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Construction of exact solitary-wave solutions for the K(2,2,1) and K(3,3,1) equations by a new algorithm for calculating Adomian polynomials

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Abstract. In this paper some nonlinear KdV equations are fully investigated using Adomian method. A new algorithm for calculating Adomian polynomials is suggested, and some new exact solitary solutions are obtained using the symbolic computation system, Maple.

1. Introduction
Finding exact solutions for nonlinear equations has great significance in the solitary theory. In recent years, many methods can be used to get the solitary solutions for KdV equations. The Homotopy Perturbation Method [HPM] proposed by He [2] is employed by H. Tari etc. [3] to find approximate solutions of K(2,2), KdV and Modified KdV. The variational iteration method is used by many authors [4, 5] to construct solitary solutions or compact-like solutions. Z. M. Odibat and S. Momani [6] apply variational iteration method to nonlinear differential equations of fractional order. S. D. Zhu use exp-function method to solve Hybrid-Lattice system [7], and he expend such method to the discrete mKdV lattice [8]. Solitary solutions for nonlinear equations can also be construct by Adomian decomposition method [9]. In this paper, we will extend the new algorithm for calculating Adomian polynomials [1] to the $K(m,n,1)$ equation:

$$u_t - (u^m)_x + (u^n)_{xxx} + u_{5x} = 0 \quad (1)$$

and seek explicit exact solutions for $K(2,2,1)$ and $K(3,3,1)$. And then more exact solitary solutions are presented.

2. Application of New Algorithm
Consider the nonlinear $K(m,n,1)$ initial value problem:

$$u_t - (u^m)_x + (u^n)_{xxx} + u_{5x} = 0, \quad m,n > 1, \quad u(x,0) = f(x) \quad (2)$$

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where $u(x,t)$ is an unknown function.

Adomian decomposition method decomposes the unknown function $u(x,t)$ by an infinite series

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

(3)

where the components $u_0, u_1, u_2, \ldots$ are usually determined recursively.

If we choose the nonlinear term $F(u) = -(u^m)_x + (u^n)_{xxx} + u_5$, then $F(u)$ can be decomposed into one infinite series of polynomials given by

$$F(u) = -(u^m)_x + (u^n)_{xxx} + u_5 = \sum_{k=0}^{\infty} A_k$$

(4)

where $A_k$ is the so-called Adomian polynomials. In the following, we will give how to calculate Adomian polynomials for nonlinear function $F(u) = -(u^m)_x + (u^n)_{xxx} + u_5$.

According to [1], let $u(\lambda) = \sum_{k=0}^{\infty} \lambda^k A_k$, then

$$F(u(\lambda)) = \sum_{k=0}^{\infty} \lambda^k A_k$$

(5)

let $\lambda = 0$, we can obtain $A_0 = -(u^m_0)_x + (u^n_0)_{xxx} + (u_5)_{xx}$. Give first derivative of (5) with respect to $\lambda$ and let $\lambda = 0$, we have $A_1 = -(mu_1 u_0^{m-1})_x + (nu_1 u_0^{n-1})_{xxx} + (u_1)_{xx}$. If 2th derivative of (5) with respect to $\lambda$ is given and $\lambda = 0$, then we can get $A_2 = -(mu_2 u_0^{m-1} + \frac{1}{2} m(m-1)u_1^2 u_0^{m-2})_x + (nu_2 u_0^{n-1} + \frac{1}{2} n(n-1)u_1^2 u_0^{n-2})_{xxx} + (u_2)_{xx}$. Continue this course, we can obtain $A_3, A_4, \ldots$.

So the first few Adomian polynomials for the term $-(u^m)_x + (u^n)_{xxx} + u_5$ are given:

$$A_0 = -(u^m_0)_x + (u^n_0)_{xxx} + (u_5)_{xx},$$

$$A_1 = -(mu_1 u_0^{m-1})_x + (nu_1 u_0^{n-1})_{xxx} + (u_1)_{xx},$$

$$A_2 = -(mu_2 u_0^{m-1} + \frac{1}{2} m(m-1)u_1^2 u_0^{m-2})_x + (nu_2 u_0^{n-1} + \frac{1}{2} n(n-1)u_1^2 u_0^{n-2})_{xxx} + (u_2)_{xx},$$

(6)

... Applying $L^{-1}$ to both sides of (2), and using the given initial conditions we obtain

$$u(x,t) = f(x) - L^{-1}(-(u^m)_x + (u^n)_{xxx} + u_5)$$

(7)

where

$$L^{-1}(\cdot) = \int_0^x (\cdot) dt.$$  

(8)
Substituting the decomposition series (3) for \( u(x,t) \) and the polynomials representation (4) into (7) yields

\[
\sum_{k=0}^{\infty} u_k(x,t) = f(x) - L^{-1} \left( \sum_{k=0}^{\infty} A_k \right)
\]  

(9)

The components \( u_k(x,t), k \geq 0 \) can be determined recursively by using the relation

\[
u_k(x,t) = f(x), u_1(x,t) = -L^{-1}(A_0), u_2 = -L^{-1}(A_1), \cdots
\]

(10)

By properly calculating the components, the series solution of \( u(x,t) \) follows immediately. The series solution may provide the solution in a closed form if an exact solution exists.

3. Exact solitary-wave solutions for the K(2,2,1) equation and K(3,3,1) equation

3.1. The K(2,2,1) equation

We consider the K(2,2,1) equation with the initial condition:

\[
u_t - (u^2)_x + (u^2)_{xxx} + u_{xx} = 0, u(x,0) = \frac{16c-1}{12} \sinh \left( \frac{x}{4} \right)
\]

(11)

where \( c \) is an arbitrary constant. Applying the integral operator \( L^{-1} \) to both sides of (11) yields

\[
u(x,t) = \frac{16c-1}{12} \sinh \left( \frac{x}{4} \right) - L^{-1}(-u^2_x) + (u^2)_{xxx} + u_{xx}
\]

(12)

Substituting the decomposition series (3) for \( u(x,t) \) into (12) yields

\[
\sum_{k=0}^{\infty} u_k(x,t) = \frac{16c-1}{12} \sinh \left( \frac{x}{4} \right) - L^{-1} \left( \sum_{k=0}^{\infty} A_k \right)
\]

(13)

Adomian polynomials \( A_k \) for \( m=n=2 \) in (6) are obtained by

\[
A_0 = -(u_2^3)_x + (u_2^3)_{xxx} + (u_0^{3+5}), A_1 = -(2u_1u_0)_x + (2u_1u_0)_{xxx} + (u_1)_{xx}, \cdots
\]

(14)

Substituting (14) into (13) gives

\[
u_0(x,t) = \frac{16c-1}{12} \sinh \left( \frac{x}{4} \right),
\]

\[
u_1(x,t) = -L^{-1} A_0 = -L^{-1} \left( \frac{2}{3} c^2 \cosh \left( \frac{x}{4} \right) \sinh \left( \frac{x}{4} \right) - \frac{1}{24} c \sinh \left( \frac{x}{4} \right) \cosh \left( \frac{x}{4} \right) \right)
\]

\[
= \frac{2}{3} c^2 t \sinh \left( \frac{x}{4} \right) \cosh \left( \frac{x}{4} \right) + \frac{1}{24} c t \sinh \left( \frac{x}{4} \right) \cosh \left( \frac{x}{4} \right),
\]

\[
u_2(x,t) = -L^{-1} A_1 = -L^{-1} \left( -\frac{1}{96} c^2 t + \frac{1}{6} c^3 t + \frac{1}{48} c^4 t \cosh 2 \frac{x}{4} - \frac{1}{3} c^5 t \cosh \frac{x}{4} \right)
\]

\[
= \frac{1}{576} c^3 t^3 \sinh \left( \frac{x}{4} \right) \cosh \left( \frac{x}{4} \right) - \frac{1}{36} c^4 t^3 \sinh \left( \frac{x}{4} \right) \cosh \left( \frac{x}{4} \right),
\]

\[
\cdots
\]
Thus this gives the solution of (11) in a series form
\[
u(x,t) = \frac{16c-1}{12} \sinh^2 \left( \frac{x}{4} \right) - \frac{2}{3} c^2 t \sinh \left( \frac{x}{4} \right) \cosh \left( \frac{x}{4} \right) + \frac{1}{24} c t \sinh \left( \frac{x}{4} \right) \cosh \left( \frac{x}{4} \right) \\
+ \frac{1}{576} c^3 t^3 \sinh \left( \frac{x}{4} \right) \cosh \left( \frac{x}{4} \right) - \frac{1}{36} c^4 t^3 \sinh \left( \frac{x}{4} \right) \cosh \left( \frac{x}{4} \right) \ldots
\]
(15)

Using Taylor series into (15), we find the closed form solution
\[
u(x,t) = \frac{16c-1}{12} \sinh^2 \left( \frac{x-c t}{4} \right).
\]
(16)

It is an exact solitary solution for the nonlinear K(2,2,1) equation.

In addition, we can develop another exact solution for the K(2,2,1) equation. Now we consider another initial value problem of K(2,2,1) equation
\[
u_t - (u^2)_x + (u^2)_{xxx} + u_{5x} = 0, u(x,0) = -\frac{16c-1}{12} \cosh^2 \left( \frac{x}{4} \right),
\]
(17)

Using the manner as discussed above, we get another exact solution given by
\[
u(x,t) = -\frac{16c-1}{12} \cosh^2 \left( \frac{ct-x}{4} \right).
\]
(18)

3.2. The K(3,3,1) equation
We now consider the initial value problem K(3,3,1)
\[
u_t - (u^3)_x + (u^3)_{xxx} + u_{5x} = 0, u(x,0) = \sqrt{\frac{81c-1}{54}} \sinh \left( \frac{x}{3} \right),
\]
(19)

Applying the inverse operator $L^{-1}$ to both sides of (19) gives rise to
\[
u(x,t) = \sqrt{\frac{81c-1}{54}} \sinh \left( \frac{x}{3} \right) - L^{-1} \left( -(u^3)_x + (u^3)_{xxx} + u_{5x} \right).
\]
(20)

Using the decomposition series (3) yields
\[
\sum_{k=0}^{\infty} u_k(x,t) = \sqrt{\frac{81c-1}{54}} \sinh \left( \frac{x}{3} \right) - L^{-1} \left( \sum_{k=0}^{\infty} A_k \right),
\]
(21)

where $A_k$ are Adomian polynomials that represent the nonlinear term $(u^3)_x - (u^3)_{xxx} + u_{5x}$. Thus we have recursive relation
\[
u_0(x,t) = \sqrt{\frac{81c-1}{54}} \sinh \left( \frac{x}{3} \right), u_{k+1} = -L^{-1} (A_k), k \geq 0.
\]
(22)

To obtain Adomian polynomials $A_k$, we take $m=n=3$ in (6) such that we have
\[
A_0 = -(u_0^3)_x + (u_0)^3_+ + (u_0)_{5x}, A_1 = -3(u_0 u_0^2)_x + (3u_0 u_0^2)_{xxx} + (u_0)_{5x}, \ldots
\]
(23)

Substituting (23) into (22) gives
Consequently, the solution of (19) in a series form is

\[ u(x,t) = \sqrt{\frac{81c-1}{54}} \sinh\left(\frac{x}{3}\right) - \frac{1}{54} \sqrt{48c-6c t} \cosh\left(\frac{x}{3}\right) - \frac{1}{324} c^2 t^2 \sinh\left(\frac{x}{3}\right) + \cdots \]  \hspace{1cm} (25)

Using Taylor series into (25), we have the closed form solution

\[ u(x,t) = \sqrt{\frac{81c-1}{54}} \sinh\left(\frac{x-ct}{3}\right). \]  \hspace{1cm} (26)

To obtain another exact solution for K(3,3,1), we consider the initial value problem of K(3,3,1) equation

\[ u_t - (u^3)_x + (u^3)_{xxx} + u_{5x} = 0, u(x,0) = -\sqrt{\frac{81c-1}{54}} \sinh\left(\frac{x}{3}\right). \]  \hspace{1cm} (27)

According to the similar steps as discussed above, we have another exact solution given by

\[ u(x,t) = -\sqrt{\frac{81c-1}{54}} \sinh\left(\frac{x-ct}{3}\right). \]  \hspace{1cm} (28)

4. More exact solitary solutions

4.1. The K(2,2,1) type

By combining (16) and (18), we find that

\[ u(x,t) = \frac{16c-1}{12} a \sinh^2\left(\frac{x-ct}{4}\right) - \frac{16c-1}{12} b \cosh^2\left(\frac{x-ct}{4}\right) \] \hspace{1cm} (29)

satisfies the K(2,2,1) equation, where a and b are constants if \( a = b, 1 - b \)

When \( a = b \), we can obtain the trivial solution \( u(x,t) = -\frac{b(16c-1)}{12} \).

When \( a = 1 - b \), we can obtain the new exact solution

\[ u(x,t) = \frac{16c-1}{12} (1 - b) \sinh^2\left(\frac{ct-x}{4}\right) - \frac{16c-1}{12} b \cosh^2\left(\frac{ct-x}{4}\right). \]

In addition, adding a constant to the arguments in (16) and (18) will exhibit more exact solutions. In other words, we can have the exact solutions

\[ u(x,t) = \frac{16c-1}{12} \sinh^2\left(\frac{ct-x}{4} + r\right), u(x,t) = -\frac{16c-1}{12} \cosh^2\left(\frac{ct-x}{4} + r\right). \] \hspace{1cm} (30)
where $r$ is a constant.

4.2. K(3,3,1) type
As discussed before, we can also obtain a new exact solution by combining (26) and (28). And we found that

$$u(x,t) = 81c - 1 \sqrt{\frac{a}{54}} \sinh\left(\frac{x - ct}{3}\right)^2 a \sinh\left(\frac{x - ct}{3}\right) - 81c - 1 \sqrt{\frac{b}{54}} \sinh\left(\frac{x - ct}{3}\right)$$

satisfies the K(3,3,1) equation if $a = b, 1 + b, -1 + b$.

Moreover, adding a constant to the arguments in (26) and (28) will exhibit more exact solutions

$$u(x,t) = 81c - 1 \sqrt{\frac{a}{54}} \sinh\left(\frac{x - ct}{3} + r\right), u(x,t) = -81c - 1 \sqrt{\frac{b}{54}} \sinh\left(\frac{x - ct}{3} + r\right)$$

where $r$ is a constant.

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