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On Solutions of Stochastic Oscillatory Quadratic Nonlinear Equations Using Different Techniques, a Comparison Study

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Abstract. In this paper, nonlinear oscillators under quadratic nonlinearity with stochastic inputs are considered. Different methods are used to obtain first order approximations, namely: the WHEP technique, the perturbation method, the Pickard approximations, the Adomian decompositions and the homotopy perturbation method (HPM). Some statistical moments are computed for the different methods using mathematica 5. Comparisons are illustrated through figures for different case-studies.

1. Introduction
Quadratic oscillation arises through many applied models in applied sciences and engineering when studying oscillatory systems [1]. These systems can be exposed to a lot of uncertainties through the external forces, the damping coefficient, the frequency and/or the initial or boundary conditions. These input uncertainties cause the output solution process to be also uncertain. For most of the cases, getting the probability density function (p.d.f.) of the solution process may be impossible. So, developing approximate techniques (through which approximate statistical moments can be obtained) is an important and necessary work. There are many techniques which can be used to obtain statistical moments of such problems. The main goal of this paper is to compare some of these methods when applied to a quadratic nonlinearity problem.

2. Problem Formulation
In this paper, the following quadratic nonlinear oscillatory equation is considered as a comparison prototype equation for the application of the different solution techniques:

$$\ddot{x}(t; \omega) + 2\omega \zeta \dot{x} + \omega^2 x + \omega \zeta^2 x^2 = F(t; \omega), \quad t \in [0, T]$$

(1)

under stochastic excitation $F(t; \omega)$ with deterministic initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0,$$

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where
\( w \): frequency of oscillation,
\( \zeta \): damping coefficient
\( \varepsilon \): deterministic nonlinearity scale
\( \omega \in (\Omega, \sigma, P) \): a triple probability space with \( \Omega \) as the sample space, \( \sigma \) is a \( \sigma \)-algebra on events in \( \Omega \) and \( P \) is a probability measure.

**Lemma:**

The solution of equation (1), if exists, is a power series of \( \varepsilon \).

**Proof**

Rewriting equation (1), it can take the following form

\[
\ddot{x}(t; \omega) + 2w \zeta \dot{x} + w^2 x = F(t) - \varepsilon w^2 x^2 
\]

Following Pickard approximation, the equation can be rewritten as

\[
\ddot{x}_{n+1}(t) + 2w \zeta \dot{x}_n + w^2 x_{n+1} = F(t) - \varepsilon w^2 x^2_n, \ n \geq 0 
\]

where the solution at \( n = 0 \), \( x_0 \), is linear and corresponding for the case \( \varepsilon = 0 \).

At \( n = 1 \), the iteration takes the form:

\[
\ddot{x}_1(t) + 2w \zeta \dot{x}_1 + w^2 x_1 = F(t) - \varepsilon w^2 x^2_0, 
\]

which has the following general solution

\[
x_1(t) = \psi(t) - \varepsilon w^2 \int_0^t h(t - s) x_0^2(s) ds, 
\]

or

\[
x_1(t) = x_1^{(0)} + \varepsilon x_1^{(1)}. 
\]

At \( n = 2 \), the iteration takes the form:

\[
\ddot{x}_2(t) + 2w \zeta \dot{x}_2 + w^2 x_2 = F(t) - \varepsilon w^2 x^2_1, 
\]

which has the following general solution

\[
x_2(t) = x_2^{(0)} + \varepsilon x_2^{(1)} + \varepsilon^2 x_2^{(2)} + \varepsilon^3 x_2^{(3)}. 
\]

Proceeding like this, one can get the following

\[
x_n(t) = x_n^{(0)} + \varepsilon x_n^{(1)} + \varepsilon^2 x_n^{(2)} + \varepsilon^3 x_n^{(3)} + \ldots + \varepsilon^{n+m} x_n^{(n+m)}. 
\]

Assuming the solution exists, it will be

\[
x(t) = \lim_{n \to \infty} x_n(t) = \sum_{j=0}^{\infty} \varepsilon^j x_j, 
\]

which is a power series of \( \varepsilon \).

As a direct result of this lemma, it is expected that the average, the variance as well as the covariance are also power series of \( \varepsilon \).

3. WHEP Technique
Since Meecham and his co-workers [2] developed a theory of turbulence involving a truncated Wiener-Hermite expansion (WHE) of the velocity field, many authors studied problems concerning turbulence [3-8]. A lot of general applications in fluid mechanics was also studied in [9,10,11]. Scattering problems attracted the WHE applications through many authors [12-16]. The nonlinear oscillators were considered as an opened area for the applications of WHE as can be found in [17-23]. There are a lot of applications in boundary value problems [24,25] and generally in different mathematical studies [26-29].

The application of the WHE aims at finding a truncated series solution to the solution process of differential equations. The truncated series composes of two major parts; the first is the Gaussian part which consists of the first two terms, while the rest of the series constitute the non-Gaussian part. In nonlinear cases, there exists always difficulties of solving the resultant set of deterministic integro-differential equations got from the applications of a set of comprehensive averages on the stochastic integro-differential equation obtained after the direct application of WHE. Many authors introduced different methods to face these obstacles. Among them, the WHEP technique was introduced in [22] using the perturbation technique to solve perturbed nonlinear problems.

The WHE method utilizes the Wiener-Hermite polynomials which are the elements of a complete set of statistically orthogonal random functions [30]. The Wiener-Hermite polynomial \( H^{(i)}(t_1,t_2,...,t_i) \) satisfies the following recurrence relation:

\[
H^{(i)}(t_1,t_2,...,t_i) = H^{(i-1)}(t_1,t_2,...,t_{i-1}).H^{(1)}(t_i) - \sum_{m=1}^{i-1} H^{(i-2)}(t_h,t_i,...t_{i-2}).\delta(t_{i-m} - t_i), i \geq 2
\]

where

\[
H^{(0)} = 1,
\]

\[
H^{(1)}(t) = n(t),
\]

\[
H^{(2)}(t_1,t_2) = H^{(1)}(t_1).H^{(1)}(t_2) - \delta(t_1 - t_2),
\]

\[
H^{(3)}(t_1,t_2,t_3) = H^{(2)}(t_1,t_2).H^{(1)}(t_3) - H^{(1)}(t_1).H^{(1)}(t_2).\delta(t_2 - t_3)
\]

\[
- H^{(1)}(t_2).\delta(t_1 - t_3),
\]

\[
H^{(4)}(t_1,t_2,t_3,t_4) = H^{(3)}(t_1,t_2,t_3).H^{(1)}(t_4) - H^{(2)}(t_1,t_2).H^{(1)}(t_3).\delta(t_3 - t_4)
\]

\[
- H^{(2)}(t_1,t_3).\delta(t_2 - t_4) - H^{(2)}(t_2,t_3).\delta(t_1 - t_4),
\]

in which \( n(t) \) is the white noise with the following statistical properties

\[
E \ n(t) = 0,
\]

\[
E \ n(t_1)n(t_2) = \delta(t_1 - t_2),
\]

where \( \delta(\cdot) \) is the Dirac delta function and \( E \) denotes the ensemble average operator. The Wiener-Hermite set is a statistically orthogonal set, i.e.

\[
E \ H^{(i)}.H^{(j)} = 0 \ \forall \ i \neq j.
\]

The average of almost all \( H \) functions vanishes, particularly,

\[
E \ H^{(i)} = 0 \ \text{for} \ i \geq 1.
\]
Due to the completeness of the Wiener-Hermite set, any random function $G(t, \omega)$ can be expanded as

$$G(t, \omega) = G(0, t) + \int_{-\infty}^{\infty} G^{(1)}(t; t_1)H^{(1)}(t_1)dt_1 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^{(2)}(t; t_1, t_2)H^{(2)}(t_1, t_2)dt_1dt_2 + \ldots \quad (7)$$

where the first two terms are the Gaussian part of $G(t, \omega)$. The rest of the terms in the expansion represent the non-Gaussian part of $G(t, \omega)$. The average of $G(t, \omega)$ is

$$\mu_{G} = E G(t, \omega) = G^{(0)}(t) \quad (8)$$

The covariance of $G(t, \omega)$ is

$$\text{Cov}(G(t, \omega), G(\tau, \omega)) = E (G(t, \omega) - \mu_{G}(t))(G(\tau, \omega) - \mu_{G}(\tau))$$

$$= \int_{-\infty}^{\infty} G^{(1)}(t; t_1)G^{(1)}(\tau, t_1)dt_1 + 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^{(2)}(t; t_1, t_2)G^{(2)}(\tau, t_1, t_2)dt_1dt_2 \quad (9)$$

The variance of $G(t, \omega)$ is

$$\text{Var} G(t, \omega) = E (G(t, \omega) - \mu_{G}(t))^2$$

$$= \int_{-\infty}^{\infty} [G^{(1)}(t; t_1)]^2dt_1 + 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [G^{(2)}(t; t_1, t_2)]^2dt_1dt_2 \quad (10)$$

The WHEP technique can be applied on linear or nonlinear perturbed systems described by ordinary or partial differential equations. The solution can be modified in the sense that additional parts of the Wiener-Hermite expansion can always be taken into considerations and the required order of approximations can always be made depending on the computing tool. It can be even run through a package if it is coded in some sort of symbolic languages. The technique was successfully applied to several nonlinear stochastic equations, see [20,22,23,25].

### 3.1 Case-study:

The quadratic nonlinear oscillatory problem, equation (1), is solved using WHEP technique. The first order approximation of the solution process takes the following form:

$$x(t; \omega) = x^{(0)}(t) + \int_{-\infty}^{\infty} x^{(1)}(t; t_1)H^{(1)}(t_1)dt_1 \quad (11)$$

Applying the WHEP technique, the following equations in the deterministic kernels are obtained:

$$Lx^{(0)}(t) + \varepsilon \omega^2 (x^{(0)}(t))^2 + \varepsilon \omega^2 \int_{-\infty}^{\infty} (x^{(1)}(t; t_1))^2dt_1 = F^{(0)}(t)$$

$$Lx^{(0)}(t, t_1) + 2\varepsilon \omega^2 x^{(0)}(t)x^{(0)}(t, t_1) = F^{(0)}(t, t_1) \quad (12)$$

Let us take the simple case of evaluating the only Gaussian part (first order approximation) of the solution process. The ensemble average is

$$\mu_x(t) = x^{(0)}(t), \quad (14)$$

and the variance is

$$\sigma^2_x(t) = \int_{-\infty}^{\infty} [x^{(1)}(t; t_1)]^2dt_1 \quad (15)$$

The WHEP technique uses the following expansion for its deterministic kernels,

$$x^{(i)}(t) = x_0^{(i)} + \varepsilon x_1^{(i)} + \varepsilon^2 x_2^{(i)} + \varepsilon^3 x_3^{(i)} + \ldots, i = 0,1 \quad (16)$$
where the first two terms consider the first correction (up to $\varepsilon$), the first three terms represent the second correction (up to $\varepsilon^2$) and so on. This means that we have a lot of corrections possibilities within each order of approximation.

3.1.1 Example. Let us apply $F(t; \omega) = e^{-t} + \varepsilon n(t; \omega)$, in the previous case-study and then solve using the WHEP technique. Some results are illustrated in figures 1 and 2.

![Figure 1-a](image1.png)  
**Figure 1-a.** The first order approximation of the mean at $\varepsilon = .1$ for different correction levels.

![Figure 1-b](image2.png)  
**Figure 1-b.** The first order approximation of the mean at $\varepsilon = .3$ for different correction levels.

![Figure 2-a](image3.png)  
**Figure 2-a.** The first order approximation of the mean at $\varepsilon = .5$ for different correction levels.

![Figure 2-b](image4.png)  
**Figure 2-b.** The first order approximation of the mean at $\varepsilon = 1$ for different correction levels.

4. The Homotopy Perturbation Method (HPM)
In this technique [31-34], a parameter $p \in [0,1]$ is embedded in a homotopy function $v(r, p) : \phi \times [0,1] \rightarrow \mathbb{R}$ which satisfies
\[ H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \]  \hspace{1cm} (18)

where \( u_0 \) is an initial approximation to the solution of the equation

\[ A(u) - f(r) = 0, \quad r \in \phi \]  \hspace{1cm} (19)

with boundary conditions

\[ B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma \]  \hspace{1cm} (20)

in which \( A \) is a nonlinear differential operator which can be decomposed into a linear operator \( L \) and a nonlinear operator \( N \), \( B \) is a boundary operator, \( f(r) \) is a known analytic function and \( \Gamma \) is the boundary of \( \phi \). The homotopy introduces a continuously deformed solution for the case of \( p=0 \), \( L(v) - L(u_0) = 0 \), to the case of \( p=1 \), \( A(v) - f(r) = 0 \), which is the original equation (1). This is the basic idea of the homotopy method which is to continuously deform from a simple problem (and easy to solve) into the difficult problem under study [35]. The HPM is widely used in solving many scientific problems, for example see [36-45].

The basic assumption of the HPM method is that the solution of the original equation (1) can be expanded as a power series in \( p \) as:

\[ v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots \]  \hspace{1cm} (21)

Now, setting \( p=1 \), the approximate solution of equation (21) is obtained as:

\[ u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots \]  \hspace{1cm} (22)

The rate of convergence of the method depends greatly on the initial approximation \( v_0 \) which is considered as the main disadvantage of HPM.

4.1 Example

Considering the same previous example as sub-section 3.1, one can get the following results w.r.t. homotopy perturbation:

\[ A(x) = L(x) + \varepsilon w x^2, \quad L(x) = \dot{x} + 2w \xi \dot{x} + w^2 x, \quad N(x) = \varepsilon x^2, \quad f(r) = F(t; \omega). \]

The homotopy function takes the following form:

\[ H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \]

or equivalently,

\[ L(v) - L(u_0) + p[L(u_0) + \varepsilon w \gamma v^2 - F(t; \omega)] = 0. \]  \hspace{1cm} (23)

Letting \( v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots \), substituting in equation (23) and equating the equal powers of \( p \) in both sides of the equation, one can get the following results:

i) \( L(v_0) = L(y_0) \), in which one may consider the following simple solution:

\[ v_0 = y_0, \quad y_0(0) = x_0, \quad \dot{y}_0(0) = \dot{x}_0. \]

ii) \( L(v_1) = F(t; \omega) - L(v_0) - \varepsilon w \gamma v_0^2, \quad v_1(0) = 0, v'_1(0) = 0. \)

iii) \( L(v_2) = -2\varepsilon w \gamma v_0 v_1, \quad v_2(0) = 0, v'_2(0) = 0. \)

iv) \( L(v_3) = -\varepsilon w^2 (v_1^2 + 2v_0 v_2), \quad v_3(0) = 0, v'_3(0) = 0. \)
The approximate solution is

\[ x(t; \omega) = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots \]

The following first order approximation expression is got.

\[ x(t; \omega) \cong x_1 = v_0 + \int_0^t h(t - s) F(s; \omega) - L(v_0)(s) - \epsilon w^2 v_0^2(s) ds \]  \hspace{1cm} (24)

or zero initial conditions, we can choose  \( v_0 = 0 \) which leads to the following results at  \( w = 1 \)  and \( \zeta = 0.5 \):

\[ x(t; \omega) \cong x_1 = t^2 \]

We can choose  \( v_0 = t^2 \) which leads to the following results at  \( w = 1 \)  and \( \zeta = 0.5 \):

One can notice high deteriorations in the mean.
5. Pickard Approximation

In this technique, the linear part of the differential operator is kept in the left hand side of the equation whereas the rest of the nonlinear terms are moved to the right part. The successive Pickard approximation are processed according to let the L.H.S. as the $n+1$ approximation for the solution process depending on the $n^{th}$ approximation in the R.H.S, $n \geq 0$. Let us illustrate the method through the following example.

5.1 Example:

Solving the quadratic nonlinear oscillatory problem in equation (1) with using Pickard technique, the following successive approximations are obtained:

$$L x_{n+1}(t;\omega) = F(t;\omega) - \varepsilon \omega^2 x_n^2(t;\omega)$$

(25)

which has the general iterative formula:

$$x_{n+1}(t;\omega) = x_{n+1}(0)\phi_1 + x_{n+1}(0)\phi_2 + \int_0^t h(t-s)F(s)ds - \varepsilon \omega^2 \int_0^t h(t-s)x_n^2(s)ds$$

(26)

If the convergence of the process is insured, one can obtain the solution as an $\varepsilon$ series in stochastic terms. Following the iterative formula (26), the first approximation is

$$x_1(t;\omega) = x_1(0)\phi_1 + x_1(0)\phi_2 + \int_0^t h(t-s)F(s)ds - \varepsilon \omega^2 \int_0^t h(t-s)x_0^2(s)ds$$

(27)

where $x_0(t;\omega) = x_0(0)\phi_1 + x_0(0)\phi_2 + \int_0^t h(t-s)F(s)ds$. The ensemble average is

$$E x_1(t;\omega) = x_1(0)\phi_1 + x_1(0)\phi_2 + \int_0^t h(t-s)EF(s)ds - \varepsilon \omega^2 \int_0^t h(t-s)Ex_0^2(s)ds$$

(28)

The covariance is

$$Cov(x_1(t),x_1(\tau)) = \int_0^t h(t-s)h(\tau-z)Cov(F(s),F(z))dz ds$$

(29)

The variance is

$$Var(x_1(t)) = \int_0^t h(t-s)h(t-z)Cov(F(s),F(z))dz ds$$

(30)

The second approximation is obtained in a similar way.

Let us take $F(t;\omega) = e^{-\varepsilon t} + \varepsilon n(t;\omega)$. In this case, the following results are obtained:
6. The direct Perturbation Method
The direct expansion of the solution process is the most conventional and direct one among all the approximation techniques. The basic assumption is

\[ x(t; \omega) = x^{(0)}(t; \omega) + \varepsilon x^{(1)}(t; \omega) + \varepsilon^2 x^{(2)}(t; \omega) + \varepsilon^3 x^{(3)}(t; \omega) + \ldots \]  

(31)

Substituting in the original equation (1) and equating the equal powers of \( \varepsilon \) in both sides of the resulting equation one can get a set of linear differential equations to be solved with their corresponding deterministic initial conditions.

6.1. Example
Applying in the prototype example of this paper, the following results are obtained:
7. The Adomian Decomposition Method

In this method, the differential operator is decomposed such that equation (1) is rewritten in the following form:

\[ Lx(t; \omega) = F(t; \omega) - R(x) - \varepsilon w^2 x^2(t; \omega) , \]  
where \( \frac{Lx(t; \omega)}{dt^2} = \frac{d^2x}{dt^2} , \)

\[ R(x) = (2w \xi \frac{d}{dt} + w^2)(x) \]  ---- \( (34) \)

These decompositions transform the problem into an easier one. The general solution procedure is got using the following:

\[ x = x(0) + x(0)t + \int_0^t F(t)dt + \int_0^t R(x)dt - \varepsilon \int_0^t x^2(t)dt \]  \( (35) \)

The method also decomposes the solution process into

\[ x = x^{(0)}(t; \omega) + x^{(1)}(t; \omega) + x^{(2)}(t; \omega) + ... \]  \( (36) \)

Substituting from equation (36) into (35), one can get the following iterative equations in the unknown kernels of equation (36):

\[ x^{(0)}(t; \omega) = x(0) + x(0)t + \int_0^t F(t; \omega)dt \]  \( (37) \)

\[ x^{(1)}(t; \omega) = -\int_0^t R(x^{(0)})dt - \varepsilon \int_0^t (x^{(0)})^2 dt \]  \( (38) \)

7.1 Example

solving the prototype example, we get the following results:

![Figure 7-a. The first order approximation of the mean.](image1)

![Figure 7-b. The first order variance at different values of \( \varepsilon \).](image2)
One can notice how completely far the behaviour of the obtained results than that of the previous techniques.

8. Conclusions
Concerning the quadratic nonlinearity problem and the prototype example used for illustrating the efficiency of the processed approximation techniques, one may suggest the use of the Pickard approximation which is very rapidly convergent to the solution, if convergent, and when using an efficient computer with an efficient symbolic program. The direct perturbation method produces good results. The WHEP technique seems an efficient one because of its corrections possibilities in spite of being analytically lengthy. The HPM is the easiest in computations, but expectedly depends highly on the initial guess. Concerning only first order approximation, the Adomian decompositions method is the worst among all other executed techniques in this paper.

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