Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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To cite this article: A A Isayev 2017 *J. Phys.: Conf. Ser.* **934** 012039

IOP Conf. Series: Journal of Physics: Conf. Series **934** (2017) 012039 doi:10.1088/1742-6596/934/1/012039

Relativistic anisotropic stars with the polytropic equation of state in general relativity

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Abstract. Spherically symmetric relativistic stars with the polytropic equation of state, which possess the local pressure anisotropy, are considered in the context of general relativity. The modified Lane-Emden equations are derived for the special ansatz for the anisotropy parameter Δ in the form of the differential relation between Δ and the metric function ν. The analytical solutions of the obtained equations are found for incompressible fluid stars. The dynamical stability of incompressible anisotropic fluid stars against radial oscillations is studied.

1. Introduction. Basic equations

It was suggested in Ref. [1] that, despite the spherically symmetric distribution of matter inside a compact stellar object, it can be characterized by the local pressure anisotropy. The analysis of the generalized equations of hydrostatic equilibrium, allowing for the pressure anisotropy, shows that anisotropy may have the substantial effect on the maximum equilibrium mass and gravitational surface redshift. The pressure anisotropy can be caused by different physical reasons, such as, e.g., availability of superfluid states with the finite orbital momentum of Cooper pairs [2, 3] or finite superfluid momentum [4, 5], or the presence of strong magnetic fields inside a star [6, 7, 8, 9, 10, 11, 12]. In the present work, we will study spherical relativistic anisotropic stars with the polytropic equation of state, aiming to obtain the modified Lane-Emden (LE) equations for the special ansatz for the anisotropy parameter Δ in the form of the differential relation between Δ and the metric function ν. In general case, the obtained LE equations can be integrated only numerically, but the analytic solutions can be found for incompressible fluid stars. Then we apply the Chandrasekhar variational procedure [13] to study the dynamical stability of incompressible anisotropic fluid stars with respect to radial oscillations.

For spherically symmetric stars, the line element is written in the form (in units with $c = 1$):

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (1)$$

While the matter distribution inside a star is spherically symmetric, we allow the existence of the local pressure anisotropy in its interior with the different values of the radial p_r and transverse $p_\theta = p_\varphi \equiv p_t$ pressures. The energy–momentum tensor for a spherical anisotropic star reads

$$T_i^k = (\varepsilon + p_t)u_i u^k - p_t\delta_i^k + (p_r - p_t)s_i s^k, \quad (2)$$

where ε is the energy density of the system, $u^i = \frac{dx^i}{ds}$ is the fluid four-velocity, and s^i is the unit space-like vector with the properties $s^i u_i = 0$, $s^i s_i = -1$. For the motions in the radial

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direction, the four-vectors u^i and s^i have the structure $u^i = \left(\frac{e^{-\frac{\nu}{2}}}{\sqrt{1-v^2 e^{\lambda-\nu}}}, \frac{v e^{-\frac{\nu}{2}}}{\sqrt{1-v^2 e^{\lambda-\nu}}}, 0, 0\right)$, $s^i = \left(\frac{v e^{\frac{\lambda}{2}-\nu}}{\sqrt{1-v^2 e^{\lambda-\nu}}}, \frac{e^{-\frac{\lambda}{2}}}{\sqrt{1-v^2 e^{\lambda-\nu}}}, 0, 0\right)$,where $v = \frac{dr}{dt}$ is the radial velocity. In the spherically symmetric case, allowing for the motions in the radial direction, Einstein equations were written in Ref. [14]. The radial component of the equation $T^k_{i;k} = 0$ can be represented as

$$\dot{T}^0_1 + T^{1\prime}_1 + \frac{1}{2}T^0_1(\dot{\nu} + \dot{\lambda}) + \frac{\nu'}{2}(T^1_1 - T^0_0) + \frac{2}{r}(T^1_1 - T^2_2) = 0, \tag{3}$$

where $\dot{\nu} \equiv \frac{\partial \nu}{\partial t}$, $\nu' \equiv \frac{\partial \nu}{\partial r}$, etc. From Eq. (3), one can get in the static limit the equation for the hydrostatic equilibrium in the presence of the pressure anisotropy in the form

$$p_r' = -\frac{\nu'}{2}(\varepsilon + p_r) + \frac{2\Delta}{r}, \quad \Delta \equiv p_t - p_r, \tag{4}$$

where Δ is the anisotropy parameter. For the static configuration, the interior metric function λ reads [1]

$$e^{-\lambda(r)} = 1 - \frac{2G}{r}m(r), \quad r < R, \tag{5}$$

where R is the radial coordinate at the surface of a star, and $m(r) = 4\pi \int_0^r \varepsilon r^2 dr$ is the mass enclosed in the sphere of radius r. From Einstein equations in the static limit, one can find

$$\nu' = 2G\frac{m(r) + 4\pi p_r r^3}{r(r - 2Gm(r))}. \tag{6}$$

Hence, the equation of hydrostatic equilibrium for a spherical anisotropic star takes the form:

$$p_r' = -G\frac{(\varepsilon + p_r)(m(r) + 4\pi p_r r^3)}{r(r - 2Gm(r))} + \frac{2\Delta}{r}. \tag{7}$$

As the boundary condition to Eq. (7), we set the radial pressure at the center of a star $p_r(0) = p_{r0}$, and determine the radial coordinate R at the surface from the condition $p_r(R) = 0$. The total mass then can be calculated as $M = m(R)$. After finding the radial pressure distribution $p_r(r)$ (together with the mass distribution $m(r)$), the metric functions $\lambda(r), \nu(r)$ can be determined from Eqs. (5), (6). At the boundary $r = R$, the metric functions are matchable to the exterior vacuum Schwarzschild metric:

$$\lambda(R) = -\nu(R) = -\ln\left(1 - \frac{2GM}{R}\right). \tag{8}$$

In order to solve Eq. (7), it is necessary to set the equation of state (EoS) of the system. Further, as the EoS of the system, we choose the polytropic EoS in the form [15]:

$$p_r = K\varrho^\gamma \equiv K\varrho^{1+\frac{1}{n}}, \tag{9}$$

where ϱ is the mass density, K is some constant, γ is the polytropic exponent, n is the polytropic index. For the EoS (9) the energy density ε is related to the mass density ϱ and the radial pressure p_r by the equation $\varepsilon = \varrho + \frac{p_r}{\gamma - 1}$ [15].

It is convenient to introduce the auxiliary dimensionless LE function θ according to equations:

$$p_r = p_{r0}\theta^{n+1}, \quad \varrho = \varrho_0\theta^n, \tag{10}$$

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where ϱ_0 is the central mass density. It follows from the boundary conditions for the radial pressure p_r that

$$\theta(0) = 1, \quad \theta(R) = 0. \tag{11}$$

Then Eq. (4) of hydrostatic equilibrium can be rewritten as

$$2q_0(n+1)d\theta - \frac{4\Delta dr}{\varrho_0 r\theta^n} + \big(1 + (n+1)q_0\theta\big)d\nu = 0, \tag{12}$$

where $q_0 \equiv \frac{p_{r0}}{\varrho_0}$. In order to solve the equation of hydrostatic equilibrium, one needs to specify the anisotropy parameter Δ. We will suppose that the presence of the anisotropy parameter Δ doesn't change the general form of LE equations for relativistic isotropic stars [15], but only can change the coefficients in these equations. Specifically, we will assume that the parameter Δ and the metric function ν are related by the differential equation

$$-\frac{4\Delta dr}{\varrho_0 r\theta^n} + \big(1 + (n+1)q_0\theta\big)d\nu = (1 + \beta q_0\theta)d\nu, \tag{13}$$

where β is some real constant. Substituting Eq. (13) into Eq. (12) and integrating it, one can obtain the metric function $\nu(r)$ in the form

$$\nu(r) = \ln\frac{1 - \frac{2GM}{R}}{\big(1 + \beta q_0\theta\big)^{\frac{2(n+1)}{\beta}}}. \tag{14}$$

Let us define the auxiliary function

$$u(r) \equiv \frac{m(r)}{M} = \frac{r}{2GM}\left(1 - e^{-\lambda(r)}\right),\ u(0) = 0, u(R) = 1, \tag{15}$$

which satisfies the differential equation $Mu' = 4\pi\varepsilon r^2$. Introducing the dimensionless variable ξ and dimensionless function η by the equations

$$r = \alpha\xi, \quad \eta = \frac{M}{4\pi\varrho_0\alpha^3}u, \tag{16}$$

where $\alpha^2 = \frac{q_0(n+1)}{4\pi G\varrho_0}$, equations for the functions θ and η can be obtained in the form

$$\frac{\xi - 2(n+1)q_0\eta}{1 + \beta q_0\theta}\xi\frac{d\theta}{d\xi} + \eta + q_0\xi^3\theta^{n+1} = 0, \tag{17}$$

$$\frac{d\eta}{d\xi} = \xi^2\theta^n(1 + nq_0\theta). \tag{18}$$

Eqs. (17) and (18) represent the modified LE equations for relativistic anisotropic polytropes with the EoS (9) and ansatz (13) for the anisotropy parameter Δ, after solving which one can find from Eqs. (10), (16) the radial distribution of the radial pressure and mass in the interior of a spherical anisotropic star. One can see that the obtained LE equations formally look as in the isotropic case [15], but with that difference that the impact of the anisotropy parameter is reflected in the coefficient β (substituting the multiplier $(n+1)$). As follows from Eqs. (11), (16), the boundary conditions for the functions $\theta(\xi)$ and $\eta(\xi)$ read

$$\theta(0) = 1,\ \theta(\xi_R) = 0, \quad \xi_R \equiv R/\alpha \tag{19}$$

$$\eta(0) = 0,\ \eta(\xi_R) = \frac{M}{4\pi\varrho_0\alpha^3}. \tag{20}$$

In general case, the LE equations (17) and (18) can be integrated only numerically, but the analytical solutions can be found for incompressible anisotropic fluid stars, characterized by the constant density $\varrho = \text{const}$. At $n = 0$, solutions for the functions $\theta(\xi)$ and $\eta(\xi)$, satisfying the boundary conditions (19) and (20), are given by

$$\theta(\xi) = \frac{1}{q_0}\frac{(1+3q_0)\big(1-\frac{2q_0}{3}\xi^2\big)^{\frac{3-\beta}{4}} - (1+\beta q_0)}{3(1+\beta q_0) - \beta(1+3q_0)\big(1-\frac{2q_0}{3}\xi^2\big)^{\frac{3-\beta}{4}}}, \quad \eta(\xi) = \frac{\xi^3}{3}. \tag{21}$$

The positive root of the $\theta(\xi)$ is $\xi_R = \sqrt{\frac{3}{2q_0}\left[1-\left(\frac{1+\beta q_0}{1+3q_0}\right)^{\frac{4}{3-\beta}}\right]}$.

2. Dynamical stability of incompressible anisotropic fluid stars

Let us consider the stability of spherically symmetric anisotropic stars with respect to radial oscillations, assuming that they do not violate the spherical symmetry. Further we will study the small radial oscillations and will represent the unknown quantities as $\varepsilon = \varepsilon^0 + \delta\varepsilon,\ p_r = p_r^0 + \delta p_r,\ p_t = p_t^0 + \delta p_t, \nu = \nu^0 + \delta\nu,\ \lambda = \lambda^0 + \delta\lambda$, where $\delta\varepsilon, \delta p_r, \delta p_t, \delta\nu, \delta\lambda$ are small perturbations with respect to the corresponding values at the state of hydrostatic equilibrium, denoted by the upper index "0". Also, for the small radial oscillations we will consider that $v \ll 1$. Following Ref. [13], it is convenient to introduce a "Lagrange displacement" ψ by the equation $v = \dot{\psi}$. We will assume that all perturbations depend on time only through the exponential factor $e^{i\omega t}$, where ω is the frequency of radial oscillations. Then, using the linearized form of Einstein equations [14], and equation for the conservation of the total baryon number, the variations $\delta\varepsilon, \delta p_r, \delta\nu, \delta\lambda$ can be expressed through the Lagrange displacement ψ. Substituting these expressions to the linearized form of Eq. (3), in the case of the polytropic EoS one gets

$$\begin{aligned} \omega^2 e^{\lambda^0-\nu^0}\big(\varepsilon^0+p_r^0\big)\psi &= \frac{2\psi}{r}p_r^{0\prime} - \frac{2\psi}{r}\Big(\gamma\big(\nu^{0\prime}+\frac{\lambda^{0\prime}}{2}+\frac{2}{r}\big)+\frac{2}{r}\Big)\big(p_t^0-p_r^0\big) \\ &+8\pi G e^{\lambda^0}p_t^0\big(\varepsilon^0+p_r^0\big)\psi - \gamma\frac{d}{dr}\Big(\frac{2}{r}\big(p_t^0-p_r^0\big)\psi\Big) - \frac{\psi}{\varepsilon^0+p_r^0}\Big(p_r^{0\prime}-\frac{2}{r}\big(p_t^0-p_r^0\big)\Big)^2 \\ &-\gamma e^{-(\nu^0+\frac{\lambda^0}{2})}\frac{d}{dr}\Big(e^{\frac{3\nu^0+\lambda^0}{2}}\frac{p_r^0}{r^2}\frac{d}{dr}\big(r^2e^{-\frac{\nu^0}{2}}\psi\big)\Big) - \frac{2}{r}\Big(\gamma p_r^0\frac{e^{\frac{\nu^0}{2}}}{r^2}\frac{d}{dr}\big(r^2e^{-\frac{\nu^0}{2}}\psi\big)+\delta p_t\Big). \end{aligned} \tag{22}$$

Solutions of Eq. (22) for the frequencies of radial oscillations should be sought under the boundary conditions $\psi(r=0)=0,\ \delta p_r(r=R)=0$. In order to get the variational basis for finding the frequencies ω, let us multiply both parts of Eq. (22) on $r^2\psi\exp\big(\frac{\nu^0+\lambda^0}{2}\big)$ and integrate over the range of r. We will write the corresponding equation already for incompressible fluid stars ($n=0$), when the polytropic exponent $\gamma \to \infty$. Omitting the upper indexes zero, one gets

$$\begin{aligned} \omega^2\int_0^R e^{\frac{3\lambda-\nu}{2}}\big(\varepsilon+p_r\big)r^2\psi^2\,dr &= \gamma\int_0^R e^{\frac{\lambda+3\nu}{2}}\frac{p_r}{r^2}\Big(\frac{d}{dr}\big(r^2e^{-\frac{\nu}{2}}\psi\big)\Big)^2dr \\ -\gamma\int_0^R e^{\frac{\lambda+\nu}{2}}r^2\psi\frac{d}{dr}\Big(\frac{2}{r}\big(p_t-p_r\big)\psi\Big)\,dr &- 2\gamma\int_0^R e^{\frac{\lambda+\nu}{2}}r\psi^2\Big(\nu'+\frac{\lambda'}{2}+\frac{2}{r}\Big)\big(p_t-p_r\big) \\ &-2\gamma\int_0^R e^{\frac{\lambda}{2}+\nu}\psi\frac{p_r}{r}\frac{d}{dr}\big(r^2e^{-\frac{\nu}{2}}\psi\big)\,dr. \end{aligned} \tag{23}$$

In the variational equation (23), the Lagrange displacement ψ should be chosen such that ω^2 is minimized. If all frequencies of radial oscillations are real, a spherical anisotropic star is dynamically stable; if some frequency appears to be imaginary, the configuration is unstable.

Table 1. The critical values of the parameter q_0 for the appearance of the dynamical instability of an incompressible anisotropic fluid star at different values of the parameter β.

β	q_{0c} evaluated with the trial function	
	$\chi_1 = e^{-\frac{\nu}{2}}\xi^2$	$\chi_2 = \xi$
0.2	2.069	-
0.4	2.986	-
0.6	5.189	-
0.8	16.421	-

A sufficient condition for the occurrence of the dynamical instability is vanishing of the r.-h. s. of Eq. (23) for some trial form of the Lagrange displacement ψ satisfying the boundary conditions.

Let us introduce, following Ref. [15], the auxiliary function $\chi = e^{-\frac{\nu}{2}}\psi$. After changing the integration variable in Eq. (23) according to Eq. (16), substituting $p_r = q_0\varrho_0\theta$, $\varepsilon = \varrho_0$, and using the anisotropy parameter Δ from Eq. (13) and expressions (14), (15) for the metric functions at $n = 0$, Eq. (23) takes the form

$$\frac{\omega^2}{\omega_0^2}\frac{1}{1-\frac{2GM}{R}}\int_0^{\xi_R}\frac{(1+q_0\theta)\xi^2\chi^2}{\left(1-\frac{2q_0\xi^2}{3}\right)^{\frac{3}{2}}}\frac{d\xi}{(1+\beta q_0\theta)^{\frac{1}{\beta}}} = \gamma\int_0^{\xi_R}\frac{\theta\left(\frac{d}{d\xi}\left(\xi^2\chi\right)\right)^2}{\xi^2\left(1-\frac{2q_0\xi^2}{3}\right)^{\frac{1}{2}}}\frac{d\xi}{(1+\beta q_0\theta)^{\frac{3}{\beta}}} \tag{24}$$
$$-\frac{\gamma(1-\beta)}{2}\int_0^{\xi_R}\frac{\xi^2\chi}{\left(1-\frac{2q_0\xi^2}{3}\right)^{\frac{1}{2}}}\frac{d}{d\xi}\left(\frac{\nu'\chi\theta}{(1+\beta q_0\theta)^{\frac{1}{\beta}}}\right)\frac{d\xi}{(1+\beta q_0\theta)^{\frac{2}{\beta}}}$$
$$-\frac{\gamma(1-\beta)}{2}\int_0^{\xi_R}\frac{\xi^2\chi^2\nu'\theta\left(\nu'+\frac{\lambda'}{2}+\frac{2}{\xi}\right)}{\left(1-\frac{2q_0\xi^2}{3}\right)^{\frac{1}{2}}}\frac{d\xi}{(1+\beta q_0\theta)^{\frac{3}{\beta}}} - 2\gamma\int_0^{\xi_R}\frac{\chi\theta\frac{d}{d\xi}\left(\xi^2\chi\right)}{\xi\left(1-\frac{2q_0\xi^2}{3}\right)^{\frac{1}{2}}}\frac{d\xi}{(1+\beta q_0\theta)^{\frac{3}{\beta}}},$$

where $\omega_0^2 = 4\pi\varrho_0 G$. Let us use the trial functions of the form $\chi_1 = e^{-\frac{\nu}{2}}\xi^2, \chi_2 = \xi$. Then for each given β we will try to find such q_{0c} at which the r.-h. s. of Eq. (24) vanishes, and, hence, the dynamical instability for an incompressible anisotropic fluid star occurs at $q_0 > q_{0c}$.

The results of calculations are presented in Table 1. The most important conclusion is that there are solutions for q_{0c} in the case of the trial function χ_1 at $\beta < 1$, i.e., for $\Delta = p_t - p_r > 0$ (and there are no solutions at $\beta > 1$). This means that the local pressure anisotropy with $p_t > p_r$ can affect the dynamical stability of spherical incompressible fluid stars, unlike to incompressible isotropic fluid stars with the polytropic EoS (9), which are stable against radial oscillations [15].

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