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General decay in a viscoelastic system

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Abstract. In this paper we consider a viscoelastic system in a bounded domain, and establish a general decay result.

1. Introduction
In this paper we consider the following problem

$$
\begin{aligned}
&u_{tt} + A u (x, t) - \int_0^t g (t - \tau) A u (x, \tau) d \tau = 0, \quad \text{in } \Omega \times (0, \infty) \\
u (x, t) = 0, \quad x \in \partial \Omega, \quad t \geq 0 \\
u (x, 0) = u_0 (x), \quad u_1 (x, 0) = u_1 (x), \quad x \in \Omega,
\end{aligned}
$$

(1.1)

where $A = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( a_{ij} (x) \frac{\partial}{\partial x_i} \right), \Omega$ is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\partial \Omega, a_{ij}$ are bounded functions satisfying conditions to be specified later and $g$ is a positive nonincreasing function defined on $\mathbb{R}^+$. For $a_{ij} = \delta_{ij},$ Cavalcanti et al. [1] studied (1.1) in the presence of a localized damping cooperating with the dissipation induced by the viscoelastic term and obtained exponential rate of decay. Berrimi et al. [2] improved Cavalcanti’s result by showing that the viscoelastic dissipation alone is enough to stabilize the system. This result has been later extended to a situation, where a source is competing with the viscoelastic dissipation, by Berrimi et al. [3] and Cavalcanti et al. [4]. Also, Cavalcanti et al. [5] have also studied, in a bounded domain, the following equation

$$
\left| u_t \right|^\rho u_{tt} - \Delta u - \Delta u_\alpha + \int_0^t g (t - \tau) \Delta u (\tau) d \tau - \gamma \Delta u_t = 0,
$$

(1.2)

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for $\rho > 0$, and proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. Messaoudi and Tatar [6], [7] studied (1.2), for $\gamma = 0$, and proved exponential and polynomial decay results in the absence, as well as in the presence, of a source term.

In this work we generalize the existing decay result in literature. Precisely, we show that the solution energy decays at a similar rate of decay of the relaxation function, which is not necessarily decaying in a polynomial or exponential fashion.

2. Decay of solutions

Before we state and prove our main result, we make the following assumptions

(A1) $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a differentiable function satisfying

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = \ell > 0$$

(A2) There exists a positive differentiable function $\xi$ such that

$$g'(t) \leq -\xi(t)g(t), \quad t \geq 0, \quad \frac{\xi'(t)}{\xi(t)} \leq k, \quad \xi(t) > 0, \quad \xi'(t) \leq 0, \quad \forall t > 0.$$

(A3) $A$ is symmetric; i.e.

$$a_{ij} = a_{ji}, \quad \forall i, j = 1, 2, \ldots n, \quad \text{a.e. } x \in \Omega.$$

(A4) $A$ is positive definite; i.e. there exists a constant $\alpha_0 > 0$, for which

$$\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \alpha_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega$$

(A5) $A$ is bounded; i.e.

$$|a_{ij}(x)| \leq M, \quad \forall i, j = 1, 2, \ldots n, \text{a.e. } x \in \Omega.$$

We introduce the “modified” energy functional

$$E(t) := \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \left(1 - \int_0^t g(s)ds\right) B(u(t)) + \frac{1}{2} \left(g \circ \nabla u\right)(t),$$

(2.1)

where

$$B(u(t)) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u(t)}{\partial x_i} \frac{\partial u(t)}{\partial x_j} dx,$$

(2.2)

$$(g \circ \nabla u)(t) = \int_0^t g(t-s) B(u(t)-u(s)) ds.$$

(2.3)

We also set

$$F(t) := E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t),$$

(2.4)

where $\varepsilon_1$ and $\varepsilon_2$ are positive constants and
Lemma 2.1. If \( u \) is a solution of (1.1), then the “modified” energy satisfies

\[
E'(t) = \frac{1}{2} (g \circ \nabla u)(t) - \frac{1}{2} g(t) B(u(t)) \leq \frac{1}{2} (g \circ \nabla u)(t) \leq 0.
\]  

Proof. By multiplying equation in (1.1) by \( u \) and integrating over \( \Omega \), using integration by parts, hypotheses (A1)-(A5) and some manipulations as in [3], we obtain (2.6).

By using (A4) and Cauchy-Schwarz and Poincaré’s inequalities, we easily prove

Lemma 2.2. For \( u \in H^1(\Omega) \), we have

\[
\int_{\Omega} \left[ \int_0^t g(t-\tau) \left( u(t) - u(\tau) \right) d\tau \right]^2 dx \leq \frac{(1-\ell)c_p^2}{\alpha_0} \frac{\left(g \circ \nabla u \right)(t)}{\alpha_0},
\]

where \( C_p \) is the Poincaré constant.

Lemma 2.3. For \( \varepsilon_1 \) and \( \varepsilon_2 \) small enough, we have

\[
\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t)
\]

holds for two positive constants \( \alpha_1 \) and \( \alpha_2 \).

Next, we establish some estimates on \( \Psi' \) and \( \chi' \).

Lemma 2.4. Under the assumptions (A1)-(A5), the functional \( \Psi(t) \) satisfies, along solutions of (1.1),

\[
\Psi'(t) \leq \left[ 1 + \frac{k^2 C_p^2}{\ell \alpha_0} \right] \xi(t) \int_{\Omega} u^2 dx - \frac{\ell}{4} \xi(t) B(u(t)) + \gamma_1 \xi(t) \left(g \circ \nabla u \right)(t),
\]

where \( \gamma_1 \) is a constant depending only on \( M, n, \ell, \) and \( \alpha_0 \).

Proof. By using equation (1.1), we easily see that

\[
\Psi'(t) = \xi(t) \int_{\Omega} u^2 dx + \xi'(t) \int_{\Omega} uu_i dx - \xi(t) B(u(t)) + \xi(t) \sum_{i,j=1}^n \frac{\partial u(t)}{\partial x_i} \int_0^t \frac{\partial u(s)}{\partial x_j} \int_{\Omega} g(t-s) \mu_i(x) dx ds
\]

We now estimate the third term in the RHS of (2.9) as follows:
\[ 
\sum_{i,j=1}^{n} \int_{\Omega} \partial u(t) \int_{0}^{t} g(t-s) a_{ij}(x) \frac{\partial u(s)}{\partial x_{i}} \, ds \, ds 
\leq \frac{1}{2} B(u(t)) + \frac{1}{2} B \left( \int_{0}^{t} g(t-s) u(s) \, ds \right) 
\leq \frac{1}{2} B(u(t)) + \frac{1}{2} B \left( \int_{0}^{t} g(t-s)(u(s)-u(t)) \, ds \right) + \frac{1}{2} B \left( \int_{0}^{t} g(t-s)u(t) \, ds \right) 
+ \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) \left( \int_{0}^{t} g(t-s) \left( \frac{\partial u(s)}{\partial x_{i}} - \frac{\partial u(t)}{\partial x_{j}} \right) \, ds \right) \left( \int_{0}^{t} g(t-s) \frac{\partial u(t)}{\partial x_{j}} \, ds \right) \, dx 
\]

(2.10)

We then use Young’s inequality and (A5) to estimate the terms of (2.10). For the second term, we have

\[ 
\frac{1}{2} B \left( \int_{0}^{t} g(t-s)(u(s)-u(t)) \, ds \right) 
\leq \frac{M}{4} \left[ \sum_{i,j=1}^{n} \int_{\Omega} \left( \int_{0}^{t} g(t-s) \left( \frac{\partial u(s)}{\partial x_{i}} - \frac{\partial u(t)}{\partial x_{j}} \right) \, ds \right)^{2} \, dx + \sum_{i,j=1}^{n} \int_{\Omega} \left( \int_{0}^{t} g(t-s) \left( \frac{\partial u(s)}{\partial x_{j}} - \frac{\partial u(t)}{\partial x_{j}} \right) \, ds \right)^{2} \, dx \right] 
\]

By using Cauchy-Schwarz inequality and (A4), we get

\[ 
\sum_{i,j=1}^{n} \int_{\Omega} \left( \int_{0}^{t} g(t-s) \left( \frac{\partial u(s)}{\partial x_{i}} - \frac{\partial u(t)}{\partial x_{j}} \right) \, ds \right)^{2} \, dx 
\leq \frac{n}{\alpha_{0}} \left( 1 - \ell \right)(g \circ \nabla u)(t). 
\]

Therefore, we arrive at

\[ 
\frac{1}{2} B \left( \int_{0}^{t} g(t-s)(u(s)-u(t)) \, ds \right) \leq \frac{nM}{2\alpha_{0}} \left( 1 - \ell \right)(g \circ \nabla u)(t). 
\]

(2.11)

The third term of (2.10) can be handled as follows

\[ 
\frac{1}{2} \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) \left( \int_{0}^{t} g(t-s) \frac{\partial u(t)}{\partial x_{j}} \, ds \right) \left( \int_{0}^{t} g(t-s) \frac{\partial u(t)}{\partial x_{j}} \, ds \right) \, dx 
= \frac{1}{2} \left( \int_{0}^{t} g(t) \, ds \right)^{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(t)}{\partial x_{i}} \frac{\partial u(t)}{\partial x_{j}} \, dx \leq \frac{(1-\ell)^{2}}{2} B(u(t)). 
\]

(2.12)

As for the fourth term of (2.10), similar calculations and using the fact that \( \int_{0}^{t} g(s) \, ds \leq 1-\ell \),
gives, for \( \eta > 0 \),
\[
\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) \left( \int_{0}^{t} g(t-s) \left( \frac{\partial u(s)}{\partial x_j} - \frac{\partial u(t)}{\partial x_j} \right) ds \right) \left( \int_{0}^{t} g(t-s) \frac{\partial u(t)}{\partial x_j} ds \right) dx \\
\leq \frac{nM}{2\eta\alpha_0} \left( 1 - \ell \right) (g \circ B u)(t) + \frac{nM \eta (1 - \ell)^2}{2\alpha_0} B(u(t)).
\]

(2.13)

By inserting (2.11)-(2.13) in (2.10), we get

\[
\sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial u(t)}{\partial x_j} \left( \int_{0}^{t} g(t-s)a_{ij}(x) \frac{\partial u(s)}{\partial x_j} dx ds \right) \leq Mn \left( 1 + \frac{1}{\eta} \right) (1 - \ell) (g \circ \nabla u)(t) \\
+ \left[ \frac{1}{2} + \frac{nM + \alpha_0}{2\alpha_0} (1 - \ell)^2 \right] B(u(t)).
\]

(2.14)

By combining (2.9) and (2.14), using

\[
\int_{\Omega} uu_{dx} \leq \frac{\ell}{4k} B(u(t)) + \frac{kC^2}{\ell\alpha_0} \int_{\Omega} u^2 dx, \quad \alpha > 0,
\]

and choosing \( \eta = \alpha_0 \ell / nM (1 - \ell) \), (2.8) is established.

Similar computations yield the following:

**Lemma 2.5.** Under the assumptions (A1)-(A5), the functional \( \chi(t) \) satisfies, along the solution of (1.1),

\[
\chi(t) \leq \delta_{234} \xi(t) \gamma_{2} B(u(t)) + \frac{\gamma_{3}}{\delta} \xi(g \circ \nabla u)(t) - \frac{\gamma_{4}}{\delta} \xi(g \circ \nabla u)(t) \\
+ \left[ \delta (k + 1) - \int_{0}^{t} g(s) ds \right] \xi(t) \int_{\Omega} u^2 dx, \quad \delta > 0,
\]

(2.15)

where \( \gamma_{2}, \gamma_{3}, \gamma_{4} \) are constants depending only on \( M, n, \ell, k, g(0), C_p, \) and \( \alpha_0 \).

Now we are ready to state and prove our main result.

**Theorem 2.6.** Let \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\) be given Assume that (A1)-(A5) hold. Then, for each \( t_0 > 0 \), there exist strictly positive constants \( K \) and \( \lambda \) such that the solution of (1.1) satisfies

\[
\epsilon(t) \leq Ke^{-\lambda t}, \quad t \geq t_0
\]

(2.16)

**Proof.** Since \( g \) is positive and \( g(0) > 0 \) then for any \( t_0 > 0 \) we have

\[
\int_{0}^{t} g(s) ds \geq \int_{0}^{t_0} g(s) ds = g_0 > 0, \quad \forall t \geq t_0.
\]

(2.17)

By using (2.4), (2.6), (2.8), (2.15) and (2.17), we obtain for \( t \geq t_0 \)
\begin{align}
F'(t) & \leq -\left[ \varepsilon_2 \{ g_0 \delta (1+k) \} - \varepsilon_1 \left( 1 + \frac{k^2 C^2 \ell}{p} \right) \right] \xi(t) \int_\Omega u^2 \, dx \\
& + \left( \frac{1}{2} - \varepsilon_2 \frac{\gamma_4}{\delta} \right) (g \circ \nabla u)(t) - \left[ \frac{\varepsilon_1 \ell}{2} - \varepsilon_2 \delta \gamma_2 \right] \xi(t) B(u(t)) \\
& + \left( \varepsilon_1 \gamma_1 + \varepsilon_2 \frac{\gamma_3}{\delta} \right) \xi(t)(g \circ \nabla u)(t).
\end{align} 

(2.18)

At this point we choose \( \delta \) so small that
\[
g_0 - \delta (1+k) > \frac{1}{2} g_0, \quad \frac{4 \delta \gamma_2}{1 + \frac{k^2 C^2 \ell}{p}} g_0.
\]

Whence \( \delta \) is fixed, the choice of any two positive constants \( \varepsilon_1 \) and \( \varepsilon_2 \) satisfying
\[
\frac{g_0}{4 \left( 1 + \frac{k^2 C^2 \ell}{p} \right)} \varepsilon_2 < \varepsilon_1 < \frac{g_0}{2 \left( 1 + \frac{k^2 C^2 \ell}{p} \right)} \varepsilon_2
\]

will make
\[
k_1 = \varepsilon_2 \left[ g_0 - \delta (1+k) \right] - \varepsilon_1 \left( 1 + \frac{k^2 C^2 \ell}{p} \right) > 0,
\]
\[
k_2 = \frac{\varepsilon_1 \ell}{4} - \varepsilon_2 \gamma_2 > 0.
\]

We then pick \( \varepsilon_1 \) and \( \varepsilon_2 \) so small that (2.7) and (2.19) remain valid and, further,
\[
k_2 > 0.
\]

Hence, by using the fact that \( \xi \) is nonincreasing, we obtain
\[
\left( \frac{1}{2} - \varepsilon_2 \frac{\gamma_4}{\delta} \right) (g \circ \nabla u)(t) + \left( \varepsilon_1 \gamma_1 + \varepsilon_2 \frac{\gamma_3}{\delta} \right) \xi(t)(g \circ \nabla u)(t)
\]
\[
\leq -\left( \frac{1}{2} - \varepsilon_2 \frac{\gamma_4}{\delta} \right) \int_\Omega \int_0^t \xi(t - \tau) g(t - \tau) B(u(t) - u(s)) ds \, dx
\]
\[
+ \left( \varepsilon_1 \gamma_1 + \varepsilon_2 \frac{\gamma_3}{\delta} \right) \xi(t)(g \circ \nabla u)(t) \leq k_3 \xi(t)(g \circ \nabla u)(t).
\]

(2.20)

Therefore, by using (2.1), (2.7) and (2.18) we arrive at
\[
F'(t) \leq -\beta_1 \xi(t) e(t) \leq -\beta_1 \alpha_1 \xi(t) F(t), \quad \forall t \geq t_0,
\]

(2.21)
A simple integration of (2.21) leads to

\[ F(t) \leq F(t_0) e^{-\beta \alpha_1 \int_0^t \xi(s) ds}, \quad \forall t \geq t_0. \]  

(2.22)

Thus (2.7), (2.22) yield

\[ \varepsilon(t) \leq \alpha_2 F(t_0) e^{-\beta \alpha_1 \int_0^t \xi(s) ds} = Ke^{-\beta \int_0^t \xi(s) ds}, \quad \forall t \geq t_0. \]  

(2.23)

This completes the proof.

**Remark 3.1.** This result generalizes and improves the results of [1-4]. In particular, it allows some relaxation functions which satisfy \( g' \leq -ag^\rho, 1 \leq \rho < 2 \) instead of \( 1 \leq \rho < 3/2 \).

**Remark 3.2.** Note that the exponential and the polynomial decay estimates are only particular cases of (2.23). More precisely, we obtain exponential decay for \( \xi(t) \equiv a(1+t)^{-1} \), where \( a > 0 \) is a constant.

**Remark 3.3.** Estimate (2.23) is also true for \( t \in [0,t_0] \) by virtue of continuity and boundedness of \( \varepsilon(t) \) and \( \zeta(t) \).

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**References**