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Double Periodic Wave Solutions and Breather-Soliton Solutions of The (n+1)-Dimensional sinh-Gordon Equation

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Abstract. In this paper, using a new method named binary F-expansion method and a simple transformation technique, we discussed the (n+1)-dimensional sinh-Gordon equation. Many double periodic wave solutions and breather-soliton solutions are obtained.

1. Introduction
Very recently, J.B. Li and M. Li studied the following (n+1)-dimensional sine-Gordon equation

\[ \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial^2 \phi}{\partial t^2} = \sin \phi \] (1)

and sinh-Gordon equation (see[1] and references cited therein)

\[ \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial^2 \phi}{\partial t^2} = \sinh \phi, \] (2)

where \( n \) is a positive integer. It is well known that sine-Gordon equation admits a geometric interpretation as the differential equation which determines surfaces of constant negative curvature in the Euclidean space, while the sinh-Gordon equation admits geometric interpretation as the differential equation which determines time-like surfaces of constant positive curvature in the same spaces. They appear in wide range of physical applications including relativistic field theory, string dynamics, hydrodynamics, thermodynamics, solid-state physics and nonlinear optics [2-9]. In Ref. [1], J.B. Li et al. obtained 21 exact bounded traveling wave solutions of Eq. (1) including all kinds of double-periodic wave solutions and breather-soliton solutions, however the solutions of Eq. (2) were not discussed in this literature.

In recent years, the Exp-function method [10-19], F-expansion method and the integral bifurcation method [20] have been used to study the problem of all kinds of traveling wave solutions in the domain of nonlinear wave equation. Very recently, the F-expansion method and extended F-expansion method have been proposed to obtain not only the single non-degenerative Jacobi elliptic function solutions, but also the combined non-degenerative Jacobi elliptic function solutions [20-24]. In this paper, we will introduce a new extended method named binary F-expansion method. Using this new

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extended method, we will study the Eq. (2). Many new traveling wave solutions will be figured out. Next, we introduce binary F-expansion method.

2. Binary F-expansion method
For a given \((n+1)\)-dimensional nonlinear partial differential equation:
\[
E[t, x_i, f(\phi), \phi_x, \phi_{xx}, \cdots] = 0, \quad i = 1, 2, \cdots, n,
\]  
where \(f(\phi)\) is compound function. The binary F-expansion method simply proceeds as follows:

**Step1.** Making a transformation
\[
\phi = g[U(\xi)V(\eta)],
\]
\[
\xi = \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n + \omega t, \eta = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + ct,
\]
where \(\gamma_i, \beta_i, (i = 1, 2, \cdots, n)\) are arbitrary constants. The \(\omega, c\) are unknown parameters to be further determined and \(\gamma_i, \beta_i, \omega, c\) satisfy the following relationship:
\[
\gamma_1 \beta_1 + \gamma_2 \beta_2 + \cdots + \gamma_n \beta_n - \omega c = 0.
\]

**Step2.** Substituting (4) into (3), then using (6) yields
\[
E[\xi, U', U'', \cdots; \eta, V', V'', \cdots] = 0.
\]

**Step3.** Supposing that \(U(\xi)\) and \(V(\eta)\) can be expanded as follows:
\[
U(\xi) = \sum_{i=1}^{N_1} a_i F^i(\xi), V(\eta) = \sum_{j=1}^{N_2} b_j G^j(\eta),
\]
where \(F(\xi), G(\eta)\) satisfy the following relations:
\[
F'^2 = P_1 F^4 + Q_1 F^2 + R_1,
\]
\[
G'^2 = P_2 G^4 + Q_2 G^2 + R_2.
\]
The \(P_{1,2}, Q_{1,2}, R_{1,2}, a_i, b_j; i, j = 1, 2, \cdots, N_{1,2}\) are real parameters to be further determined.

**Step4.** Substituting (8) into (7), then using (9) and (10), we can obtain a series in \(F^n, G^m\). Equating each coefficient of \(F^n, G^m\) to zero yields a group of algebraic equations. Solving these algebraic equations, we can obtain the explicit parameter expressions of the constants \(a_i, b_j, \omega, c\).

**Step5.** By using the Appendix A of Ref. [25], all kinds of double periodic solutions of Jacobi elliptic function type can be obtained. When the modulus approach to 1 or 0, using these double-periodic solutions of Jacobi elliptic functions, we obtained simultaneously many breather-soliton solutions of kink and anti-kink wave types.

3. Double periodic solutions and breather-soliton solutions of Eq. (2)
We make a transformation
\[
\phi = 2 \ln \left[ \frac{[V(\eta) + U(\xi)]/[V(\eta) - U(\xi)]}{\left[\frac{[V(\eta) + U(\xi)]}{[V(\eta) - U(\xi)]}\right]} \right],
\]
where \(\xi\) and \(\eta\) are given by (5). Substituting (11) into (2), then using (6), we get
\[ A(V^2 U - U^3)V_{\eta\eta} - B(V^3 - VU^2)U_{\eta\xi} - 2AVUV^2 - 2BVUU^2 + (VU^3 + V^3 U) = 0, \]  
(12)

where \( A = \beta^2 - c^2 \), \( B = \gamma^2 - \omega^2 \) with \( \beta^2 = \beta_1^2 + \beta_2^2 + \cdots + \beta_n^2 \), \( \gamma^2 = \gamma_1^2 + \gamma_2^2 + \cdots + \gamma_n^2 \). Using the method of homogeneous balance, we take \( N_1 = N_2 = 1 \). Thus we suppose that

\[ U(\xi) = a_1 F(\xi), \quad V(\eta) = b_1 G(\eta), \]  
(13)

where \( a_1, b_1 \) are unknown constants to be further determined and the functions \( F(\xi), G(\eta) \) satisfy the equations (9) and (10). Substituting (13), (9), (10) into (12) and equating each coefficient of \( F^3 G^2 \) to zero yields

\[
F^3 G^2 : -2b_1 a_1 (a_1^2 \beta^2 P_2 - b_1^2 \omega^2 P_1 + b_1^2 \gamma^2 Q_1 - a_1^2 c^2 P_2) = 0,
\]

\[
FG^3 : -b_1^3 a_1 (\gamma^2 Q_1 - 1 - \omega^2 Q_1 + \beta^2 Q_2 - c^2 Q_2) = 0,
\]

\[
F^3 G : -b_1 a_1^3 (\gamma^2 Q_1 - 1 - \omega^2 Q_1 + \beta^2 Q_2 - c^2 Q_2) = 0,
\]

\[
FG : -2b_1 a_1 (a_1^2 \gamma^2 R_1 + b_1^2 \beta^2 R_2 - b_1^2 c^2 R_2 - a_1^2 \omega^2 R_1) = 0.
\]

When \( a_1 / b_1 \neq 0 \), solving the above algebraic equations, we obtain

\[
a_1 / b_1 = \pm [(P_1 R_2) / (P_2 R_1)]^{1/4}
\]  
(14)

and

\[
\omega = \pm \sqrt{\gamma^2 R_1 Q_2 \sqrt{P_1 R_2 / (P_2 R_1) - \gamma^2 R_2 Q_1 + R_2} / [R_1 Q_2 \sqrt{P_1 R_2 / (P_2 R_1) - R_2 Q_1}]},
\]

\[
c = \pm \sqrt{R_1 (\beta^2 Q_2 - 1) \sqrt{P_1 R_2 / (P_2 R_1) - \beta^2 R_2 Q_1} / [R_1 Q_2 \sqrt{P_1 R_2 / (P_2 R_1) - R_2 Q_1}]}.
\]  
(15)

Using Appendix A of Ref. [25] and equations (14), (13), (11), we obtain many double periodic wave solutions of elliptic function type. When the modulus \( m_1, m_2 \) approach to 1 or 0, we obtain simultaneously many solutions of other type.

For example, when \( P_1 = m_1^2, Q_1 = -(1 + m_1^2), R_1 = 1, P_2 = m_2^2, Q_2 = -(1 + m_2^2), R_2 = 1 \), we obtain a double periodic wave solution of Eq. (2),

\[
\phi_1 = 2 \ln \left[ \left\{ \text{sn}(\eta, m_1) + \sqrt{m_1 / m_2} \ \text{sn}(\xi, m_1) \right\} / \left\{ \text{sn}(\eta, m_2) - \sqrt{m_1 / m_2} \ \text{sn}(\xi, m_1) \right\} \right],
\]  
(16)

where \( c = \pm \sqrt{\beta^2 + m_1 / [(1 - m_1 m_2) (m_1 - m_2)]} \) and \( \omega = \pm \sqrt{\gamma^2 - m_2 / [(1 - m_1 m_2) (m_1 - m_2)]} \) which is given by (15). The \( m_1, m_2 \) (\( 0 < m_1, m_2 < 1 \)) are two different modules of Jacobi elliptic function.

When \( m_1 \to 1 \) (i.e. \( P_1 = R_1 = 1, Q_1 = -2 \)), we obtain a breather-soliton solution of kink wave type,

\[
\phi_2 = 2 \ln \left[ \left\{ \text{sn}(\eta, m_2) + \sqrt{1 / m_2} \ \text{tanh}(\xi) \right\} / \left\{ \text{sn}(\eta, m_2) - \sqrt{1 / m_2} \ \text{tanh}(\xi) \right\} \right],
\]  
(17)

where \( c = \pm \sqrt{\beta^2 + 1 / (1 - m_2^2)} \) and \( \omega = \pm \sqrt{\gamma^2 - m_2 / (1 - m_2^2)} \). When \( m_2 \to 1 \) (i.e. \( P_2 = R_2 = 1, Q_2 = -2 \)), we also obtain another breather-soliton solution of kink wave type,

\[
\phi_3 = 2 \ln \left[ \left\{ \text{tanh}(\eta) + \sqrt{m_1 \ \text{sn}(\xi, m_1)} \right\} / \left\{ \text{tanh}(\eta) - \sqrt{m_1 \ \text{sn}(\xi, m_1)} \right\} \right].
\]  
(18)
where $c = \pm \sqrt{\beta^2 - m_1^2 / (1 - m_1^2)}$ and $\omega = \pm \sqrt{\gamma^2 + 1 / (1 - m_1^2)}$. When $m_1 \rightarrow 1$ and $m_2 \rightarrow 1$ (i.e. $P_{1,2} = R_{1,2} = 1$), $c \rightarrow \pm \omega \rightarrow \pm \omega$. In this case, Eq. (2) has no traveling wave solution. The similar cases are not explained any more in the below discussions. Similarly, under different parametric conditions, we obtain all kinds of traveling wave solutions of Eq. (2). Firstly, we list all kinds of double periodic wave solutions of Jacobian elliptic-function-type as follows:

3.1. When $P_1 = m_1^2, Q_1 = -(1 + m_1^2), R_1 = 1, P_2 = -1, Q_2 = 2 - m_2^2, R_2 = m_2^2 - 1$,

$$\phi_0 = 2 \ln \left| \frac{dn(\eta, m_1) + \sqrt{m_1^2(1 - m_2^2)} sn(\xi, m_1)}{dn(\eta, m_1) - \sqrt{m_1^2(1 - m_2^2)} sn(\xi, m_1)} \right|. \quad (19)$$

where $c, \omega$ are offered by (15), substituting the above parametric constants into (15), we can always obtain their expressions, so we need not to write them down every time.

3.2. When $P_1 = m_1^2, Q_1 = -(1 + m_1^2), R_1 = 1, P_2 = m_2^2 - 1, Q_2 = 2 - m_2^2, R_2 = -1$,

$$\phi_0 = 2 \ln \left| \frac{nd(\eta, m_1) + \sqrt{m_2^2(1 - m_2^2)} sn(\xi, m_1)}{nd(\eta, m_1) - \sqrt{m_2^2(1 - m_2^2)} sn(\xi, m_1)} \right|. \quad (20)$$

3.3. When $P_1 = m_1^2, Q_1 = -(1 + m_1^2), R_1 = 1, P_2 = -1 - m_2^2, Q_2 = 2 - m_2^2, R_2 = -1$,

$$\phi_0 = 2 \ln \left| \frac{sc(\eta, m_2) + \sqrt{m_1^2(1 - m_2^2)} sn(\xi, m_2)}{sc(\eta, m_2) - \sqrt{m_1^2(1 - m_2^2)} sn(\xi, m_2)} \right|. \quad (21)$$

3.4. When $P_1 = m_1^2, Q_1 = -(1 + m_1^2), R_1 = 1, P_2 = 1 - m_2^2, Q_2 = 2 - m_2^2, R_2 = 1$,

$$\phi_0 = 2 \ln \left| \frac{cs(\eta, m_2) + \sqrt{m_1^2(1 - m_2^2)} sn(\xi, m_2)}{cs(\eta, m_2) - \sqrt{m_1^2(1 - m_2^2)} sn(\xi, m_2)} \right|. \quad (22)$$

3.5. When $P_1 = m_1^2, Q_1 = -(1 + m_1^2), R_1 = 1, P_2 = 1, Q_2 = 2 - m_2^2, R_2 = 1 - m_2^2$,

$$\phi_0 = 2 \ln \left| \frac{sn(\eta, m_2) + \sqrt{m_1^2(1 - m_2^2)} sn(\xi, m_2)}{sn(\eta, m_2) - \sqrt{m_1^2(1 - m_2^2)} sn(\xi, m_2)} \right|. \quad (23)$$

3.6. When $P_1 = -m_1^2, Q_1 = 2m_2^2 - 1, R_1 = -m_1^2, P_2 = -m_2^2, Q_2 = 2m_2^2 - 1, R_2 = 1 - m_2^2$,

$$\phi_0 = 2 \ln \left| \frac{cn(\eta, m_2) + \sqrt{m_1^2(1 - m_2^2)} cn(\xi, m_2)}{cn(\eta, m_2) - \sqrt{m_1^2(1 - m_2^2)} cn(\xi, m_2)} \right|. \quad (24)$$

3.7. When $P_1 = -m_1^2, Q_1 = 2m_2^2 - 1, R_1 = 1 - m_1^2, P_2 = 1 - m_2^2, Q_2 = 2m_2^2 - 1, R_2 = -m_2^2$,

$$\phi_0 = 2 \ln \left| \frac{nc(\eta, m_2) + \sqrt{m_1^2(1 - m_2^2)} cn(\xi, m_2)}{nc(\eta, m_2) - \sqrt{m_1^2(1 - m_2^2)} cn(\xi, m_2)} \right|. \quad (25)$$

3.8. When $P_1 = -m_1^2, Q_1 = 2m_2^2 - 1, R_1 = 1 - m_1^2, P_2 = -m_2^2 + m_2^4, Q_2 = 2m_2^2 - 1, R_2 = 1$,

$$\phi_0 = 2 \ln \left| \frac{sd(\eta, m_2) + \sqrt{m_1^2(1 - m_2^2)} cn(\xi, m_2)}{sd(\eta, m_2) - \sqrt{m_1^2(1 - m_2^2)} cn(\xi, m_2)} \right|. \quad (26)$$

3.9. When $P_1 = -m_1^2, Q_1 = 2m_2^2 - 1, R_1 = 1 - m_1^2, P_2 = 1, Q_2 = 2m_2^2 - 1, R_2 = -m_2^2 + m_2^4$. 


\[ \phi_2 = 2 \ln \left| ds(\eta, m_2) + \sqrt{m_2^2 (1 - m_2^2)} \, (1 - m_2^2) \, cn(\xi, m_2) \right| \left| ds(\eta, m_2) - \sqrt{m_2^2 (1 - m_2^2)} \, (1 - m_2^2) \, cn(\xi, m_2) \right| \]. \quad (27)

3.10. When \( P_1 = -1, Q_1 = 2 - m_2^2, R_1 = m_2^2 - 1, P_2 = -1, Q_2 = 2 - m_2^2, R_2 = m_2^2 - 1, \)
\[ \phi_3 = 2 \ln \left| \left[ dn(\eta, m_2) + \sqrt{1 - m_2^2} \, dn(\xi, m_2) \right| / \left[ dn(\eta, m_2) - \sqrt{1 - m_2^2} \, dn(\xi, m_2) \right| \right| . \quad (28) \]

3.11. When \( P_1 = -1, Q_1 = 2 - m_2^2, R_1 = m_2^2 - 1, P_2 = 1, Q_2 = -(1 + m_2^2), R_2 = m_2^2, \)
\[ \phi_4 = 2 \ln \left| \left[ ns(\eta, m_2) + \sqrt{m_2^2 / (1 - m_2^2)} \, dn(\xi, m_2) \right| / \left[ ns(\eta, m_2) - \sqrt{m_2^2 / (1 - m_2^2)} \, dn(\xi, m_2) \right| \right| . \quad (29) \]

3.12. When \( P_1 = -1, Q_1 = 2 - m_2^2, R_1 = m_2^2 - 1, P_2 = 1 - m_2^2, Q_2 = 2 - m_2^2, R_2 = 1, \)
\[ \phi_5 = 2 \ln \left| \left[ sc(\eta, m_2) + \sqrt{(1 - m_2^2) / (1 - m_2^2)} \, dn(\xi, m_2) \right| / \left[ sc(\eta, m_2) - \sqrt{(1 - m_2^2) / (1 - m_2^2)} \, dn(\xi, m_2) \right| \right| . \quad (30) \]

3.13. When \( P_1 = -1, Q_1 = 2 - m_2^2, R_1 = m_2^2 - 1, P_2 = 1, Q_2 = 2 - m_2^2, R_2 = 1 - m_2^2, \)
\[ \phi_6 = 2 \ln \left| \left[ cs(\eta, m_2) + \sqrt{(1 - m_2^2) / (1 - m_2^2)} \, dn(\xi, m_2) \right| / \left[ cs(\eta, m_2) - \sqrt{(1 - m_2^2) / (1 - m_2^2)} \, dn(\xi, m_2) \right| \right| . \quad (31) \]

3.14. When \( P_1 = 1, Q_1 = -(1 + m_2^2), R_1 = m_2^2, P_2 = 1, Q_2 = -(1 + m_2^2), R_2 = m_2^2, \)
\[ \phi_7 = 2 \ln \left| \left[ ns(\eta, m_2) + \sqrt{m_2^2 / m_1} \, ns(\xi, m_1) \right| / \left[ ns(\eta, m_2) - \sqrt{m_2^2 / m_1} \, ns(\xi, m_1) \right| \right| . \quad (32) \]

3.15. When \( P_1 = 1, Q_1 = -(1 + m_2^2), R_1 = m_2^2, P_2 = m_2^2 - 1, Q_2 = 2 - m_2^2, R_2 = 1, \)
\[ \phi_8 = 2 \ln \left| \left[ nd(\eta, m_2) + \sqrt{1 / (m_2^2 (1 - m_2^2))} \, ns(\xi, m_1) \right| / \left[ nd(\eta, m_2) - \sqrt{1 / (m_2^2 (1 - m_2^2))} \, ns(\xi, m_1) \right| \right| . \quad (33) \]

3.16. When \( P_1 = 1, Q_1 = -(1 + m_2^2), R_1 = m_2^2, P_2 = 1 - m_2^2, Q_2 = 2 - m_2^2, R_2 = 1, \)
\[ \phi_9 = 2 \ln \left| \left[ sc(\eta, m_2) + \sqrt{1 / (m_2^2 (1 - m_2^2))} \, ns(\xi, m_1) \right| / \left[ sc(\eta, m_2) - \sqrt{1 / (m_2^2 (1 - m_2^2))} \, ns(\xi, m_1) \right| \right| . \quad (34) \]

3.17. When \( P_1 = 1, Q_1 = -(1 + m_2^2), R_1 = m_2^2, P_2 = 1, Q_2 = 2 - m_2^2, R_2 = 1 - m_2^2, \)
\[ \phi_{10} = 2 \ln \left| \left[ cs(\eta, m_2) + \sqrt{(1 - m_2^2) / m_1^2} \, ns(\xi, m_1) \right| / \left[ cs(\eta, m_2) - \sqrt{(1 - m_2^2) / m_1^2} \, ns(\xi, m_1) \right| \right| . \quad (35) \]

3.18. When \( P_1 = 1 - m_2^2, Q_1 = 2m_2^2 - 1, R_1 = -m_2^2, P_2 = 1 - m_2^2, Q_2 = 2m_2^2 - 1, R_2 = -m_2^2, \)
\[ \phi_{11} = 2 \ln \left| \left[ nc(\eta, m_2) + \sqrt{m_2^2 (1 - m_2^2)}/(m_2^2 (1 - m_2^2)) \, nc(\xi, m_1) \right| / \left[ nc(\eta, m_2) - \sqrt{m_2^2 (1 - m_2^2)}/(m_2^2 (1 - m_2^2)) \, nc(\xi, m_1) \right| \right| . \quad (36) \]

3.19. When \( P_1 = 1 - m_2^2, Q_1 = 2m_2^2 - 1, R_1 = -m_2^2, P_2 = 1, Q_2 = 2m_2^2 - 1, R_2 = -m_2^2 + m_2^4, \)
\( \phi_2 = 2\ln |[\text{ds}(\eta, m_2) + \frac{1}{\sqrt{2}} m_2^2 (1-m_2^2)/(1-m_2^2) \text{nd}(\xi, m_1)]/|\text{ds}(\eta, m_2) - \frac{1}{\sqrt{2}} m_2^2 (1-m_2^2)/(1-m_2^2) \text{nd}(\xi, m_1)|]|. \)  \( (37) \)

3.20. When \( P_1 = m_1^2 - 1, Q_1 = 2 - m_2^2, R_1 = -1, P_2 = m_2^2 - 1, Q_2 = 2 - m_2^2, R_2 = -1, \)
\( \phi_3 = 2\ln |[\text{nd}(\eta, m_2) + \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{nd}(\xi, m_1)]/|\text{nd}(\eta, m_2) - \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{nd}(\xi, m_1)|]|. \)  \( (38) \)

3.21. When \( P_1 = m_1^2 - 1, Q_1 = 2 - m_2^2, R_1 = -1, P_2 = 1 - m_2^2, Q_2 = 2 - m_2^2, R_2 = 1, \)
\( \phi_4 = 2\ln |[\text{sc}(\eta, m_2) + \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{sc}(\xi, m_1)]/|\text{sc}(\eta, m_2) - \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{sc}(\xi, m_1)|]|. \)  \( (39) \)

3.22. When \( P_1 = m_1^2 - 1, Q_1 = 2 - m_2^2, R_1 = -1, P_2 = 1, Q_2 = 2 - m_2^2, R_2 = 1 - m_2^2, \)
\( \phi_5 = 2\ln |[\text{cs}(\eta, m_2) + \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{sc}(\xi, m_1)]/|\text{cs}(\eta, m_2) - \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{sc}(\xi, m_1)|]|. \)  \( (40) \)

3.23. When \( P_1 = 1 - m_1^2, Q_1 = 2 - m_2^2, R_1 = 1, P_2 = 1 - m_2^2, Q_2 = 2 - m_2^2, R_2 = 1, \)
\( \phi_6 = 2\ln |[\text{sc}(\eta, m_2) + \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{sc}(\xi, m_1)]/|\text{sc}(\eta, m_2) - \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{sc}(\xi, m_1)|]|. \)  \( (41) \)

3.24. When \( P_1 = 1 - m_1^2, Q_1 = 2 - m_2^2, R_1 = 1, P_2 = 1, Q_2 = 2 - m_2^2, R_2 = 1 - m_2^2, \)
\( \phi_7 = 2\ln |[\text{cs}(\eta, m_2) + \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{cs}(\xi, m_1)]/|\text{cs}(\eta, m_2) - \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{cs}(\xi, m_1)|]|. \)  \( (42) \)

3.25. When \( P_1 = -m_1^2 + m_1^4, Q_1 = 2m_1^2 - 1, R_1 = 1, P_2 = -m_2^2 + m_2^4, Q_2 = 2m_2^2 - 1, R_2 = 1, \)
\( \phi_8 = 2\ln |[\text{sd}(\eta, m_2) + \frac{1}{\sqrt{2}} m_2^2 (1-m_2^2)/(1-m_2^2) \text{sd}(\xi, m_1)]/|\text{sd}(\eta, m_2) - \frac{1}{\sqrt{2}} m_2^2 (1-m_2^2)/(1-m_2^2) \text{sd}(\xi, m_1)|]|. \)  \( (43) \)

3.26. When \( P_1 = -m_1^2 + m_1^4, Q_1 = 2m_1^2 - 1, R_1 = 1, P_2 = 1, Q_2 = 2m_2^2 - 1, R_2 = -m_2^2 + m_2^4, \)
\( \phi_9 = 2\ln |[\text{ds}(\eta, m_2) + \frac{1}{\sqrt{2}} m_2^2 (1-m_2^2)/(1-m_2^2) \text{ds}(\xi, m_1)]/|\text{ds}(\eta, m_2) - \frac{1}{\sqrt{2}} m_2^2 (1-m_2^2)/(1-m_2^2) \text{ds}(\xi, m_1)|]|. \)  \( (44) \)

3.27. When \( P_1 = 1, Q_1 = 2 - m_2^2, R_1 = 1 - m_2^2, P_2 = 1, Q_2 = 2 - m_2^2, R_2 = 1 - m_2^2, \)
\( \phi_{10} = 2\ln |[\text{cs}(\eta, m_2) + \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{cs}(\xi, m_1)]/|\text{cs}(\eta, m_2) - \frac{1}{\sqrt{2}} (1-m_2^2)/(1-m_2^2) \text{cs}(\xi, m_1)|]|. \)  \( (45) \)

3.28. When \( P_1 = 1, Q_1 = 2 - m_2^2, R_1 = -m_1^2 + m_2^4, P_2 = 1, Q_2 = 2m_2^2 - 1, R_2 = -m_2^2 + m_2^4, \)
\( \phi_{11} = 2\ln |[\text{ds}(\eta, m_2) + \frac{1}{\sqrt{2}} m_2^2 (1-m_2^2)/(1-m_2^2) \text{ds}(\xi, m_1)]/|\text{ds}(\eta, m_2) - \frac{1}{\sqrt{2}} m_2^2 (1-m_2^2)/(1-m_2^2) \text{ds}(\xi, m_1)|]|. \)  \( (46) \)

Second, when \( m_1 \to 1 \) or \( m_2 \to 1 \), we list all kinds of breather-soliton solutions of Eq. (2) as follows:
\phi_{s_4} = \lim_{m_2 \to 1} \phi_4 = 2 \ln \left[ \frac{dn(\eta, m_2) + \sqrt{1 - m_2^2} \tanh(\xi)}{dn(\eta, m_2) - \sqrt{1 - m_2^2} \tanh(\xi)} \right], \quad (47)

\phi_{s_5} = \lim_{m_2 \to 1} \phi_5 = 2 \ln \left[ \frac{sn(\eta, m_2) + \sqrt{m_2 \tanh(\xi)}}{sn(\eta, m_2) - \sqrt{m_2 \tanh(\xi)}} \right], \quad (48)

\phi_{s_6} = \lim_{m_2 \to 1} \phi_6 = 2 \ln \left[ \frac{coth(\eta) + \sqrt{m_1 sn(\xi, m_1)}}{coth(\eta) - \sqrt{m_1 sn(\xi, m_1)}} \right], \quad (49)

\phi_{s_7} = \lim_{m_2 \to 1} \phi_7 = 2 \ln \left[ \frac{nd(\eta, m_2) + \sqrt{1/(1 - m_2^2)} \tanh(\xi)}{nd(\eta, m_2) - \sqrt{1/(1 - m_2^2)} \tanh(\xi)} \right], \quad (50)

\phi_{s_8} = \lim_{m_2 \to 1} \phi_8 = 2 \ln \left[ \frac{sc(\eta, m_2) + \sqrt{1/(1 - m_2^2)} \tanh(\xi)}{sc(\eta, m_2) - \sqrt{1/(1 - m_2^2)} \tanh(\xi)} \right], \quad (51)

\phi_{s_9} = \lim_{m_2 \to 1} \phi_9 = 2 \ln \left[ \frac{cs(\eta, m_2) + \sqrt{1/(1 - m_2^2)} \tanh(\xi)}{cs(\eta, m_2) - \sqrt{1/(1 - m_2^2)} \tanh(\xi)} \right], \quad (52)

\phi_{s_10} = \lim_{m_2 \to 1} \phi_{10} = 2 \ln \left[ \frac{coth(\eta) + \sqrt{1/m_1} \tanh(\xi)}{coth(\eta) - \sqrt{1/m_1} \tanh(\xi)} \right], \quad (53)

\phi_{s_11} = \lim_{m_2 \to 1} \phi_{11} = 2 \ln \left[ \frac{sn(\eta, m_2) + \sqrt{m_2 coth(\xi)}}{sn(\eta, m_2) - \sqrt{m_2 coth(\xi)}} \right], \quad (54)

\phi_{s_12} = \lim_{m_2 \to 1} \phi_{12} = 2 \ln \left[ \frac{cs(\eta, m_2) + \sqrt{1/(1 - m_2^2)} \tanh(\xi)}{cs(\eta, m_2) - \sqrt{1/(1 - m_2^2)} \tanh(\xi)} \right], \quad (55)

\phi_{s_13} = \lim_{m_2 \to 1} \phi_{13} = 2 \ln \left[ \frac{nd(\eta, m_2) + \sqrt{1/(1 - m_2^2)} coth(\xi)}{nd(\eta, m_2) - \sqrt{1/(1 - m_2^2)} coth(\xi)} \right], \quad (56)

\phi_{s_14} = \lim_{m_2 \to 1} \phi_{14} = 2 \ln \left[ \frac{sc(\eta, m_2) + \sqrt{1/(1 - m_2^2)} coth(\xi)}{sc(\eta, m_2) - \sqrt{1/(1 - m_2^2)} coth(\xi)} \right], \quad (57)

\phi_{s_15} = \lim_{m_2 \to 1} \phi_{15} = 2 \ln \left[ \frac{cs(\eta, m_2) + \sqrt{1/(1 - m_2^2)} coth(\xi)}{cs(\eta, m_2) - \sqrt{1/(1 - m_2^2)} coth(\xi)} \right], \quad (58)

In the end, when \( m_1 \to 0 \) or \( m_2 \to 0 \), we obtain all kinds of double periodic wave solutions of elliptic-trigonometric function type of Eq. (2). For example,

\phi_{s_16} = \lim_{m_2 \to 1} \phi_16 = 2 \ln \left[ \frac{\tan(\eta) + \sqrt{m_2 sn(\xi, m_1)}}{\tan(\eta) - \sqrt{m_2 sn(\xi, m_1)}} \right], \quad (59)

\phi_{s_17} = \lim_{m_2 \to 1} \phi_17 = 2 \ln \left[ \frac{\tan(\eta) + sn(\xi, m_1)}{\tan(\eta) - sn(\xi, m_1)} \right]. \quad (60)

The other double periodic wave solutions of this type can be similarly discussed, here we omit them.

In order to describe the dynamic properties of above traveling wave solutions intuitively, taking \( \xi = 3x_1 - 2x_2 + \omega t, \eta = 2x_1 - 3x_2 + ct \), when \( t = 1 \), we give wave-form figures of solutions (16), (29), (47) and (56) which are shown in Figure 1. (a), (b), (c) and (d):
Figure 1. (a). The 3D-graph of double periodic wave solution (16): \( m_1 = 0.9, m_2 = 0.6 \). (b). The 3D-graph of double periodic wave solution (29): \( m_1 = 0.9, m_2 = 0.6 \). (c). The 3D-graph of breather-soliton solution (47) of kink wave type: \( m_1 = 1, m_2 = 0.6 \). (d). The 3D-graph of breather-soliton solution (56) of kink wave type: \( m_1 = 1, m_2 = 0.3 \).

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References