Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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To cite this article: Francisco Delgado 2017 *J. Phys.: Conf. Ser.* **936** 012083

IOP Conf. Series: Journal of Physics: Conf. Series **936** (2017) 012083 doi:10.1088/1742-6596/936/1/012083

Generalized Bell states map physical systems' quantum evolution into a grammar for quantum information processing

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Abstract. Quantum information processing should be generated through control of quantum evolution for physical systems being used as resources, such as superconducting circuits, spin-spin couplings in ions and artificial anyons in electronic gases. They have a quantum dynamics which should be translated into more natural languages for quantum information processing. On this terrain, this language should let to establish manipulation operations on the associated quantum information states as classical information processing does. This work shows how a kind of processing operations can be settled and implemented for quantum states design and quantum processing for systems fulfilling a $SU(2)$ reduction in their dynamics.

1. Introduction

Synergy between quantum mechanics and computer science is generating disruptive developments. In the gate array version of quantum processing, appropriate logical gates are essential. They adopt forms inspired by classical computation, despite their construction is based on the behavior of quantum systems and normally is based on universal sets of quantum gates with which any processing task can be expressed [1, 2]. Commonly, some universal gates still should be constructed as iterative steps of controlled physical evolutions. Methods as Cosine-Sine decomposition [3], multiplexor gates [4], Gray codes basis [5] or $q-$deformed algebras [6] are used as approaches. In the $SU(2)$ reduction approach [7, 8] for two qubits, which has been extended to multipartite systems [9], unitary factorization and natural gates for two-qubits has been obtained, letting to define basic operations closer to classical computation. This paper exploits the $SU(2)$ decomposition [10] to address the evolution into basic processing tasks for state design. The second section presents the technical details of $SU(2)$ reduction to give a generic processing gate. The third section presents key operations for quantum processing. The fourth section presents examples of quantum state design. Last section set the conclusions.

2. $SU(2)$ decomposition technical details

$SU(2)$ decomposition was developed for a generic Hamiltonian combining $2d$ two-level interacting subsystems resembling the Heisenberg-Ising interaction and including local driven operations [11]. It is reached with an appropriate basis to reduce the Hamiltonian into a 2×2-block matrix and as a consequence, the evolution matrix in a block matrix with blocks in $SU(2)$:

$$H = \left(\begin{array}{c|c|c|c} \mathbf{S}_{H1} & \mathbf{0} & ... & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{S}_{H2} & ... & \mathbf{0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0} & \mathbf{0} & ... & \mathbf{S}_{H2^{2d-1}} \end{array} \right) \quad \Rightarrow \quad U = \left(\begin{array}{c|c|c|c} \mathbf{S}_{U1} & \mathbf{0} & ... & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{S}_{U2} & ... & \mathbf{0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0} & \mathbf{0} & ... & \mathbf{S}_{U2^{2d-1}} \end{array} \right) \tag{1}$$

the inheritance for U is due to the group properties of block matrices via time ordered integral $\mathbf{S}_{Ui} = \tau\{e^{-\frac{i}{\hbar}\int_0^t \mathbf{S}_{Hi} dt'}\}$. Basis elements should be rearranged to show the blocks as in (1). For:

$$H^{(j,k')} = H_D + H_{ND}^{(j,k')} \quad \text{with}: \quad H_{ND}^{(j,k')} = \sum_{t'=0}^{1} h_{(j4^{k'+dt'-1})_4^{2d}} \bigotimes_{s=1}^{2d} \sigma_{(j4^{k'+dt'-1})_{4,s}^{2d}} \tag{2}$$
$$H_D \equiv \sum_{i'=1}^{3}\sum_{k=1}^{d} h_{(i'(4^{k-1}+4^{k+d-1}))_4^{2d}} \bigotimes_{s=1}^{2d} \sigma_{(i'(4^{k-1}+4^{k+d-1}))_{4,s}^{2d}}$$

as Hamiltonian ($h_{\{i_k\}}$ are time-dependent real functions), including a) non-local interactions H_D (spin-spin couplings like) between all paired parts $[k, k+d]$ (called *correspondent*) in three directions $i' = 1, 2, 3$, giving the diagonal entries; and b) $H_{ND}^{(j,k')}$, a couple of local interactions on the correspondent pair $[k', k'+d]$ in direction j, giving the diagonal-off entries [9]. For:

$$\left|\Psi_{\mathcal{I}_4^d}\right\rangle = \frac{1}{\sqrt{2^d}} \sum_{\mathcal{E},\mathcal{D}=0}^{2^d-1} (\tilde{\sigma}_{i_1} \otimes ... \otimes \tilde{\sigma}_{i_d})_{\mathcal{E}_2^d,\mathcal{D}_2^d} \left|\mathcal{E}_2^d\right\rangle \otimes \left|\mathcal{D}_2^d\right\rangle \tag{3}$$

the generalized Bell states (GBS) basis with $\tilde{\sigma}_{i\neq 2} = \sigma_i, \tilde{\sigma}_2 = i\sigma_2$ [12], the decomposition is achieved. $\mathcal{E}_2^d, \mathcal{D}_2^d$ are base-2 numbers with d digits ($\mathcal{E}, \mathcal{D} \in \{0, 1, ..., 2^d - 1\}$) representing $\{\epsilon_1, ..., \epsilon_d\}, \{\delta_1, ..., \delta_d\}$, with $\epsilon_s, \delta_s \in \{0, 1\}$ (digits appear reversed in $\mathcal{E}_2^d$ or $\mathcal{I}_4^d$), giving:

$$\left\langle \Psi_{\mathcal{I}_4^d} | H^{(j,k')} | \Psi_{\mathcal{K}_4^d} \right\rangle = \delta_{\mathcal{I}\mathcal{K}} \sum_{i'=1}^{3}\sum_{k''=1}^{d} c_{i',i'}^{i_{k''},i_{k''}} h_{(i'(4^{k''-1}+4^{k''+d-1}))_4^{2d}} +$$
$$\sum_{t'=0}^{1} \delta_{\mathcal{I}\mathcal{K}}^{\{k'\}} \mathcal{F}_{j,k'}^{j\delta_{0,t'},j\delta_{1,t'}} h_{(j4^{k'+dt'-1})_4^{2d}} \equiv H_{D\mathcal{I}\mathcal{K}} + H_{ND\ \mathcal{I}\mathcal{K}}^{(j,k')} \tag{4}$$

with the coefficients $c_{i',i'}^{i_{k''},i_{k''}}$, $\mathcal{F}_{j,k'}^{j\delta_{0,t'},j\delta_{1,t'}}$ reported in [11]. The most important aspect is the establishment of blocks. Due to the coefficient properties, they are located in the rows $\mathcal{I} = \{i_1 i_2 ... i_n\}$ and $\mathcal{I}' = \{i_1' i_2' ... i_n'\}$ differing in only one subscript $i_k' \neq i_k$ with the correspondence rule: $0 \leftrightarrow j$, $i \leftrightarrow k$, being j the direction of local driven interactions in (2) and i, j, k a permutation of $1, 2, 3$. This is the relation rule for the GBS basis elements in the blocks. Hilbert space $\mathcal{H}^{2d}$ becomes the direct sum of 2^{2d-1} subspaces generated by these GBS basis elements. There, the dynamics mixes the probabilities only inside each subspace. Because states in the basis are entangled (non-maximal), each block $\mathbf{S}_{Ui}$ in (1) includes the controlled gate operations. For the time-independent case, in terms of $\mathbf{S}_{H\mathcal{I},\mathcal{I}'} = \{h_{ij}\}$ in (4):

$$\mathbf{S}_{U\mathcal{I},\mathcal{I}'} = e^{i\frac{h_{11}+h_{22}}{2\hbar}t} \begin{pmatrix} \cos\omega t + i\frac{h_{11}-h_{22}}{2\hbar\omega}\sin\omega t & i\frac{h_{12}}{\hbar\omega}\sin\omega t \\ i\frac{h_{12}^*}{\hbar\omega}\sin\omega t & \cos\omega t - i\frac{h_{11}-h_{22}}{2\hbar\omega}\sin\omega t \end{pmatrix} \tag{5}$$

with: $\hbar\omega = \sqrt{|h_{12}|^2 + \frac{1}{4}|h_{11} - h_{22}|^2}$. Clearly $\mathbf{S}_{U\mathcal{I},\mathcal{I}'} \in U(1) \times SU(2)$. $\mathcal{F}_{j,k'}^{j_d,j_{d+s}}$ in (4) is imaginary only if j_d or j_{d+s} are 2, then only one n_1 or n_2 is non-zero.

IOP Conf. Series: Journal of Physics: Conf. Series **936** (2017) 012083 doi:10.1088/1742-6596/936/1/012083

3. Stating notable manipulation operations in quantum information

Note $\mathbf{S}_{U\mathcal{I},\mathcal{I}'}$ byself is a general $SU(2)$ operation in the form of a mixing matrix. Other notable operations are easily achievable. The first one with $\frac{h_{11}+h_{22}}{2\hbar\omega} = (\frac{\alpha}{m} - 1)\pi, \omega t = m\pi; n, m \in \mathbf{Z}$, the quasi-identity gate $\mathbf{S}_{U\mathcal{I},\mathcal{I}'} = e^{i\alpha\pi}\mathbf{I}_{\mathcal{I},\mathcal{I}'} \equiv I^{\alpha}_{\mathcal{I},\mathcal{I}'}$. The second one is a family of operations achieved with $\omega t = \frac{2n+1}{2}\pi$, $\frac{h_{11}-h_{22}}{2\hbar\omega} = \epsilon$, $\frac{h_{12}}{\hbar\omega} = i^c\delta$, $\frac{h_{11}+h_{22}}{2\hbar\omega} = 2(m - \frac{1}{2}); n, m \in \mathbf{Z}, c \in \{0, 1\}, \delta \in \mathbf{R}$, where $\epsilon^2 + \delta^2 = 1$. c depends on the direction of local interaction. We get for $\mathbf{S}_{U\mathcal{I},\mathcal{I}'}$ on the GBS basis:

$$\mathbf{H}^c_m(\delta, \epsilon)_{\mathcal{I},\mathcal{I}'} = (-1)^m \begin{pmatrix} \epsilon & i^c\delta \\ (-i)^c\delta & -\epsilon \end{pmatrix} \tag{6}$$

It is widely versatile: a) if $s_\epsilon\epsilon = \delta = \frac{1}{\sqrt{2}}$ ($s_\epsilon = \mathrm{sign}(\epsilon)$, $\pm$ in the notation), a Hadamard-like gate $H^{m,c,\mathrm{sign}(\epsilon)}_{\mathcal{I},\mathcal{I}'}$ is obtained; b) if $\delta = 1$, the exchange-like gate $E^{m,c}_{\mathcal{I},\mathcal{I}'}$ is obtained [7], a limit case for the independent-time case when $h_{12} \gg h_{11} - h_{22}$ (5). Despite there are only two independent blocks in the evolution matrix, they have $3d + 3$ parameters (including time and Hamiltonian terms involved). By rearranging the parts of correspondent pairs and controlling non-local and local strengths as well as their directions, global operations could be made on the entire quantum information space of the system.

4. Quantum states design: maximal entangled states

Quantum design and manipulation become reduced to operations transforming indexes and changing probability amplitudes in the GBS basis elements. Depending on the path of the state to manipulate, this task could be still hard. Despite one index exchange on the states are limited to reach general states on $2d-$qubits, by rearranging the pairing of qubits, we can extend the entanglement manipulation. To illustrate the procedure for four qubits ($d = 2$), we show the construction of $|GHZ\rangle^4$ and $|W\rangle^4$, which could be expressed in the GBS basis as:

$$|GHZ\rangle^4 = \frac{1}{\sqrt{2}}(|\Psi_{0,0}\rangle + |\Psi_{3,3}\rangle) = \frac{1}{\sqrt{2}} \sum_{\mathcal{I}\in\{0,15\}} |\Psi_\mathcal{I}\rangle \tag{7}$$

$$|W\rangle^4 = \frac{1}{2}(\,|\Psi_{1,0}\rangle + |\Psi_{0,1}\rangle + |\Psi_{3,1}\rangle + |\Psi_{1,3}\rangle) = \frac{1}{2} \sum_{\mathcal{I}\in\{1,4,7,13\}} |\Psi_\mathcal{I}\rangle \tag{8}$$

In terms of the previous operations and departing from $|0000\rangle$, the steps to achieve $|GHZ\rangle^4$ are:

$$\begin{aligned}
|0000\rangle \quad & \underset{H^{0,0,+}_{0,3} \oplus H^{0,0,+}_{12,15}}{\xrightarrow{H^{(3,1)}}} \quad \frac{1}{\sqrt{2}}|\Psi_0\rangle_1 \otimes (|\Psi_0\rangle_2 + |\Psi_3\rangle_2) \\
& \underset{H^{0,0,+}_{0,12}}{\xrightarrow{H^{(3,2)}}} \quad |\Psi_0\rangle_1 \otimes |\Psi_0\rangle_2 = |\Psi_0\rangle = \frac{1}{2}\sum_{i=0}^{3} |\Psi_i\rangle_{1'} \otimes |\Psi_i\rangle_{2'} \\
& \underset{H^{0,0,+}_{0,3} \oplus H^{0,0,+}_{12,15} \oplus H^{0,0,+}_{5,6} \oplus H^{0,0,+}_{9,10}}{\xrightarrow{H^{(3,1')}}} \quad \frac{1}{\sqrt{2}}(|\Psi_3\rangle_1 \otimes |\Psi_0\rangle_2 + |\Psi_1\rangle_1 \otimes |\Psi_1\rangle_2) \\
& \underset{E^{0,0}_{0,3} \oplus I^0_{5,6}}{\xrightarrow{H^{(3,1)}}} \quad \frac{1}{\sqrt{2}}(|\Psi_0\rangle_1 \otimes |\Psi_0\rangle_2 + |\Psi_1\rangle_1 \otimes |\Psi_1\rangle_2) = \frac{1}{\sqrt{2}}(|\Psi_0\rangle + |\Psi_5\rangle) \\
& \underset{E^{0,1}_{5,7} \oplus I^0_{0,2}}{\xrightarrow{H^{(2,1)}}} \quad \frac{1}{\sqrt{2}}(|\Psi_0\rangle_1 \otimes |\Psi_0\rangle_2 + i\,|\Psi_3\rangle_1 \otimes |\Psi_1\rangle_2) \\
& \underset{E^{1,1}_{7,15} \oplus I^0_{0,8}}{\xrightarrow{H^{(2,2)}}} \quad \frac{1}{\sqrt{2}}(|\Psi_0\rangle_1 \otimes |\Psi_0\rangle_2 + |\Psi_3\rangle_1 \otimes |\Psi_3\rangle_2) = |GHZ\rangle^4
\end{aligned} \tag{9}$$

where symbol above the arrow point to the Hamiltonian being used, while symbol below points to the concrete operations being used. Two notations for the kets are used by convenience: $|\Psi_k\rangle_j$ is the Bell state $|\Psi_k\rangle$ on the j^{th} correspondent pair, while $|\Psi_{i_1,i_2,\ldots,i_d}\rangle = |\Psi_{\mathcal{I}}\rangle =$ is the $\mathcal{I} = 4^{d-1}i_d + \ldots + 4i_2 + i_1$ element in the GBS basis. Note that pairs 1 and 2 contains the parts $[1,2]$ and $[3,4]$ respectively, while $1'$ and $2'$ contains the parts $[1,4]$ and $[2,3]$ respectively. This temporary rearrangement lets enlarge the entanglement into the whole system. Finally, departing from $\frac{1}{\sqrt{2}}(|\Psi_0\rangle + |\Psi_5\rangle)$ (included as step in the previous process), we can achieve $|W\rangle^4$:

$$\begin{aligned}
\frac{1}{\sqrt{2}}(|\Psi_0\rangle + |\Psi_5\rangle) \quad & \underset{E^{0,0}_{0,4}\oplus E^{0,0}_{1,5}}{\xrightarrow{H^{(1,2)}}} \quad \frac{1}{\sqrt{2}}(|\Psi_0\rangle_1 \otimes |\Psi_1\rangle_2 + |\Psi_1\rangle_1 \otimes |\Psi_0\rangle_2) \\
& \underset{H^{0,0,+}_{4,7}\oplus I^{2p}_{1,2}}{\xrightarrow{H^{(3,1)}}} \quad \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|\Psi_0\rangle_1 + |\Psi_3\rangle_1) \otimes |\Psi_1\rangle_2 + |\Psi_1\rangle_1 \otimes |\Psi_0\rangle_2) \\
& \underset{I^{2q}_{4,8}\oplus I^{2r}_{7,11}\oplus H^{0,0,+}_{1,13}}{\xrightarrow{H^{(3,2)}}} \quad \frac{1}{2}((|\Psi_0\rangle_1 + |\Psi_3\rangle_1) \otimes |\Psi_1\rangle_2 + |\Psi_1\rangle_1 \otimes (|\Psi_0\rangle_2 + |\Psi_3\rangle_2)) \\
& \qquad = \frac{1}{2}(|\Psi_4\rangle + |\Psi_7\rangle + |\Psi_1\rangle + |\Psi_{13}\rangle) = |W\rangle^4
\end{aligned} \tag{10}$$

with $p, q, r \in \mathbf{Z}$. Block independence has been applied justifying simultaneous operations, despite only two are possible (but repeated through different blocks). This couple of examples shows briefly the manipulation model of quantum information applied to state design.

5. Conclusions

While quantum system is the object where interactions work, quantum information is the associated element containing the sensible data of the history and the existence properties of it. The generic Hamiltonian in this work comprises representative systems in quantum information processing. $SU(2)$ reduction lets understand quantum processing as a simple series of operations manipulating directly the underlying quantum information while GBS basis works as a universal grammar there. Still, the general state design process is an open problem, in particular those containing the complexity of entanglement creation in all their possible levels between separability and maximal entanglement.

Acknowledgments

The support from Tecnológico de Monterrey to develop this research is acknowledged.

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