades, various investigations of wave-induced seabed liquefaction have been carried out. Although the protection of marine structures has been extensively studied in recent years, understanding of their interaction with waves and the seabed is far from complete. Damage of marine structures still occurs from time to time, with two general failure modes evident. The first mode is that of structural failure, caused by wave forces acting on and damaging the structure itself. The second mode is that of foundation failure, caused by liquefaction or erosion of the seabed in the vicinity of the structure, resulting in collapse of the structure as a whole. Numerous research studies have been carried out on this topic in the last decade \cite{3, 4}. Figure 1 illustrates the change of a seabed due to wave action (Figure (1a): without liquefaction, Figure (1b): with liquefaction).

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Bjerrum [5] was possibly the first author that considered wave-induced liquefaction occurring in saturated seabed sediments. Recently, Rahman [6] established the relationship between liquefaction and characteristics of wave and soil. He concluded that liquefaction potential increases in degree of saturation and with an increase of wave period. Jeng [7] examined a wave-induced liquefied state for several different cases, together with Zen and Yamasaki’s [8] field data. He found that no liquefaction occurs in a saturated seabed, except in very shallow water, for large waves and seabed with very low permeability.

After introducing applications of wave equation, it can be known importance of investigating wave equation. In this paper, we apply the homotopy perturbation method (HPM) [9-16] to second-order non-linear wave equation. The HPM deforms a difficult problem in to a simple problem which can be easily solved.

2. Basic idea of homotopy-perturbation method

To explain this method, let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Omega$$

(2)
With the boundary conditions of:

\[ B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma, \quad (3) \]

where \( A, B, f(r) \) and \( \Gamma \) are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain \( \Omega \), respectively. Generally speaking the operator \( A \) can be divided into a linear part \( L \) and a nonlinear part \( N(u) \). Equation (2) can therefore, be written as:

\[ L(u) + N(u) - f(r) = 0, \quad (4) \]

By the homotopy technique, we construct a homotopy \( v(r, p) : \Omega \times [0, 1] \rightarrow R \) which satisfies

\[ H(v, p) = (1 - p) \left[ L(v) - L(u_0) \right] + p \left[ A(v) - f(r) \right] = 0, \quad p \in [0, 1], r \in \Omega, \quad (5) \]

Or

\[ H(v, p) = L(v) - L(u_0) + pL(u_0) + p \left[ N(v) - f(r) \right] = 0, \quad (6) \]

Where \( p \in [0, 1] \) is an embedding parameter, while \( u_0 \) is an initial approximation of Eq. (2), which satisfies the boundary conditions. Obviously, from Equations (5) and (6) we will have:

\[ H(v, 0) = L(v) - L(u_0) = 0, \quad (7) \]

\[ H(v, 1) = A(v) - f(r) = 0, \quad (8) \]

The changing process of \( p \) from zero to unity is just that of \( v(r, p) \) from \( u_0 \) to \( u(r) \). In topology, this is called deformation, while \( L(v) - L(u_0) \) and \( A(v) - f(r) \) are called homotopy.

According to the HPM, we can first use the embedding parameter \( p \) as a “small parameter”, and assume that the solutions of Equations (5) and (6) can be written as a power series in \( p \):

\[ v = v_0 + pv_1 + p^2v_2 + ..., \quad (9) \]

Setting \( p = 1 \) yields in the approximate solution of Eq. (2) to:

\[ u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + ..., \quad (10) \]

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantage. The series (10) is convergent for most cases. However, the convergent rate depends on the nonlinear operator \( A(v) \). Moreover, He made the following suggestions [16]:
• The second derivative of \( N(v) \) with respect to \( v \) must be small because the parameter may be relatively large, i.e. \( p \to 1 \).
• The norm of \( L^{-1} \frac{\partial N}{\partial v} \) must be smaller than one so that the series converges.

3. Application of Homotopy-perturbation method
We consider Equation (1) with initial and boundary conditions as follows:

\[
\begin{align*}
    u(x,0) &= u_0(x,0) = e^x, \quad 0 < x < 1, \\
    u(0,t) &= e^t, t > 0,
\end{align*}
\]

with the exact solution:

\[
    u(x,t) = e^{x+t},
\]

To solve Equation (1) by means of HPM, we consider the following process after separating the linear and nonlinear parts of the equation.
A homotopy can be constructed as follows:

\[
    H(v, p) = (1-p)(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v_0}{\partial t^2}) + p(\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^2 v(x,t)}{\partial x^2}v(x,t) + v(x,t)^2 - v(x,t)(\frac{\partial}{\partial x} v(x,t))) = 0, \quad (13)
\]

Substituting \( v = v_0 + pv_1 + ... \) in to Equation (13) and rearranging the resultant equation based on powers of \( p \)-terms, one has:

\[
    p^0 : \frac{\partial^2}{\partial t^2} v_0(x,t) = 0, \quad (14)
\]

\[
    p^1 : \frac{\partial^2}{\partial t^2} v_1(x,t) + (\frac{\partial^2}{\partial t^2} v_0(x,t)) - v_0(x,t)(\frac{\partial}{\partial x} v_0(x,t)) + v_0(x,t)^2 = 0, \quad (15)
\]

\[
    p^2 : \frac{\partial^2}{\partial t^2} v_2(x,t) + 2v_0(x,t)v_1(x,t) - v_1(x,t)(\frac{\partial}{\partial x} v_0(x,t)) - v_0(x,t)(\frac{\partial}{\partial x} v_1(x,t)) - \frac{\partial^2}{\partial x^2} v_1(x,t) = 0, \quad (16)
\]

with the following conditions:

\[
    v_0(x,0) = \frac{\partial}{\partial t} v_0(x,0) = e^x, \quad v_0(0,t) = e^t, \quad 0 < x < 1, t > 0
\]

\[
    v_i(x,0) = \frac{\partial}{\partial t} v_i(x,0) = 0, \quad v_i(0,t) = 0 \quad i = 1,2,........
\]
With the effective initial approximation for $v_0$ from the conditions (17) and solutions of Equations (14), (15) and (16) may be written as follows:

$$v_0(x,t) = e^x(t+1),$$  \hspace{1cm} (18)

$$v_1(x,t) = \frac{1}{6} e^x t^3 + \frac{1}{2} e^x t^2,$$  \hspace{1cm} (19)

$$v_2(x,t) = \frac{1}{6} e^x \left( \frac{1}{20} t^5 + \frac{1}{4} t^4 \right),$$  \hspace{1cm} (20)

In the same manner, the rest of components were obtained using the maple package.

According to the HPM, we can conclude that:

$$u(x,t) = \lim_{p \to 1} v(x,t) = v_0(x,t) + v_1(x,t) + \ldots,$$  \hspace{1cm} (21)

Therefore, substituting the values of $v_0(x,t)$, $v_1(x,t)$ and $v_2(x,t)$ from Equations (18), (19) and (20) in to Equation (21) yields:

$$u(x,t) = e^x \left( 1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \frac{1}{24} t^4 + \frac{1}{120} t^5 + \ldots \right) = e^x e^t = e^{x+t},$$  \hspace{1cm} (22)

As it can be seen, using HPM in solving this equation leads to the exact solution (Figure 2).

![Figure 2](image.png)

**Figure 2.** A 3D scheme of the obtained results of $U(x,t)$ by HPM and exact solution

4. Conclusions
The homotopy perturbation method has been successfully used to study second-order non-linear wave equation. This equation describes the propagation of a wave (disturbance), and it arises in a wide variety of physical problems.

The results obtained here were compared with the exact solutions. The results revealed that the homotopy perturbation method is a powerful mathematical tool for solutions of nonlinear differential equations in terms of accuracy and efficiency.

References