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Finite volume method for a Keller-Segel problem

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Abstract. In this paper, we are interested in the numerical simulation of the mathematical model of Keller-Segel Elliptic-Parabolic problem using finite volume scheme. The finite volume scheme is applied to the elliptic-parabolic model's problem and we have shown under certain assumptions, the existence of a unique and positive approximate solution. Moreover, under adequate regularity assumption of the exact solution, the finite volume scheme is the first order accurate. A good agreement between our numerical simulation and the theoretical results has been obtained.

1. Introduction

Chemotaxis is an important means for cellular communication. It is the influence of chemical substances in the environment on the movement of mobile species. This can lead to strictly oriented movement or to partially oriented and partially tumbling movement. The movement towards a higher concentration of the chemical substance is called positive chemotaxis whereas the movement towards regions of lower chemical concentration is called negative chemotactical movement.

The classical chemotaxis model of the so-called Keller–Segel model defined in system (1) was first introduced by Paltak [9], E. Keller and L. Segel [7]

$$\begin{array}{rr} u_t - \nabla(a\nabla u) + \nabla(\chi u \nabla v) = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \\ \alpha v_t - \Delta v + \tau v + \beta u = 0 & x \in \mathbb{R}^d \end{array} \tag{1}$$

where $u(t,x)$ denotes the density of bacteria in the position $x \in \mathbb{R}^d$ at time t, v the concentration of chemical signal substance, $\alpha \geq 0$ the relaxation time, the parameter χ the sensitivity of cells to the chemoattractant and $a, \tau,\ \beta$ are given smooth functions. As it can be seen, when $\alpha \neq 0$ the model is called Parabolic-Parabolic while it is an Elliptic-Parabolic model when $\alpha = 0$. This model is very simple. It exhibits a profound mathematical structure and only dimension 2 is well understood, especially chemotactic collapse. It has been extensively studied in the last few years, see [5, 6, 11, 10] for a recent status of the problem.

Finite volume method is a class of discretization schemes that has been proven highly successful in approximating the solution of a wide variety of conservation law systems. It is extensively used in fluid mechanics, meteorology, electromagnetic, semi-conductor device simulation, models of biological processes and many other engineering areas governed by conservative systems that can be written in an integral control volume form. Numerical simulations for biological Keller-Segel model's problems were investigated in the following references [1, 2, 4].

The aim of this paper is to study the finite volume scheme applied to the elliptic-parabolic model's problem defined as

$$(\mathrm{P})\begin{cases}(\mathrm{P}_1)\begin{cases}u_t-\Delta u+\operatorname{div}(u\nabla v)=0 & (t,x)\in\mathbb{R}^+\times\Omega\\ u=0 & \partial\Omega\\ u(0,x)=u_0 & x\in\Omega\end{cases}\\ \\ (\mathrm{P}_2)\begin{cases}-\Delta v+\tau v=0 & x\in\Omega\\ v=g & \partial\Omega.\end{cases}\end{cases}\tag{2}$$

where
Assumption 1

(i) Ω is an open bounded polygonal subset of $\mathbb{R}^2$ or $\mathbb{R}^3$
(ii) $\tau\geq 0$,
(iii) $g\in C(\partial\Omega,\mathbb{R})$
(iv) $u_0\in C^2(\overline{\Omega},\mathbb{R})$

First we state the existence and uniqueness of the solution.

Theorem 1.1. *Based on [8]. If $g\in L^{\frac{3}{2}}(\Omega)$, $u_0\in L^2(\Omega)$. Then the problem (P) has a unique solution.*

Remark 1.2. *If the function g is positive on $\partial\Omega$, then the third condition of the assumption assure existence and uniqueness as well as positive of variational solution to problem (P_2) via Lax- Milgran's Theorem and Maximum principle*

To obtain a finite volume discretization of problem (P), we introduce some important notations and definitions:

(i) size(T) =sup{diam(K), $K\in T$}.
(ii) For any $K\in T$ and $\sigma\in\Xi$, $m(K)$ is the d-dimensional Lebesgue measure of K and $m(\sigma)$ is the (d-1)-dimensional measure of σ.
(iii) Denoting the set of interior (resp. boundary) edges by Ξ_{int} (resp. byΞ_{ext})).
(iv) Let $N(K)$ be the set of neighbors of K. If $\sigma=K|L$, we denote d_σ or $d_{K,L}$ the Euclidean distance between x_K and x_L and $d_{K,\sigma}$ is the distance from x_K to σ. If $\sigma\in\Xi_K\cap\Xi_{ext}$, d_σ or $d_{K,\sigma}$ denote the Euclidean distance between x_K and y_σ, where y_σ is the point of intersection of the straight line σ and the straight line which is orthogonal to σ and contains the point x_K.
(v) The transmissibility through σ is defined by $\tau_\sigma=\frac{m(\sigma)}{d_\sigma}$ and $n_{K,\sigma}$ resp (n_K) denotes the outward normal unit vector to σ resp (∂K) .
We note that throughout this paper we consider T is an admissible mesh in the sense of Definition 3.1 in [3].

For the time discretization, we define a temporal partition $t^n=nk$, for $n\in\{0,...,N_k+1\}$ where $k\in Ţ$ is the time step and $N_k=max\{n\in\mathcal{N},nk<Ţ\}$. The value (u_K^n,v_K) is an approximation of $(u(nk,x_K),v(x_K))$ at $(t^n,x_K)\in(0\,Ţ)\times K$.

Definition 1.3. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, and T is an admissible mesh. Define $X(T)$ as the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh.*

Definition 1.4. *(Discrete H_0^1 norm) Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, and T an admissible finite volume mesh. For $v \in X(T)$, define the discrete H_0^1 norm by*

$$\|v\|_{1,T} = \left(\sum_{\sigma\in\Xi} \tau_\sigma \left(D_\sigma v\right)^2 \right)^{\frac{1}{2}} \tag{3}$$

where

$$\begin{aligned} D_\sigma v &= |v_L - v_K| \;\; \forall\sigma \in \Xi_{int}, \quad \sigma = K\,|L\,, \\ D_\sigma v &= |v_K| \quad\;\; if \;\; \sigma \in \Xi_K \cap \Xi_{ext}. \end{aligned}$$

The paper is organized as follows. In section 2, we give some results related to the existence and the uniqueness of the approximate solution of problem (P_2). Under adequate regularity condition on the exact solution of this problem, the finite volume scheme is the first order accurate. In section 3, we prove the same results as in section 2 (existence, the uniqueness, and C^2 error estimate) for problem (P_1) which is based on the proof of Theorem 2.3 and Theorem 4.1 in Ref. [3]. We conclude in section 4 the same results obtained in the previous sections for problem (p). Finally, in Section 5 we illustrate the methods performance against test problems from the literature, and verify that the results are in agreement with our numerical analysis.

2. Finite volume method for a elliptic problem ($\mathbf{P}_2$)

A finite volume scheme for the problem (P_2) is given by

$$\sum_{\sigma\in\Xi_K} F_{K,\sigma} + \tau m\left(K\right) v_K = 0 \quad\;\; \forall K \in T \tag{4}$$

$$F_{K,\sigma} = -\tau_{K|L}\left(v_L - v_K\right) \qquad \forall\sigma \in \Xi_{int} \quad \text{if } \sigma = K|L \tag{5}$$

$$F_{K,\sigma} = -\tau_\sigma\left(v_\sigma - v_K\right) \qquad \forall\sigma \in \Xi_{ext} \cap \partial\Omega \;\; \text{ where } \;\; v_\sigma = g\left(y_\sigma\right) \tag{6}$$

and y_σ is the point of intersection of the straight line σ and the straight line which is orthogonal to σ and contains the point x_K.
We recall some important results of the approximate solution of elliptic problem (P2) (for proof we refer to [3]).

Proposition 2.1. *Based on [3]. Under assumption 1. Let Tbe an admissible mesh . If $g\left(y_\sigma\right) \geq 0$ for all $\sigma \in \Xi_{ext}$, then the solution $(v_K)_{K\in T}$ of (4)-(6) is positive for any $K \in T$.*

Lemma 2.2. *(Existence and uniqueness [3]). Under assumptions 1, let T be an admissible mesh ; there exists a unique solution $(v_K)_{K\in T}$ to the equations (4)-(6).*

Theorem 2.3. *(C^2 error estimate) [3]. Under assumption 1, let T be an admissible mesh, $(v_K)_{K\in T}$ is the solution to (4)-(6). Assume that the unique variational solution v of problem (P_2) satisfied $v \in C^2(\overline{\Omega})$. Let, $e_T \in X(T)$ defined by $e_T = e_K = (v(x_K) - v_K)$ for a.e. $x \in K$ and for all $K \in T$. Then, there exists $C > 0$ depends only on v, τ and Ω such that*

$$\|e_T\|_{1,T} \leq C size(T), \tag{7}$$

$$\|e_T\|_{L^2(\Omega)} \leq C size(T), \tag{8}$$

and

$$\begin{aligned}
&\sum_{\substack{\sigma\in\Xi_{int}\\ \sigma=K|L}} m(\sigma)\, d_\sigma \left(\frac{(v_L-v_K)}{d_\sigma}-\frac{1}{m(\sigma)}\int_\sigma \nabla v(x).n_{K,\sigma} d\delta(x)\right)^2\\
&+\sum_{\substack{\sigma\in\Xi_{ext}\\ \sigma=\overline{K}\cap\partial\Omega}} m(\sigma)\, d_\sigma \left(\frac{(g(y_\sigma)-v_K)}{d_\sigma}-\frac{1}{m(\sigma)}\int_\sigma \nabla v(x).n_{K,\sigma} d\delta(x)\right)^2\\
\leq\ & C\,(size(T))^2.
\end{aligned}$$

3. Finite volume method for a parabolic problem ($\mathbf{P}_1$)

In this section, we applied the following implicit finite volume scheme for the discretization of parabolic problem (P_1) which can be written as follows.

$$m(K)\,\frac{u_K^{n+1}-u_K^n}{k}+\sum_{\sigma\in\Xi_K} F_{K,\sigma}^{n+1}+\sum_{\sigma\in\Xi_K} V_{K,\sigma}u_{\sigma,+}^{n+1}=0, \qquad \forall K\in T,\ \forall n\in\{0,\ldots,N_k\}, \tag{9}$$

$$u_K^0=u_0(x_K) \qquad \forall K\in T, \tag{10}$$

where

$$F_{K,\sigma}^n=-\tau_{K|L}\left(u_L^n-u_K^n\right) \text{ for all } \sigma\in\Xi_{int} \text{ such that } \sigma=K\,|L\,, \text{ for } n\in\{1,...,N_k+1\}\,, \tag{11}$$

$$F_{K,\sigma}^n=\tau_\sigma\left(u_K^n\right) \qquad \forall\sigma\in\Xi_{ext}\cap\partial\Omega, \text{ for } n\in\{0,...,N_k+1\}\,, \tag{12}$$

$$\begin{cases} u_{\sigma,+}^n=u_K^n, & \text{if } V_{K,\sigma}\geq 0,\\ u_{\sigma,+}^n=u_L^n, & \text{if } V_{K,\sigma}<0,\end{cases} \quad \text{for all } \sigma\in\Xi_{int} \quad \text{such that } \sigma=K\,|L\,, \tag{13}$$

$$\begin{cases} u_{\sigma,+}^n=u_K^n & \text{if } V_{K,\sigma}\geq 0,\\ u_{\sigma,+}^n=0 & \text{if } V_{K,\sigma}<0,\end{cases} \quad \text{for all } \sigma\in\Xi_K \quad \text{such that } \ \sigma\subset\partial\Omega, \tag{14}$$

$$V_{K,\sigma}=\tau_\sigma\left(v_L-v_K\right), \quad \text{for all } \sigma\in\Xi_{int} \text{ such that } \sigma=K\,|L\,, \tag{15}$$

and

$$V_{K,\sigma}=\tau_\sigma\left(g(y_\sigma)-v_K\right), \quad \text{for all} \sigma\in\Xi_{ext} \text{ such that } \sigma\subset\partial\Omega, \tag{16}$$

with v_K satisfies the equations (4)-(6) for all $K\in T$.

3.1. Error estimation

In the following Theorem we give a L^∞ estimate as well as the estimate error.

Theorem 3.1. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $\underset{\sim}{T}>0$. Let $u\in C^2(\mathbb{R}_+\times\overline{\Omega},\mathbb{R})$ be defined in (P_1) and $v\in C^2(\overline{\Omega},\mathbb{R})$ defined in (P_2) . Let $u_0\in C^2(\overline{\Omega},\mathbb{R})$ and $g\in C^0(\partial\Omega,\mathbb{R}_+)$. Let T be an admissible mesh and $k\in(0,\underset{\sim}{T})$. Then there exists a unique vector $(u_K)_{K\in T}$ satisfying (9-16). In addition to that, there exists C depends only on u_0, such that*

$$\sup\left\{\left|u_K^n\right|,\ K\in T,\ n\in\{1,...,N_k+1\}\right\}\leq C. \tag{17}$$

Furthermore, let $e_{K,}^n=u(t^n,x_K)-u_K^n$, for $K\in T$, $n\in\{1,...,N_k+1\}$ and $h=size(T)$. Then there exists $C>0$ depends only on u, v, Ω and $\underset{\sim}{T}$ such that

$$\left(\sum_{K\in T} m(K)\left(e_K^n\right)^2\right)^{\frac{1}{2}}\leq C(h+k) \qquad \forall K\in T,\ \ \forall n\in\{1,...,N_k+1\}\,. \tag{18}$$

i) **Existence and uniqueness and L$^\infty$estimate**
For the existence and uniqueness of the solution of finite volume scheme , we can follow the proof of Lemma 3.2 page 42 in[3].
Let us now prove the estimate (17). Let $X_K \in K$, $n \in \{0, ..., N_k\}$ for all $K \in T$. Then, we claim that

$$\min\left\{u_K^{n+1}, \quad K \in T\right\} \geq \min\left\{\min\left\{u_K^n, \quad K \in T\right\}, \quad 0\right\} \tag{19}$$

Indeed, if $\min\left\{u_K^{n+1}, \quad K \in T\right\} < 0$, let $K_0 \in T$ such that $u_{K_0}^{n+1} = \min\left\{u_K^{n+1}, \quad K \in T\right\}$. Since $u_{K_0}^{n+1} < 0$ and replacing in the relation (9) K by K_0 to get

$$m(K)\frac{u_{K_0}^{n+1} - u_{K_0}^n}{k} = -\sum_{\sigma \in \Xi_{K_0}} F_{K_0,\sigma}^{n+1} - \sum_{\sigma \in \Xi_{K_0}} V_{K_0,\sigma}\left(u_{\sigma,+}^{n+1} - u_{K_0}^{n+1}\right) - \sum_{\sigma \in \Xi_{K_0}} V_{K_0,\sigma} u_{K_0}^{n+1}$$

with $K_0 \in T$ and $\forall n \in \{0, ..., N_k\}$.
From the definition of $F_{K_0,\sigma}^{n+1}$ and $u_{\sigma,+}^{n+1}$, we deduce that

$$\sum_{\sigma \in \Xi_{K_0}} F_{K_0,\sigma}^{n+1} \leq 0 \text{ and } \sum_{\sigma \in \Xi_{K_0}} V_{K_0,\sigma}\left(u_{\sigma,+}^{n+1} - u_{K_0}^{n+1}\right) \leq 0.$$

Next, in view the Proposition 2.1, the function v is positive verifying -$\Delta v + \tau v = 0$ then we find

$$\sum_{\sigma \in \Xi_{K_0}} V_{K_0,\sigma} u_{K_0}^{n+1} = u_{K_0}^{n+1} \int_{\partial K_0} \nabla v.n_{K_0}(x)\gamma(x) = \tau u_{K_0}^{n+1} \int_{K_0} v \, dx \leq 0.$$

Thus, we have

$$u_{K_0}^{n+1} \geq u_{K_0}^n,$$

this proves (19) and by induction, we deduce that:

$$\min\left\{u_K^{n+1}, K \in T\right\} \geq \min\left\{\min\left\{u_K^0, K \in T\right\}, 0\right\}. \tag{20}$$

This implies that

$$\max\left\{u_K^{n+1}, K \in T\right\} \leq \max\left\{\max\left\{u_K^0, K \in T\right\}, 0\right\}.$$

So, the inequality (17) is satisfied with C depends only on u_0.
ii) **Error estimate**
One uses the regularity of the data and the solution to write an equation for the error $e_{K,}^n = u(t^n, x_K) - u_K^n$, defined for $K \in T$ and $n \in \{1, ..., N_k + 1\}$. Note that $e_K^0 = 0$ for $K \in T$. Let $n \in \{1, ..., N_k\}$. Integrating the first equation of problem (P$_1$) over each control volume K of T, at time $t = t^{n+1}$:

$$\int_K u_t\left(t^{n+1}, x\right) dx - \int_{\partial K}\left(\nabla u\left(t^{n+1}, x\right) - u\left(t^{n+1}, x\right) \nabla v(x)\right).n_K(x)\, d\gamma(x) = 0. \tag{21}$$

Subtracting (9) to (21) yields:

$$m(K)\frac{e_K^{n+1} - e_K^n}{k} + \sum_{\sigma \in \Xi_K}\left(G_{K,\sigma}^{n+1} + W_{K,\sigma}^{n+1}\right) = -\sum_{\sigma \in \Xi_K} mes(\sigma)\left(R_{K,\sigma}^n + r_{K,\sigma}^n\right) - S_K^n \quad \forall K \in T, \tag{22}$$

where

$$\begin{aligned} G_{K,\sigma}^{n+1} &= -\tau_\sigma \left(e_L^{n+1} - e_K^{n+1}\right) \quad \forall \sigma \in \Xi_{int} \cap \Xi_K, \quad \sigma = K\,|L\,, \\ G_{K,\sigma}^{n+1} &= \tau_\sigma e_K^{n+1} \quad \forall \sigma \in \Xi_{ext} \cap \Xi_K, \\ W_{K,\sigma}^{n+1} &= W_{K,\sigma,1}^{n+1} + W_{K,\sigma,2}^{n+1}, \end{aligned}$$

with

$$\begin{cases} e_K = (v\,(x_K) - v_K) \\ W_{K,\sigma,1}^{n+1} = \tau_\sigma\,(e_L - e_K)\,u_{\sigma,+}^{n+1} = \tau_\sigma D_\sigma\,(e)\,u_{\sigma,+}^{n+1}, \\ W_{K,\sigma,2}^{n+1} = \tau_\sigma\,(v(x_L) - v(x_K))\left(u\left(t^{n+1}, x_{\sigma,+}\right) - u_{\sigma,+}^{n+1}\right) = \tau_\sigma D_\sigma\,(v)\left(e_{\sigma,+}^{n+1}\right), \end{cases}$$

where $x_{\sigma,+} = x_K$ (resp. x_L) $\sigma \in \Xi_{int}$, $\sigma = K\,|L$ and $\nabla v.n_{K,\sigma} \geq 0$ (resp. $\nabla v.n_{K,\sigma} < 0$) and $x_{\sigma,+} = x_K$ (resp. y_σ) if $\sigma \in \Xi_{ext} \cap \Xi_K$, and $\nabla v.n_{K,\sigma} \geq 0$ (resp. $\nabla v.n_{K,\sigma} < 0$).
Using a Taylor expansion and as the regularity of u is considered, then there exists C_1 and C_2 which C_1 depends only on u and Ţ and C_2 depends only on u, v and Ţ such that for all $x \in K$ and all $K \in T$ we have

$$S_K^n = \int_K u_t\left(t^{n+1}, x\right) dx - m\,(K)\,\frac{\left(u\left(t^{n+1}, x_K\right) - u(t^{n+1}, x_K)\right)}{k},$$

$$\begin{aligned} -m\,(\sigma)\,R_{K,\sigma}^n &= \tau_\sigma\left(u\left(t^{n+1}, x_K\right) - u(t^{n+1}, x_L)\right) + \int_\sigma \nabla u\left(t^{n+1}, x\right).n_{K,\sigma} d\delta(x) \;\; \text{if } \sigma = K\,|L \in \Xi_{int}, \\ -m\,(\sigma)\,R_{K,\sigma}^n &= \tau_\sigma\left(u\left(t^{n+1}, x_K\right)\right) + \int_\sigma \nabla u\left(t^{n+1}, x\right).n_{K,\sigma} d\delta(x) \;\; \text{if } \sigma \in \Xi_{ext} \cap \Xi_K, \end{aligned}$$

$$\begin{aligned} m\,(\sigma)\,r_{K,\sigma}^n &= \tau_\sigma\,(v\,(x_K) - v(x_L))u\,(x_{\sigma,+}) + \int_\sigma \nabla v(x).n_{K,\sigma} u\left(t^{n+1}, x\right) d\delta(x) \;\; \text{if } \sigma = K|L \in \Xi_{int}, \\ m\,(\sigma)\,r_{K,\sigma}^n &= \tau_\sigma(v\,(x_K) - g\,(y_\sigma))u\,(x_{\sigma,+}) + \int_\sigma \nabla v(x).n_{K,\sigma} u\left(t^{n+1}, x\right) d\delta(x) \;\; \text{if } \sigma \in \Xi_{ext} \cap \Xi_K, \end{aligned}$$

where

$$S_K^n \leq m(K)C_1\,(h + k)\,, \tag{23}$$

and

$$\left|R_{K,\sigma}^n\right| + \left|r_{K,\sigma}^n\right| \leq C_2 h \qquad \forall K \in T \text{ and } \; \sigma \in \Xi_K. \tag{24}$$

Multiplying (22) by e_K^{n+1}, summing for $K \in T$, and noting that

$$\begin{aligned} \sum_{K\in T}\sum_{\sigma\in\Xi_K} G_{K,\sigma}^{n+1} e_K^{n+1} &= \sum_{\sigma\in\Xi} \tau_\sigma \left(e_K^{n+1}\right)^2 = \left\|e_T^{n+1}\right\|_{1,\tau}^2, \\ \sum_{K\in T}\sum_{\sigma\in\Xi_k} W_{K,\sigma,2}^{n+1} e_K^{n+1} &\geq 0, \\ \sum_{K\in T} m\,(K)\left(e_K^{n+1}\right)^2 &= \left\|e_T^{n+1}\right\|_{L^2(\Omega)}^2, \end{aligned}$$

where $e_T^n \in X\,(T)$, $e_T^n\,(x) = e_K^n$ for a.e. $x \in K$, we obtain

$$\begin{aligned} \left\|e_T^{n+1}\right\|_{L^2(\Omega)}^2 + k\left\|e_T^{n+1}\right\|_{1,T}^2 \leq -k \textstyle\sum_{K\in T}\sum_{\sigma\in\Xi_K} mes(\sigma)\left(R_{K,\sigma}^n + r_{K,\sigma}^n\right) e_K^{n+1} + \sum_{K\in T} k\,|S_K^n|\,e_K^{n+1} \\ -k \textstyle\sum_{K\in T}\sum_{\sigma\in\Xi_K} W_{K,\sigma,1}\, e_K^{n+1} + \sum_{K\in T} mes\,(K)\,e_K^{n+1} e_K^n. \end{aligned} \tag{25}$$

Using (23), (24) and Cauchy-Schwarz's inequality yields, with some C_3 only depending on u, v, Ω and Ţ

$$\begin{array}{c}\left\|e_T^{n+1}\right\|_{L^2(\Omega)}^2 + \frac{k}{2}\left\|e_T^{n+1}\right\|_{1,T}^2 \\ \leq C_3 k h^2 + C_1 k\,(h+k)\sum_{K\in T} m\,(K)\left|e_K^{n+1}\right| - k\sum_{K\in T}\sum_{\sigma\in\Xi_K} W_{K,\sigma,1}e_K^{n+1} + \sum_{K\in T} m\,(K)\,e_K^{n+1}e_K^n.\end{array} \quad (26)$$

Applying again the Cauchy-Schwarz's inequality and the estimate $\left(ab \leq \frac{1}{2}\left(a^2+b^2\right)\right)$ to estimate

$$\begin{aligned}(h+k)\,k\sum_{K\in T} mes(K)e_K^{n+1} &\leq (h+k)\,k\left(\sum_{K\in T} mes(K)\right)^{\frac{1}{2}}\left(\sum_{K\in T} mes(K)\left(e_K^{n+1}\right)^2\right)^{\frac{1}{2}} \\ &= (h+k)\,k\,(mes(\Omega))^{\frac{1}{2}}\left\|e_T^{n+1}\right\|_{L^2(\Omega)}.\end{aligned}$$

and

$$\begin{aligned}\left|\sum_{K\in T} m\,(K)\,e_K^{n+1}e_K^n\right| &\leq \frac{1}{2}\sum_{K\in T} m\,(K)\left(e_K^{n+1}\right)^2 + \frac{1}{2}\sum_{K\in T} m\,(K)\,(e_K^n)^2 \\ &= \frac{1}{2}\left\|e_T^{n+1}\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|e_T^n\right\|_{L^2(\Omega)}^2.\end{aligned}$$

Using the estimation (7) of the Theorem 2.3 to find

$$\begin{aligned}\sum_{K\in T}\sum_{\sigma\in\Xi_K} W_{K,\sigma,1}e_{\sigma,+}^{n+1} &= \sum_{\sigma\in\Xi}\tau_\sigma D_\sigma\,(e)\,u_{\sigma,+}{}^{n+1}\left(e_{\sigma,+}^{n+1} - e_{\sigma,-}^{n+1}\right) \\ &\leq C\,|e_T\|_{1,T}\left\|e_T^{n+1}\right\|_{1,T} \\ &\leq C_4 h\left\|e_T^{n+1}\right\|_{1,T},\end{aligned}$$

where

$$\left\{\begin{array}{llll} e_{\sigma,-}^{n+1} = e_K^n & \text{if } V_{K,\sigma} \leq 0, & e_{\sigma,+}^{n+1} = e_K^n & \text{if } V_{K,\sigma} \geq 0 \\ e_{\sigma,-}^{n+1} = e_L & \text{if } V_{K,\sigma} > 0, & e_{\sigma,+}^{n+1} = e_L & \text{if } V_{K,\sigma} < 0\end{array}\right. \quad \text{for all } \sigma\in\Xi_K$$

and $V_{K,\sigma} = \tau_\sigma D_\delta\,(v)$. We note that if $\sigma\in\Xi_{ext}$ then $e_{\sigma,-}^{n+1} = e_{\sigma,+}^{n+1} = 0$.
Then, we obtain

$$\begin{array}{c}\frac{1}{2}\left\|e_T^{n+1}\right\|_{L^2(\Omega)}^2 + \frac{k}{2}\left\|e_T^{n+1}\right\|_{1,T}^2 \\ \leq C_3 k h^2 + C_1\,(h+k)\,k\,(mes(\Omega))^{\frac{1}{2}}\left\|e_T^{n+1}\right\|_{L^2(\Omega)} + C_4 h k\left\|e_T^{n+1}\right\|_{1,T} + \frac{1}{2}\left\|e_T^n\right\|_{L^2(\Omega)}^2.\end{array} \quad (27)$$

So, applying again the estimate $\left(ab \leq \frac{1}{2}\left(a^2+b^2\right)\right)$ to $C_4 k h\left\|e_T^{n+1}\right\|_{1,T}$ to deduce

$$\left\|e_T^{n+1}\right\|_{L^2(\Omega)}^2 \leq C_5 k h^2 + 2C_1\,(h+k)\,k\,(mes(\Omega))^{\frac{1}{2}}\left\|e_T^{n+1}\right\|_{L^2(\Omega)} + \left\|e_T^n\right\|_{L^2(\Omega)}^2. \quad (28)$$

Since, we obtain

$$\left\|e_T^{n+1}\right\|_{L^2(\Omega)}^2 \leq C_6\left(h^2 k + (h+k)\,k\left\|e_K^{n+1}\right\|_{L^2(\Omega)}\right) + \left\|e_T^n\right\|_{L^2(\Omega)}^2. \quad (29)$$

Similarly as in proof of Theorem 4.1 in [3], we get the estimate (18).

Proposition 3.2. *Under assumption 1. Let T be an admissible mesh. If $u_0\,(x) \geq 0$ for all $K\in T$, then the solution $(u_K)_{K\in T}$ of (9)-(16) satisfied $u_K \geq 0$ for all $K\in T$.*

Proof. We have $u_0\,(x) \geq 0$ and from the relation (20), we deduce that

$$\min\left\{u_K^{n+1}, K\in T\right\} \geq \min\left\{\min\left\{u_K^0, K\in T\right\}, 0\right\} = 0,$$

therefore, $u_K^{n+1} \geq 0,\ \forall K\in T$. So this proves this Proposition. □

4. Finite volume method for a Elliptic-parabolic problem (P)

A finite volume scheme of the problem (P) can be defined by the following set of equations:

$$m(K)\frac{u_K^{n+1}-u_K^n}{k}+\sum_{\sigma\in\Xi_K}F_{K,\sigma}^{n+1}-\sum_{\sigma\in\Xi_K}F_{K,\sigma}u_{\sigma,+}^{n+1}=0 \quad \forall K\in T, \tag{30}$$

$$\sum_{\sigma\in\Xi_K}F_{K,\sigma}+\tau m(K)\,v_K=0 \quad \forall K\in T, \tag{31}$$

$$u_K^0=u_0(x_K) \quad \forall K\in T, \tag{32}$$

$$F_{K,\sigma}^n=-\tau_{K|L}\left(u_L^n-u_K^n\right) \ \forall\sigma\in\Xi_{int} \quad \text{if } \sigma=K|L\,, \tag{33}$$

$$F_{K,\sigma}^n=\tau_\sigma\left(u_K^n\right) \quad \forall\sigma\in\Xi_{ext}\cap\Xi_K, \tag{34}$$

$$F_{K,\sigma}=-\tau_{K|L}\left(v_L-v_K\right) \quad \forall\sigma\in\Xi_{int} \quad \text{if } \sigma=K|L\,, \tag{35}$$

$$F_{K,\sigma}=-\tau_\sigma\left(g\left(y_\sigma\right)-v_K\right) \quad \forall\sigma\in\Xi_{ext} \ \text{ such that } \ \sigma\in\Xi_K. \tag{36}$$

with $u_{\sigma,+}^{n+1}$ is defined by the relations (13)-(14),(we note that $F_{K,\sigma}=-V_{K,\sigma}$). In this work, we assume that the unknowns u and v are constants over each control volume K of the mesh T.

Now, from sections 2 and 3 we are able to conclude that the main result is represented by the following Theorem .

Theorem 4.1. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $\underset{\cdot}{T}>0$. Let $(u,v)\in C^2(\mathbb{R}_+\times\overline{\Omega},\mathbb{R})\times C^2(\overline{\Omega},\mathbb{R})$ be defined in (P). Let $u_0\in C^2(\overline{\Omega},\mathbb{R})$ and $g\in C^0(\partial\Omega,\mathbb{R}_+)$. Let T be an admissible mesh and $k\in(0,\underset{\cdot}{T})$. Then there exists a unique vector $(u_K,v_K)_{K\in T}$ satisfying (30)-(36). Furthermore, let $e_K^n=u(t^n,x_K)-u_K^n$ and $e_K=v(x_K)-v_K$ for $K\in T$ and $n\in\{1,...,N_k+1\}$ and $h=size(T)$. Then there exists C only depending on u, v and $\underset{\cdot}{T}$ such that*

$$\left(\sum_{K\in T}m(K)\left(e_K^n\right)^2\right)^{\frac12}+\left(\sum_{K\in T}m(K)\left(e_K\right)^2\right)^{\frac12}\le C(h+k), \qquad \forall n\in\{1,...,N_k+1\}$$

The demonstration of this Theorem is a consequence of Lemma 2.2, Theorems 2.3 and 3.1.

Proposition 4.2. *Under assumption 1. Let T be an admissible mesh. If $u_0(x)\ge 0$ for all $K\in T$, then the solution $(u_K,v_K)_{K\in T}$ of (30)-(36) is positive for all $K\in T$, ($u_K\ge 0$ and $v_K\ge 0\ \forall K\in T$)*

The proof of this Proposition follows from the Propositions 2.1 and 3.2.

5. Numerical simulations

In this section we present the results of numerical simulations. Two examples of problem (P) are considered.

- In the first example u_0 and g are positive such that

$$\begin{aligned} u_0 &= \sin(\pi x)\sin(\pi y)\,,\\ g &= \begin{cases} v(0,y)=0\\ v(1,y)=1\\ v(x,0)=v(x,1)=1-x^2\end{cases}\end{aligned}$$

- In the second example we choose

$$
\begin{aligned}
u_0 &= xy(1-x)(1-y) \\
g &= \begin{cases} v(0,y) = 0 \\ v(1,y) = \sin\left(\frac{\pi}{2}y\right) \\ v(x,0) = 0 \\ v(x,1) = \sin\left(\frac{\pi}{2}x\right) \end{cases}
\end{aligned}
$$

In our numerical scheme of the finite volume method, we consider $\Omega = [01]^2$ and the time (in second) $t \in [0, 0.2]$. The domain Ω is discretized on a regular square grid of 21×21 nodes so that the spatial step size is $h = 0.05$. For the time, we have used the time step size $k = 0.02s$.

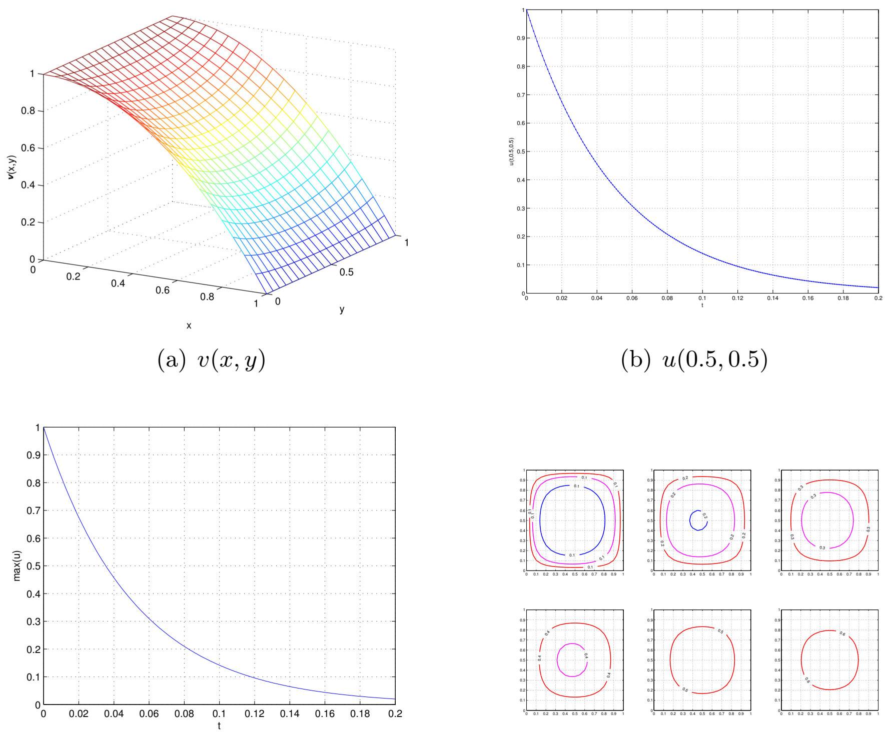
Discussion

In both examples the function g is chosen to be positive.

(i) In figures 1(a) and 2(a), the function v is also positive. Therefore, the principal of maximum is verified.

(ii) In figures 1(b) and 2(b), the initial condition u_0 is positive and the point $(x, y) = (0.5, 0.5)$. The solution is given as a function of time t. It is clear in this case that the solution u of the problem (P) is positive. Therefore, the principal of maximum is verified.

(iii) In figures 1(c) and 2(c), the initial condition u_0 is positive. It is clear from these figures that the maximum of the solution u of the problem (P) remains positive during the time evolution $t \in [0\ 1]$. Moreover, the solution u is also positive. Therefore, the principal of maximum is verified.



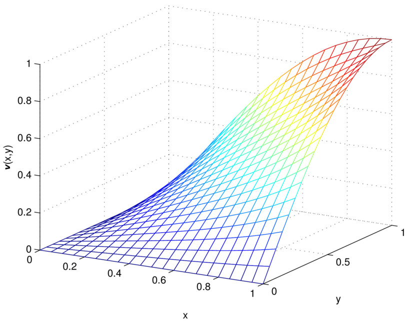
(a) $v(x,y)$

(b) $u(0.5, 0.5)$

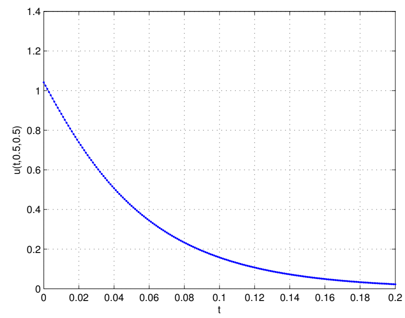
(c) Max of u

(d) Countour plot

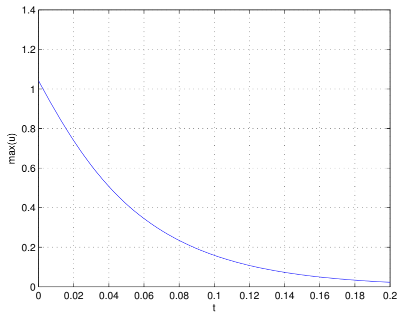
Figure 1. Numerical solution for the Example 1



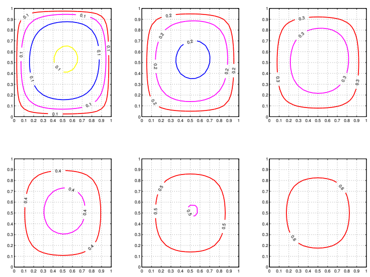
(a) $v(x, y)$



(b) $u(0.5, 0.5)$

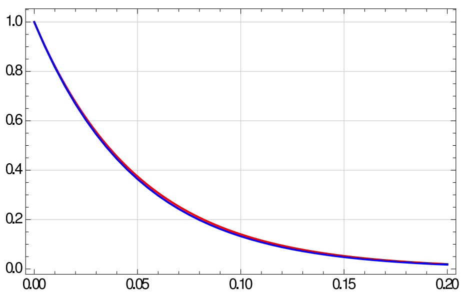


(c) Max of u

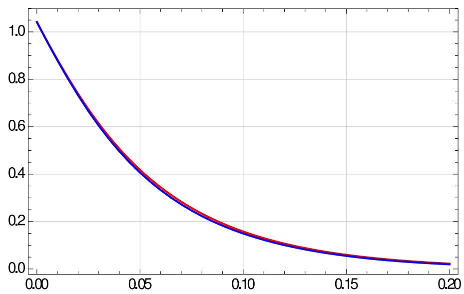


(d) Countour plot

Figure 2. Numerical solution for the Example 2



(a) Example 1: $u(t, 0.5, 0.5)$



(b) Example 2: $100u(t, 0.5, 0.5)/6$

Figure 3. Comparison between the numerical solutions obtained by finite volume method and the finite element method.

(iv) The figures 1(d) and 2(d) represent the contour plot for the solution. One can see that the solution is always positive.

(v) We remark that for positive values of u_0 and g , the solution is also positive. This means that the principle of maximum remains numerically valid .
We compare the numerical solutions obtained by finite volume method and the finite element method (from Mathematica 10.2T) at the coordinates $(0.5, 0.5)$. There is a good agreement between these two solutions. (see Fig. 3a and 3b

(vi) From the figures. 1(d) and 2(d) we see that during the evolution the amplitude of u decreases.

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References
[1] Marianne Bessemoulin-Chatard and Ansgar Jüngel. A finite volume scheme for a keller–segel model with additional cross-diffusion. *IMA Journal of Numerical Analysis*, page drs061, 2013.
[2] Adrien Blanchet, Vincent Calvez, and José A Carrillo. Convergence of the mass-transport steepest descent scheme for the subcritical patlak–keller–segel model. *SIAM Journal on Numerical Analysis*, 46(2):691–721, 2008.
[3] Robert Eymard, Thierry Gallouët, and Raphaèle Herbin. Finite volume methods. *Handbook of numerical analysis*, 7:713–1018, 2000.
[4] Ibrahim Fatkullin. A study of blow-ups in the keller–segel model of chemotaxis. *Nonlinearity*, 26(1):81, 2012.
[5] Miguel A Herrero and J J L Velázquez. A blow up mechanism for a chemotaxis model. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, 24(4):633–683, 1997.
[6] T Hillen and K Painter. Global existence for a parabolic chemotaxis model with prevention of overcrowding. *Advances in Applied Mathematics*, 26(4):280 – 301, 2001.
[7] Evelyn F Keller and Lee A Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, 26:399–415, 1970.
[8] C Messikh, A Guesmia, and S Saadi. Global existence and uniqueness of the weak solution in keller segel model. *GJSFR*, 14:146–55, 2014.
[9] Clifford S Patlak. Random walk with persistence and external bias. *The bulletin of mathematical biophysics*, 15(3):311–338, 1953.
[10] Michael Winkler. Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity. *Mathematische Nachrichten*, 283:1664–1673, 2010.
[11] Michael Winkler. Aggregation vs. global diffusive behavior in the higher-dimensional keller segel model. *Journal of Differential Equations*, 248(12):2889–2905, 2010.