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Nonlinear Vibration of Thick Stiff Fabric with Small Flexural Stiffness

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Abstract. Dynamic behaviour of fabric is very complex during weaving, dyeing and finishing processes. Thick stiff fabric vibration has great influence not only on the fabric itself but also on the performance of machine. The theoretic analysis for the nonlinear free vibration of thick stiff fabric with small flexural stiffness is put forward in the paper. The nonlinear partial differential equation is derived by applying the flexible thin plate theory, and then transformed into nonlinear ordinary differential equation by the Galerkin method. The approximate analytical solution is obtained by the homotopy perturbation method.

1. Introduction
During weaving, dyeing and finishing processes, fabric is of course a processing object but sometimes is also a driving component. There is a coupling between fabric and related mechanisms. In general, thin fabric is very flexible, and its flexural stiffness can be neglected. But, when the thick stiff fabric is processed, we must consider its flexural stiffness. Usually, people mostly concentrate on the mechanics research of fabric under stoical or small stress situation, like fabric draping, diagonal stretch and so on. Few literatures dealing with the dynamic characteristics of fabric have been published at home and abroad. This paper makes the preliminary exploration in this aspect. The authors regard the fabric as a flexible plate with small flexural stiffness, and consider its stress–strain as non-linear relations. The nonlinear partial differential equation is derived by appraising the flexible thin plate theory.

Fig. 1 Tensile force of fabric

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Recently, many concepts in mathematics and mechanics have been successfully applied to model the processing of the textile industry. For example, the conservation laws in fluid mechanics have been applied to establish a quasi-static model for two-strand yarn spinning, the homotopy perturbation method has been applied to solve the nonlinear problems arising in yarn spinning processing. Similarly, the homotopy perturbation method has also been applied to obtain the approximate analytical solution of many nonlinear problems.


We consider the fabric subjected to a uniform tension in the boundary, and ignore the gravity and damping force, as shown in Fig. 1. Supposing \( w(x, y, t) \) is the transverse displacement in the \( z \) direction. \( t \) is the time, \( \rho \) is the fabric volume density, \( N_1 \) and \( N_2 \) are the tension in the unit length in the \( x \) and \( y \) directions, respectively, \( a \) and \( b \) are the fabric spans in the \( x \) and \( y \) directions, respectively, and \( h \) is the thickness of fabric.

We consider the following dynamics equations of the fabric element based on flexible thin plate theory

\[
\begin{align*}
\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \\
\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \\
\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} - Q_x &= 0 \\
\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y &= 0 \\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} - \rho h \frac{\partial^2 w}{\partial t^2} &= 0
\end{align*}
\]

where \( N_x, N_y \) are the tension functions of the fabric element per unit length, while \( N_{xy} \) is its shear force per unit length in the \( x \) and \( y \) directions, \( M_x, M_y, M_{xy} \) are its bending moments functions per unit length, and \( Q_x, Q_y \) are its shear forces functions per unit length in the \( z \) direction.

\[
\begin{align*}
N_x &= \int_{x \min}^{x \max} \sigma_x \, dz; & N_y &= \int_{y \min}^{y \max} \sigma_y \, dz; & N_{xy} &= \int_{y \min}^{y \max} \tau_{xy} \, dz; \\
M_x &= \int_{x \min}^{x \max} \sigma_x z \, dz; & M_y &= \int_{x \min}^{x \max} \sigma_y z \, dz; & M_{xy} &= \int_{x \min}^{x \max} \tau_{xy} z \, dz.
\end{align*}
\]

In Eqs. (1), the first two are independent, so we have

\[
\begin{align*}
N_x &= N_1 \\
N_y &= N_2 \\
N_{xy} &= 0
\end{align*}
\]

The following equation can be obtained by substituting Eq. (3) into Eq. (1),

\[
\frac{\partial^3 M}{\partial x^3} + 2 \frac{\partial^3 M_{xy}}{\partial x \partial y} + \frac{\partial^3 M}{\partial y^3} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} - \rho h \frac{\partial^2 w}{\partial t^2} = 0.
\]

Consider the nonlinear stress-strain elastic relations: \( \sigma = E(\varepsilon - Bh \varepsilon^3) \), where \( E \) is Young’s modulus of the fabric and \( B \) is a new material constant, then we can have the stresses as follows.
\[ \sigma_x = \frac{E}{1-\mu^2} \left( \left[ \varepsilon_x - B \varepsilon_y \right] + \mu \left[ \varepsilon_y - B \varepsilon_x \right] \right) \]
\[ \sigma_y = \frac{E}{1-\mu^2} \left( \left[ \varepsilon_y - B \varepsilon_x \right] + \mu \left[ \varepsilon_x - B \varepsilon_y \right] \right) \]
\[ \tau_{xy} = G \left[ \gamma_{xy} - B \gamma_{yx} \right] \]

And the displacements are
\[
\begin{align*}
    u(x, y, z, t) &= -z \frac{\partial w}{\partial x} \\
    v(x, y, z, t) &= -z \frac{\partial w}{\partial y} \\
    w(x, y, z, t) &= w(x, y, t)
\end{align*}
\]

Therefore the strain can be obtained,
\[
\begin{align*}
    \varepsilon_x &= -z \frac{\partial^2 w}{\partial x^2} \\
    \varepsilon_y &= -z \frac{\partial^2 w}{\partial y^2} \\
    \gamma_{xy} &= -2z \frac{\partial^2 w}{\partial x \partial y}
\end{align*}
\]

Substituting Eq. (7) into Eq. (5), and then substituting into Eq. (2), we obtain:
\[
\begin{align*}
    M_x &= -D \left( \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) - B \left( \left( \frac{\partial^2 w}{\partial x \partial y} \right)^3 + \mu \left( \frac{\partial^2 w}{\partial x \partial y} \right)^3 \right) \\
    M_y &= -D \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) - B \left( \left( \frac{\partial^2 w}{\partial x \partial y} \right)^3 + \mu \left( \frac{\partial^2 w}{\partial x \partial y} \right)^3 \right) \\
    M_{xy} &= -D (1 - \mu) \left( \frac{\partial^2 w}{\partial x \partial y} - B \left( \frac{\partial^2 w}{\partial x \partial y} \right)^3 \right)
\end{align*}
\]

where \( D = \frac{E h^3}{12(1-\mu)} \) is the flexural rigidity of fabric, and \( B = \frac{3h^2}{20} \).

The following equation can be obtained by substituting Eq. (8) into Eq. (4)
\[
\begin{align*}
    &D \nabla^4 w - 3BD \left\{ \left( \frac{\partial^4 w}{\partial x^4} \right) + \frac{\partial^2 w}{\partial x^2} \left( \frac{\partial^2 w}{\partial x^2} \right) + \frac{\partial^2 w}{\partial y^2} \left( \frac{\partial^2 w}{\partial y^2} \right) + 2 \left( \frac{\partial^4 w}{\partial x^4} \right) \left( \frac{\partial^4 w}{\partial y^4} \right) \right\} + 2 \left( \frac{\partial^4 w}{\partial x^4} \right) \left( \frac{\partial^4 w}{\partial y^4} \right) + 2 \left( \frac{\partial^4 w}{\partial x^4} \right) \left( \frac{\partial^4 w}{\partial y^4} \right) + 4(1-\mu) \left( \frac{\partial^2 w}{\partial x \partial y} \right) \left( \frac{\partial^2 w}{\partial x \partial y} \right) + \left( \frac{\partial^2 w}{\partial x \partial y} \right) \left( \frac{\partial^2 w}{\partial x \partial y} \right) \\
    &+ 2(1-\mu) \left( \frac{\partial^2 w}{\partial x \partial y} \right) \left( \frac{\partial^2 w}{\partial x \partial y} \right) \left( \frac{\partial^2 w}{\partial x \partial y} \right) + N \frac{\partial^2 w}{\partial x^2} - N \frac{\partial^2 w}{\partial y^2} + \rho h \frac{\partial^2 w}{\partial t^2} \\
    &= L(w) - N \frac{\partial^2 w}{\partial x^2} - N \frac{\partial^2 w}{\partial y^2} + \rho h \frac{\partial^2 w}{\partial t^2} = 0
\end{align*}
\]

where \( \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \), \( \Delta = \left( \frac{\partial^2}{\partial x^2} \right) + \left( \frac{\partial^2}{\partial y^2} \right) \).

Boundary conditions are
\[
x = 0, \quad a : w = \frac{\partial w}{\partial x} = 0,
\]
3. Solution of Nonlinear Partial Differential Equation

Eq. (9) is a nonlinear partial differential equation, which can be transformed into nonlinear ordinary differential eq. by the Galerkin method. The general solution of Eq. (9) is assumed to be in the form

\[ w(x, y, t) = T(t)w_0(x, y) \]  

where \( w_0(x, y) \) are the deflection functions which satisfy all the boundary conditions.

Let \( w_0(x, y) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \), applying the Galerkin method to Eq. (9), we obtain

\[ \int \int \left[ L(w) - N_1 \frac{\partial^2 w}{\partial x^2} - N_2 \frac{\partial^2 w}{\partial y^2} + \rho k \frac{\partial^2 w}{\partial t^2} \right] w_0 \, ds = 0 \]  

Substituting Eq. (10) into Eq. (11), we can obtain the following nonlinear ordinary differential equation:

\[ T^* + \lambda T + \varepsilon T^3 = 0 \]  

where

\[ \lambda = \frac{1}{\rho h} \left\{ D \left[ \left( \frac{\pi}{a} \right)^4 + \left( \frac{\pi}{b} \right)^4 \right]^2 + N_1 \left( \frac{\pi}{a} \right)^2 + N_2 \left( \frac{\pi}{b} \right)^2 \right\}, \]

\[ \varepsilon = \frac{9 \beta D}{16 \rho h} \left[ \left( \frac{\pi}{a} \right)^4 + \left( \frac{\pi}{b} \right)^4 + \mu \left( \frac{\pi}{a} \right)^2 \left( \frac{\pi}{b} \right)^2 \right] + 2 \left( 1 - \mu \right) \left( \frac{\pi}{a} \right)^4 \left( \frac{\pi}{b} \right)^4 \]

Here, we must point out that \( \lambda \) is relating to the natural frequency of corresponding linear problem obtained by applying the Galerkin method, i.e. \( \lambda = \omega_0^2 \), while \( \varepsilon \) is a smaller constant.

Setting the amplitude \( T(0) = A \), and the initial velocity \( T'(0) = 0 \), we construct a homotopy in the form of \( T'' + \lambda T + \varepsilon p T^3 = 0 \)

\[ T = T_0 + pT_1 + \cdots, \]

\[ \lambda = \omega^2 + p\omega_1 + \cdots. \]

Substituting them into Eq. (13), and simplify it, we have

\[ T_0'' + \omega^2 T_0 = 0, \quad T_0(0) = A, T_0'(0) = 0, \]  

\[ T_1'' + \omega^2 T_1 + \varepsilon T_0^3 + \omega_1 T_0 = 0, \quad T_1(0) = 0, \quad T_1'(0) = 0. \]

The solution of Eq. (14) is

\[ T_0 = A \cos(\omega t). \]

Substituting it into Eq. (15), we obtain
\[ T_1'' + \omega^2 T_1 + A(\omega t + \frac{3}{4} \varepsilon A^2) \cos(\omega t) + \frac{1}{4} \varepsilon A^3 \cos(3\omega t) = 0. \] (6)

No secular term in \( T_1 \) requires that

\[ \omega_1 = -\frac{3}{4} \varepsilon A^2. \]

So

\[ T_1'' + \omega^2 T_1 + \frac{1}{4} \varepsilon A^3 \cos(3\omega t) = 0. \] (7)

According to initial condition \( T_1(0) = 0, \ T_1'(0) = 0 \), the general solution of Eq. (7) is

\[ T_i = \frac{\varepsilon A^3}{32 \omega^2} \left[ \cos(3\omega t) - \cos(\omega t) \right]. \] (8)

Thus we obtain the first-order approximate solution is

\[
\begin{cases}
T = A \cos(\omega t) + \frac{\varepsilon A^3}{32 \omega^2} \left[ \cos(3\omega t) - \cos(\omega t) \right] \\
\omega^2 = \omega_0^2 + \frac{3}{4} \varepsilon A^2
\end{cases}
\] (9)

Obviously, the nonlinear free vibration natural frequency of the fabric with small stiffness depends on the linear free vibration frequency \( \omega_0 \) and also on the amplitude \( A \). As shown in Fig. 2, since generally, the small parameter \( \varepsilon \) is positive, therefore the greater the amplitude \( A \) is, the higher will be the natural frequency \( \omega \).

![Fig.2 The relationship between the frequency and amplitude](image)

4. Conclusion

After considering the nonlinear stress-strain relations, the dynamic equation of fabric with the small stiffness will be the nonlinear partial differential equation. It is very hard to find its analytical solution. We may transform it into nonlinear ordinary differential eq. by the Galerkin method and then the approximate analytical solution can be obtained by the homotopy method. From the approximate analytical solution, we can obtain that when the amplitude is bigger, the natural frequency will be higher.

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