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Exp-function method for solving Fisher’s equation

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Abstract. There are many methods to solve Fisher’s equation, but each method can only lead to a special solution. In this paper, a new method, namely the exp-function method, is employed to solve the Fisher’s equation. The obtained result includes all solutions in open literature as special cases, and the generalized solution with some free parameters might imply some fascinating meanings hidden in the Fisher’s equation.

1. Introduction
The nonlinear reaction–diffusion equation
\[ u_t = Du_{xx} + mu(1-u) \]  \hspace{1cm} (1)
was first introduced by Fisher as a model for the propagation of a mutant gene [1]. It has wide application in the fields of logistic population growth [2,3], flame propagation [4], neurophysiology [5], autocatalytic chemical reactions [6], branching Brownian motion processes [7], and nuclear reactor theory [8].

In chemical media the function \( u(x,t) \) is the concentration of the reactant. \( D \) represents its diffusion coefficient, and the ositive constant \( m \) specifies the rate of chemical reaction. In media of other natures, \( u \) might be temperature or electric potential, \( D \) might be the thermal conductivity or specific electrical conductivity. The medium described by Eq. (1) is often referred to as a bistable medium because it has two homogeneous stationary states, \( u = 0 \) and \( u = 1 \). A kink-like travelling wave solution of Eq. (1) describes a constant-velocity front of transition from one homogeneous state to another.

Various powerful methods have been used to solve the equation: the inverse scattering transfer [9], the Hirota method [10], Lamb’s ansatz [11,12], Adomian decomposition method [16], etc. [13]. The problem of obtaining solutions for systems including dissipative losses, e.g., reaction–diffusion systems, turned out to be complex. Most of the above-mentioned methods do not work [14]. The usual way of treating the problem is by perturbation theory or numerical investigation.

Writing \( t^* = mt, \ x^* = \left( \frac{m}{D} \right)^{\frac{1}{2}} x \), and dropping the star, Eq. (1) becomes

\[ u_t = \left( \frac{m}{D} \right)^{\frac{1}{2}} u_{x^*x^*} + \left( \frac{m}{D} \right)^{\frac{1}{2}} mu(1-u) \]
In the spatial homogeneous situation the steady states are $u = 0$ and $u = 1$, which are respectively unstable and stable. This suggests that we should look for travelling wave front solutions to Eq. (2).

2. Basic idea of Exp-function method([15])

We first unite the independent variables $x$ and $t$ into one wave variable $\eta = kx + \omega t$ to carry out a PDE in two independent variables

$$ P(u,u_x,u_{xx},u_{xxx},\ldots) = 0, $$

into an ODE

$$ Q(u,u',u'',\ldots) = 0 $$

The Exp-function method is based on the assumption that traveling wave solutions can be expressed in the following form[1]:

$$ u(\eta) = \sum_{n=-c}^{d} a_n \exp(n\eta) - \sum_{n=-p}^{q} b_m \exp(m\eta)$$

$$ = a_c \exp(c\eta) + \cdots + a_d \exp(-d\eta) - b_p \exp(p\eta) + \cdots + b_q \exp(-q\eta) $$

where $c$, $d$, $p$ and $q$ are positive integers which are unknown to be further determined, $a_n$ and $b_m$ are unknown constants. To determine the values of $c$ and $p$, we balance the linear term of highest order in Eq.(4) with the highest order nonlinear term. Similarly to determine the values of $d$ and $q$, we balance the linear term of lowest order in Eq.(4) with the lowest order nonlinear term.

3. Application to Fisher’s equation

Using the wave variable $\eta = kx + \omega t$ convert the Fisher’s equation (2) to the ODE

$$ -\omega u' + k^2 u'' + u(1-u) = 0, \tag{6} $$

The highest linear term $u''$ is now given by

$$ u'' = \frac{c_1 \exp[3p + c] \eta + \cdots}{c_2 \exp[4p\eta] + \cdots}, \tag{7} $$

and

$$ u^2 = \frac{c_1 \exp[2c\eta] + \cdots}{c_2 \exp[2p\eta] + \cdots} = \frac{c_3 \exp[(2c + 2p)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots}. \tag{8} $$

Balancing the highest order of Exp-function in Eqs.(7) and (8), we have $3p + c = 4c$, and this gives $p = c$. Using the same method, we can also obtain that $q = d$.

For simplicity, we set $p = c = 1$ and $q = d = 1$, so Eq. (4) reduce to

$$ u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \tag{9} $$

Substituting Eq. (9) into Eq.(6), and by the help of Mathematics, we have
\[-\frac{1}{A} \left[ C_1 e^{3\eta} + C_2 e^{2\eta} + C_3 e^{\eta} + C_0 + C_{-1} e^{-\eta} + C_{-2} e^{-2\eta} + C_{-3} e^{-3\eta} \right] = 0, \]  

where

\[ A = \left( e^\eta + b_0 + b_{-1} e^{-\eta} \right)^3 \]

\[ C_1 = -a_i + a_i^2 \]

\[ C_2 = -a_0 - k^2 a_0 - \omega a_0 + 2a_o a_i - 2a_i b_0 + k^2 a_i b_0 + \omega a_i b_0 + a_i^2 b_0 \]

\[ C_3 = -a_{-1} - 4k^2 a_{-1} - 2\omega a_{-1} + a_0^2 + 2a_o a_{-1} - 2a_i a_{-1} + 4k^2 a_i b_{-1} + 2\omega a_i b_{-1} + a_i^2 b_{-1} - 2a_i b_0 + k^2 a_i b_0 - \omega a_i b_0 + 2a_o a_i b_0 - a_i b_0^2 - k^2 a_i b_0^2 + \omega a_i b_0^2 \]

\[ C_0 = 2a_o a_0 - 2a_o a_i - 6k^2 a_o b_{-1} + 2a_o a_i b_{-1} - 2a_i b_0 - 3k^2 a_i b_0 - 3\omega a_i b_0 + a_i^2 b_0 + 2a_o a_i b_0 - 2a_i b_{-1} - 3k^2 a_i b_{-1} + 5\omega a_i b_{-1} - a_i b_{-1}^2 - \omega a_i b_{-1}^2 \]

\[ C_{-1} = a_{-1}^2 - 2a_o b_{-1} + 4k^2 a_o b_{-1} - 2\omega a_o b_{-1} + a_0^2 b_{-1} + 2a_o a_i b_{-1} - 2a_i b_0 + k^2 a_i b_0 + \omega a_o b_{-1} b_0 - a_i b_{-1}^2 - k^2 a_i b_{-1}^2 - \omega a_i b_{-1}^2 \]

\[ C_{-2} = 2a_o a_{-1} b_{-1} - a_0 b_{-1}^2 - k^2 a_o b_{-1}^2 + \omega a_o b_{-1}^2 + a_i b_{-1} b_0 - 2a_i b_{-1} b_0 + k^2 a_i b_{-1} b_0 - \omega a_i b_{-1} b_0 \]

\[ C_{-3} = -a_o^2 b_{-1} - a_o b_{-1}^2 \]

Solving this system (10.1-7) by using Mathematica, we obtain the following results

**Case 1:**

\[ k = -\frac{1}{\sqrt{6}}, \quad \omega = \frac{5}{6}, \quad a_i = \frac{b_0^2}{4}, \quad a_o = 0, \quad a_{-1} = 0, \quad b_{-1} = \frac{b_0^2}{4}, \]

**Case 2:**

\[ k = \frac{1}{\sqrt{6}}, \quad \omega = \frac{5}{6}, \quad a_i = \frac{b_0^2}{4}, \quad a_o = 0, \quad a_{-1} = 0, \quad b_{-1} = \frac{b_0^2}{4}, \]

**Case 3:**

\[ k = \frac{1}{\sqrt{6}}, \quad \omega = -\frac{5}{6}, \quad a_i = 0, \quad a_o = 0, \quad a_{-1} = 1, \quad b_{-1} = \frac{b_0^2}{4}, \]

**Case 4:**

\[ k = \frac{1}{\sqrt{6}}, \quad \omega = -\frac{5}{6}, \quad a_i = 0, \quad a_o = 0, \quad a_{-1} = 1, \quad b_{-1} = \frac{b_0^2}{4}. \]

where \( b_0 \) is a free parameter.

For the case 1, inserting (12) into (9) and yielding the following solution of (2).
\[ u_t(x,t) = \frac{b_0^2}{\left(2e^{\frac{-1}{6}x} + b_0\right)^2}, \quad (16) \]

For the case 2, case 3 and case 4, we get:
\[ u_2(x,t) = \frac{b_0^2}{\left(2e^{\frac{-1}{6}x} + b_0\right)^2}, \quad u_3(x,t) = \frac{4e^{\frac{-1}{6}x} - \frac{5}{6}}{\left(2e^{\frac{-1}{6}x} + b_0\right)^2}, \quad u_4(x,t) = \frac{4e^{\frac{-1}{6}x} - \frac{5}{6}}{\left(2e^{\frac{-1}{6}x} + b_0\right)^2} \]
\[ (17) \]

To compare our results with those obtained in ([3]), if we set \( b_0 = 2 \), Eq. (16) becomes
\[ u_1(x,t) = \frac{1}{\left(1 + e^{\frac{-5}{6}}\right)^2} = \frac{1}{2} \left(1 - \tanh\left[\frac{1}{2\sqrt{6}}(x - \frac{5}{\sqrt{6}}t)\right]\right)^2 \]
\[ (18) \]
which is the kink solution obtained by the tanh-coth method in [3]. If we set \( b_0 = -2 \), Eq. (16) becomes
\[ u_2(x,t) = \frac{1}{\left(1 - e^{\frac{-5}{6}}\right)^2} = \frac{1}{2} \left(1 - \coth\left[\frac{1}{2\sqrt{6}}(x - \frac{5}{\sqrt{6}}t)\right]\right)^2 \]
\[ (19) \]
which is the travelling solution obtained by the tanh-coth method in [17].

**Conclusion**

After the Exp-function method is illustrated by He and Wu([15]), Some nonlinear equations are solved exactly, such as KdV equation, Dodd-Bullough-Mikhalov equation, Burger’s equation, Boussinesq equation, Liouville equation, and so on.

**References**

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