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A proposal for unification of fatigue crack growth law

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Abstract. In the present paper, the new fractional-differential dependences of cycles to failure for a given initial crack length upon the stress amplitude in the linear fracture approach are proposed. The anticipated unified propagation function describes the infinitesimal crack length growths per increasing number of load cycles, supposing that the load ratio remains constant over the load history. Two unification fractional-differential functions with different number of fitting parameters are proposed. An alternative, threshold formulations for the fractional-differential propagation functions are suggested. The mean stress dependence is the immediate consequence from the considered laws. The corresponding formulas for crack length over the number of cycles are derived in closed form.

1. Introduction

The functional relation between the loading properties, stress gradient and the physical time-depending characteristics of materials are essential for the statistical fatigue analysis and analytical evaluation number of the cycles to failure [1-2]. The traditional evaluations concepts based on the Palmgren-Miner's rule of damage accumulation, rain-flow counting of time-dependent loads, Wöhler curve and Basquin equation, Paris-Erdogan law and diverse extensions of the fracture and damage mechanics approaches [3]. The book [4] reviews the numerical treatment of fatigue microscopic crack propagation together with their implementation in fatigue-life prediction models. It was recently pointed out that fracture and fatigue are not scale invariant [5]. An attempt to account the scale-invariance in the traditional fatigue approach is performed in this paper. The approach is based on the introduction of fractal length that divides the crack propagation into two characteristic scales.

There are three stages of fatigue fracture commonly distinguished: initiation, propagation, and final rupture [6-7]. The first stage (stage I) of fatigue is referred to as initiation. Initiation is probably the most complex stage of fatigue fracture. The most significant factor about the initiation stage of fatigue fracture is that the irreversible alterations in the metal are caused by cyclic shear stresses. The accumulation of microscopic faults over a large number of load applications, leads to cumulative damage. At the location of a severe stress concentration, the number depends on the geometry of the part as well as on environmental, stress, metallurgical, and strength conditions, as will become apparent. During the stage I, the spread of fatigue crack per unit cycle decelerates smoothly with number of cycles and the process approaches its second stage.

The second stage of fatigue is known as crack propagation (stage II). The propagation stage of fatigue causes the microscopic crack to change direction and grow perpendicular to the tensile stress. The second, or propagation, stage of fatigue is typically the most readily recognizable area of a fatigue fracture. The Paris law describes the stage II propagation law [8]. The traditional form of Paris law pronounces the spread of fatigue crack per unit cycle as a power function of the range of stress intensity



factor. Towards the end of the stage II, the spread of fatigue crack per unit cycle begins to accelerate smoothly with number of cycles and the process turns to the third stage.

The final, third stage of fatigue is the final rupture (stage III). As the propagation of the fatigue crack endures, progressively sinking the cross-sectional area of the test specimen, it eventually deteriorates the part so significantly that final, broad fracture occurs with a couple of load cycles. The fracture mode may be either ductile (with a dimpled fracture surface) or brittle (with a cleavage, or intergranular, fracture surface). Occasionally occur the combinations of these modes, depending upon the metal concerned, the stress level and the environment. In the course of the stage III, the spread of fatigue crack per unit cycle progressively accelerates with each cycle.

Numerous load damage models, which extend the celebrated Paris and Erdogan propagation law, were proposed since the 1970s. The article [9] reviews the articles focused on the prediction of fatigue properties of structures under variable amplitude loading. The reliability and accuracy of prediction models and the physical concept of fatigue damage was discussed. The reviewed fatigue life models were based on the scientific and engineering knowledge about fatigue of material and structures under constant and variable amplitude loading.

The near-threshold deviation of the common propagation law was suggested by [10]. The observed rapidly increasing growth towards ductile tearing was accounted. The brittle fracture correction of the power law was proposed in the cited article in the analytical form.

In the work [11] several combinations of high and low amplitude of stress intensity factors values were studied. A similar approach of for transition region between high and low amplitude of stress intensity factors was proposed in [12].

A more general ‘‘unified law’’ accounts certain deviations from the power-law regime [13]. The implemented extension of the Paris law for crack propagation results in the generalized Wöhler fatigue curves.

2. The propagation laws for cyclic loads

The crack propagation is calculated as a function of the range of the stress intensity factor:

$$K = K_{\max} - K_{\min} .$$

The factors in this equation are

K_{\max} the maximum stress intensity factor and

K_{\min} the minimum stress intensity factor per cycle.

The common form of Paris' law quantifies the fatigue life of a specimen for a given particular crack size a . The range of stress intensity factor reads:

$$K = Y\sigma\sqrt{\pi a} .$$

The range of stress intensity factor depends on:

$\sigma = \sigma_{\max} - \sigma_{\min}$ the stress range,

σ_{\max} the maximum stress,

σ_{\min} the minimum stress per cycle

Y a dimensionless parameter that reflects the geometry.

The parameter Y possesses the value 1 for a center crack in an infinite sheet. Hereafter, this value is assumed for brevity: $Y = 1$.

In the present paper the traditional form for propagation law is implemented:

$$c \frac{dn}{da} = U(K) \quad \text{or} \quad c \frac{dn}{da} = U(\sigma\sqrt{\pi a}). \quad (1)$$

The coefficients in Eq. (1) are:

$c = c(R)$ the material constant for a given stress ratio R ,

$$R = \frac{K_{\min}}{K_{\max}} < 1 \quad \text{the stress ratio of cyclic load,}$$

$$K_m \equiv \frac{K_{\max} + K_{\min}}{2} = \frac{1+R}{1-R} \frac{K}{2} \quad \text{the mean value of stress intensity factor.}$$

The *propagation function* $U(K)$ describes the infinitesimal crack length growths per increasing number of load cycles n , supposing that the load ratio R remains constant over the load history. It is presumed that the cracks with the initial length δ exist in the material. The cycle count starts with the beginning of crack propagation from its initial length:

$$n|_{a=\delta} = 0. \quad (2)$$

Thus, the range K and the mean value K_m of stress intensity factor remain constant. The variations of range K and the mean value K_m of stress intensity factor over the load history could be accounted by applications of damage accumulation theories [14-15].

3. Fractal crack equation (type I)

The *propagation function* $U(K)$ of the first type

$$L_1[n(a)] \equiv \frac{dn}{da} + \kappa^{\alpha-1} \frac{d^\alpha n}{da^\alpha} \quad (3)$$

$$L_1[n(a)] = \frac{U(K)}{c}, \quad U(K) = K^{-p},$$

where the stress intensity factor, corresponds to the transition to VHCF regime is

$$\kappa = \sqrt{\pi\lambda} \sigma, \quad (4)$$

λ is the characteristic transition length.

The method is used to calculate the fractional derivative using the Davison-Essex definition [19], that is, first differentiate n times, and then integrate $n - \alpha$ times:

$$\frac{d^\alpha n(a)}{da^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^a (a-t)^{n-\alpha-1} \frac{d^n n(t)}{dt^n} dt, \quad (5)$$

where n smallest integer greater than or equal to a number α , $\alpha \leq n$. This definition handles differentiation orders $\alpha > -1$.

At first assume that $0 < \alpha < 1$. The definition of the fractional derivative in the equation (3) leads to the integral-differential equation:

$$\frac{dn}{da} + \frac{(\pi\lambda\sigma^2)^{\alpha-1}}{\Gamma(1-\alpha)} \int_0^a (a-t)^{-\alpha} \frac{d^n n(t)}{dt^n} dt - \frac{(\pi\sigma^2 a)^{-p/2}}{c} = 0 \quad (6)$$

Application of the Laplace transform to the equation (6) leads to the following linear algebraic equation

$$\begin{aligned} \mathcal{L} \left\{ \frac{dn}{da} + \frac{(\pi\lambda\sigma^2)^{\alpha-1}}{\Gamma(1-\alpha)} \int_0^a (a-t)^{-\alpha} \frac{d^n n(t)}{dt^n} dt - \frac{(\pi\sigma^2 a)^{-p/2}}{c} \right\} = \\ = s^{\frac{p-1}{2}} (\pi\sigma^2)^{-\frac{p}{2}} \Gamma\left(1 - \frac{p}{2}\right) + c n_0 \left[1 + (s\pi\lambda\sigma^2)^{\alpha-1} \right] - c \left[s + (\pi\lambda\sigma^2)^{\alpha-1} s^\alpha \right] \tilde{n}(s) = 0 \end{aligned} \quad (7)$$

in terms of a Laplace transform of the function $\tilde{n}(s) = \mathcal{L}\{n(a)\}$ with an unknown constant n_0 . The solution of the equation (7) delivers the transformed function:

$$\tilde{n}(s) = \frac{1}{c} \frac{s^{\frac{p-1}{2}} (\pi\sigma^2)^{-\frac{p}{2}} \Gamma\left(1 - \frac{p}{2}\right)}{s + (\pi\lambda\sigma^2)^{\alpha-1} s^\alpha} + \frac{n_0}{s}. \quad (8)$$

The application of Bromwich integral inverse Laplace transform of the function (8) results the solution of fractional differential equation (3):

$$\begin{aligned} n_{1,1}(a) = \mathcal{L}^{-1}\{n(s)\} = \frac{(\pi\lambda\sigma^2)^{1-p}}{c \Gamma\left(-\frac{p}{2} + \frac{3}{2}\right)} \exp\left(\frac{a}{\pi\lambda\sigma^2}\right) \cdot \\ \left[\Gamma\left(-\frac{p}{2} + \frac{3}{2}, \frac{a}{\pi\lambda\sigma^2}\right) \Gamma\left(-\frac{p}{2} + 1\right) - \Gamma\left(-\frac{p}{2} + 1, \frac{a}{\pi\lambda\sigma^2}\right) \Gamma\left(-\frac{p}{2} + \frac{3}{2}\right) \right] + n_0, \\ N_{1,1}(a_1, a_2) = n_{1,1}(a_1) - n_{1,1}(a_2) \end{aligned} \quad (9)$$

One limit case corresponds to an infinite scaling parameter λ :

$$\kappa^{\alpha-1} \frac{d^\alpha n}{da^\alpha} - \frac{(\pi\sigma^2 a)^{-p/2}}{c} \equiv \frac{(\pi\lambda\sigma^2)^{\alpha-1}}{\Gamma(1-\alpha)} \int_0^a (a-t)^{-\alpha} \frac{d^n n(t)}{dt^n} dt - \frac{(\pi\sigma^2 a)^{-p/2}}{c} = 0. \quad (10)$$

The application of the Laplace transform to the Eq. (10) and leads to the transformed function

$$\tilde{n}(s) = \frac{1}{c} \left[s^{\frac{p-1}{2}-\alpha} (\pi\sigma^2)^{-\frac{p}{2}+1-\alpha} \lambda^{1-\alpha} \Gamma\left(1 - \frac{p}{2}\right) \right] + \frac{n_0}{s}. \quad (11)$$

The inverse Laplace transform of (11) provides the expression for the asymptotic limit of the function (9) for an infinite scaling parameter λ :

$$N_{1,2}(a_1, a_2) = \frac{(\pi\sigma^2)^{\frac{1-p}{2}} \Gamma\left(-\frac{p}{2} + 1\right)}{c \Gamma\left(-\frac{p}{2} + \frac{3}{2}\right)} \sqrt{\lambda} \left[a_1^{\frac{1-p}{2}} - a_2^{\frac{1-p}{2}} \right]. \quad (12)$$

The other limit case corresponds to an infinitesimally small scaling parameter λ :

$$\frac{dn}{da} - \frac{(\pi\sigma^2 a)^{-p/2}}{c} = 0 \quad (13)$$

with an elementary solution

$$N_{1,3}(a_2, a_1) = \frac{2(\pi\sigma^2)^{\frac{p}{2}}}{c(2-p)} \left[a_2^{1-\frac{p}{2}} - a_1^{1-\frac{p}{2}} \right]. \quad (14)$$

4. Fractal crack equation (type II)

The second formulation presumes the application of the fractional differentiation operator L_2 to the right side of the Eq. (1):

$$\frac{dn}{da} = L_2 \left[\frac{K^{-p}}{c} \right], \quad L_2 \left[\frac{K^{-p}}{c} \right] = \frac{K^{-p}}{c} - \kappa^{-\alpha} \frac{d^\alpha}{da^\alpha} \left(\frac{K^{-p}}{c} \right). \quad (15)$$

The definition of the fractional derivative leads an ordinary inhomogeneous differential equation of the first order:

$$\frac{dn}{da} - \frac{(\pi\sigma^2 a)^{-p/2}}{c} + \lambda^\alpha \frac{\Gamma(1-p/2)}{\Gamma(\alpha+1-p/2)} \frac{(\pi\sigma^2 a)^{\alpha-p/2}}{c} = 0. \quad (16)$$

The Laplace transform (16) brings the following expression

$$\tilde{n}(s) = \frac{\Gamma\left(1-\frac{p}{2}\right)}{c} \left[-s^{\frac{p}{2}-2-\alpha} (\pi\sigma^2)^{\frac{p}{2}-\alpha} \lambda^{-\alpha} + s^{\frac{p}{2}-2} (\pi\sigma^2)^{\frac{p}{2}} \right] + \frac{n_0}{s}, \quad (17)$$

which results in the formula for the number of cycles

$$n_{2,1}(a) = \frac{\Gamma\left(1-\frac{p}{2}\right)}{c} \left[\frac{(\pi\sigma^2)^{-\frac{p}{2}} a^{1-\frac{p}{2}}}{\Gamma\left(2-\frac{p}{2}\right)} - \frac{(\pi\sigma^2)^{\frac{\alpha-p}{2}} a^{1+\frac{\alpha-p}{2}}}{\Gamma\left(2+\alpha-\frac{p}{2}\right)} \lambda^\alpha \right] + n_0, \quad (18)$$

$$N_{2,1}(a_2, a_1) = n_{2,1}(a_2) - n_{2,1}(a_1).$$

The limit case corresponds to an infinite scaling parameter λ :

$$\frac{dn}{da} = \lambda^\alpha \frac{\Gamma(1-p/2)}{\Gamma(\alpha+1-p/2)} \frac{(\pi\sigma^2 a)^{\alpha-p/2}}{c} = 0 \quad (19)$$

with the corresponding solution

$$N_{2,2}(a_1, a_2) = \frac{\Gamma\left(1-\frac{p}{2}\right)}{c\Gamma\left(2+\alpha-\frac{p}{2}\right)} (\pi\sigma^2)^{\frac{\alpha-p}{2}} \lambda^\alpha \left[a_1^{1+\frac{\alpha-p}{2}} - a_2^{1+\frac{\alpha-p}{2}} \right]. \quad (20)$$

The other limit case corresponds to an infinitesimally small scaling parameter λ leads once again to the equation (13) with an already known solution

$$N_{1,3}(a_2, a_1) = N_{2,3}(a_2, a_1).$$

5. Effect of stress ratio

The load ratio is defined as the ratio of the algebraically minimum over the maximum load, Eq. (1). The experiments demonstrate that the load ratio affects the fatigue crack growth and threshold behaviour. Namely, the fatigue crack propagation rate and threshold value vary with the applied load ratio (Walker, 1970). If the load ratio is positive, the experiments reveal that the necessary stress intensity factor range for growth decreases with increasing positive values. In the region of the negative load ratio the required stress intensity factor range for growth (threshold stress intensity range) decreases as load ratio decreases. Reaching a definite negative value, known as saturation point, the required stress intensity factor range for fatigue crack growth stabilizes.

Consider now the variation of load ratio for the proposed propagation laws. The closed form solution for the cycles to failure for a given initial crack length upon the stress amplitude could be found for given load ratio.

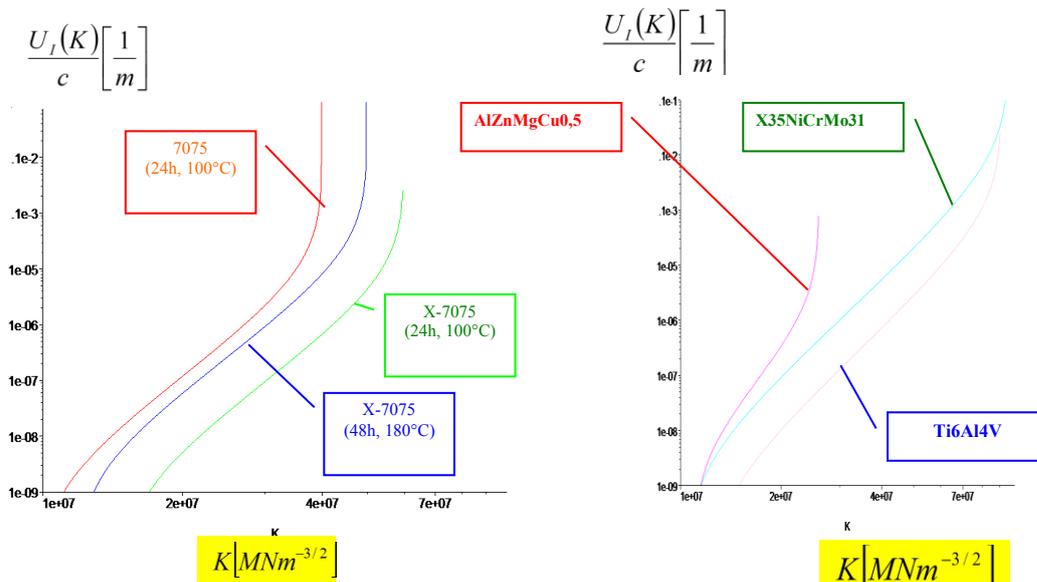


Fig. 1 The plot of the relations between the crack growth rate $dn/da = U_I(K)/c$ and the range of stress intensity factor K for different materials [17,18]

Instead of an empirical approach, in this paper we use the equations to derive the influence of the stress ratio on the crack growth rate explicitly. For this purpose consider two harmonically varying loads with equal amplitude, but with the different mean values. The mean values and amplitudes are assumed to be positive, such that:

$$K_m > 0, \quad K_m^* > 0, \quad K > 0.$$

Both loads lead to the harmonically varying stress intensity factors. The harmonically varying stress intensity factors correspondingly are:

$$K(t) = K_m + \frac{1}{2} K \sin t, \quad K^*(t) = K_m^* + \frac{1}{2} K \sin t. \quad (21)$$

For both considered cyclic loads the ratios are equal correspondingly to:

$$R = \frac{K_m - \frac{1}{2} K}{K_m + \frac{1}{2} K}, \quad R^* = \frac{K_m^* - \frac{1}{2} K}{K_m^* + \frac{1}{2} K}. \quad (22)$$

Consider at first the positive load cycles

$$K(t) > 0, \quad K^*(t) > 0.$$

For the positive load cycles the stress ratios are in the ranges:

$$-1 < R < 1, \quad -1 < R^* < 1. \quad (23)$$

The damage caused by the varying load depends linearly upon the stress intensity factor in the power p along the cycle. Consequently, the relation between fatigue coefficients, which correspond to the different stress ratios $c(R)$ and $c(R^*)$, reduces to:

$$\Lambda[R, R^*] = \frac{c(R)}{c(R^*)},$$

$$c(R) = \frac{c_0}{2\pi} \int_0^{2\pi} (K_m + \frac{1}{2} K \sin t)^p dt, \quad (24)$$

$$c(R^*) = \frac{c_0}{2\pi} \int_0^{2\pi} (K_m^* + \frac{1}{2} K \sin t)^p dt.$$

The integrals in (19) are expressed in term of the Meijer G- function [20]:

$$\frac{1}{2\pi} \int_0^{2\pi} (K_m + \frac{1}{2} K \sin t)^p dt = \mathbf{G}_{pq}^{mn} \left(\left(\frac{K}{2K_m} \right)^2 \middle| \frac{1-p}{2}, -\frac{p}{2} \right)_1 K_m^p.$$

The parameters of Eq. (24) are:

$$\frac{1-R}{1-R} = \frac{K}{2K_m}, \quad \frac{1-R^*}{1-R^*} = \frac{K}{2K_m^*}. \quad (25)$$

Consequently the relation between fatigue coefficients for two different load ratios is:

$$\Lambda[R, R^*] = \frac{\lambda[R]}{\lambda[R^*]}, \quad (26)$$

where

$$\lambda[R] = \mathbf{G}_{pq}^{mn} \left(\frac{(1-R)^2}{(1+R)^2} \middle| \frac{1-p}{2}, -\frac{p}{2} \right)_1 \frac{(1+R)^p}{(1-R)^p}.$$

6. Threshold-limited formulations for propagation functions

For some alloys there is a theoretical value for stress amplitude below which the material will not fail for any number of cycles. In this paper the method of representation of crack propagation functions through appropriate elementary functions is employed. The choice of the elementary functions is motivated by the phenomenological data and covers broad region possible parameters. With the introduced crack propagation functions differential equations describing the crack propagation are solved rigorously. The resulting closed form solutions allow the evaluation of crack propagation histories on one side, and the effects of stress ratio on crack propagation, on the other side.

Two new unified functions (type I and type II) were proposed in the paper (Kobelev, 2017). The functions were suggested in the form that incorporates the three commonly accepted stages of fatigue. The advantage of the newly proposed functions is the closed form solution of crack propagation.

The simplified representation of unified propagation function for the damage growths per cycle accounts the Paris propagation law together with transition regions at high and low amplitudes of stress intensity factors:

$$U(K) = K^{-p} \frac{1 - \left(\frac{K}{K_2}\right)^m}{1 - \left(\frac{K_1}{K}\right)^m}. \quad (27)$$

$m_2 > 1$ is the exponent at the short-term limit,

$m_1 > 1$ is the endurance limit exponent.

The function (27) delivers the closed form solution of the ordinary differential equation with the initial condition (2):

$$n = n(a, \delta, \sigma). \quad (28)$$

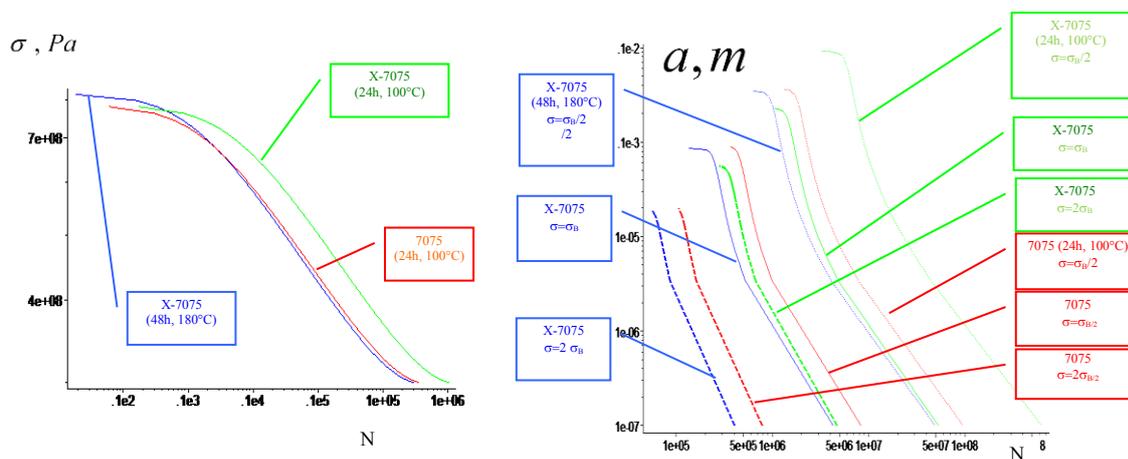


Fig. 2 The dependencies between the cycles to failure N for the given initial crack lengths upon the stress amplitude σ (s-N-curve, left) and cycles to failure N for the given stresses upon the length of crack (a-N-curve, right)

Precisely, the function $n(a, \delta, \sigma)$ delivers the number of cycles for growth of the crack length from the initial value δ to the given value a , assuming that the stress range σ and mean stress

$$\sigma_m = \frac{\sigma_{\max} + \sigma_{\min}}{2} \quad (29)$$

remain constant.

From the equation (1) two ultimate crack sizes could be determined immediately. Firstly, the value:

$$\pi a_2 = K_2^2 / \sigma^2$$

is the critical crack length at which instantaneous fracture will occur. Secondly, the value:

$$\pi a_1 = K_1^2 / \sigma^2$$

is the initial crack length at which fatigue crack growth starts for the given stress range. The number of cycles to failure at $a_1 = \delta$ is infinite. For a finite number of cycles before fracture the initial crack length δ must satisfy the condition:

$$a_1 < \delta < a_2 .$$

The relations between the crack growth rate

$$\frac{dn}{da} = \frac{U(K)}{c} \quad \text{with} \quad K = \sqrt{\pi a} \sigma$$

and the range of stress intensity factor K for simulated materials are shown on the **Fig.1**. The endurance and short time threshold exponents were experimentally acquired [17], alloys AlZnMgCu0,5, Ti6Al4V, X35NiCrMo31 and [18], aluminum alloys X-7075 (180°), X-7075 (100°), 7075 (100°).

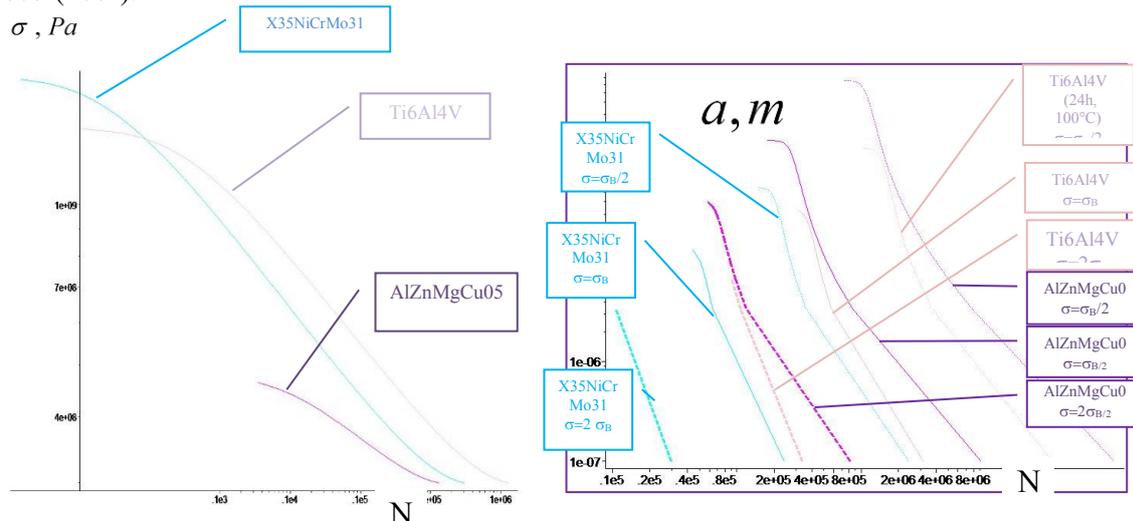


Fig. 3 The dependencies between the cycles to failure N for the given initial crack lengths upon the stress amplitude σ (s-N-curve, left) and cycles to failure N for the given stresses upon the length of crack (a-N-curve, right)

The solution of the ordinary differential equation (27) with the corresponding initial condition delivers closed form analytical expression for the remaining number of cycles to fracture:

$$n(a, \delta, \sigma) = \tilde{n}(a, \sigma) - \tilde{n}(\delta, \sigma), \quad (30)$$

where the auxiliary functions are:

$$\tilde{n}(a, \sigma) = \frac{2aK^{-p}}{cm} \left[\frac{K^m}{K_2^m} \mathbf{G}_{pq}^{mn} \left(\begin{matrix} \frac{K^m}{K_1^m} \left| 0, \frac{p-2}{m} \right. \\ \frac{K^m}{K_1^m} \left| 0, -\frac{m+p-2}{m} \right. \end{matrix} \right) - \mathbf{G}_{pq}^{mn} \left(\begin{matrix} \frac{K^m}{K_1^m} \left| 0, \frac{m+p-2}{m} \right. \\ \frac{K^m}{K_1^m} \left| 0, \frac{p-2}{m} \right. \end{matrix} \right) \right], \quad (31)$$

In the expression (31) appears the Meijer G- function [19]. The following example illustrates the behaviour of the metallic material with the unified propagation function (27). The diagrams that express the dependence of cycles to failure, S-N curves are presented on the pictures on left side of **Fig.2** and **Fig.3**. The same initial length of the crack was assumed for definiteness in all cases $\delta = 10^{-4}m$. The stress intervals for calculation depend on threshold stress intensity factors:

$$\sigma_1 < \sigma < \sigma_2, \quad \sigma_1 = K_1 \sqrt{\pi \delta}, \quad \sigma_2 = K_2 \sqrt{\pi \delta} .$$

The dependencies of crack lengths on cycles, a-N curves are drawn on the right pictures. The number of cycles was calculated as the function of length of the crack in the interval

$$10^{-7}m < a < 10^{-2}m .$$

The NSa surface N-S and curves and a-N curves are shown on the **Fig.2** for aluminium alloys X-7075 (180°), X-7075 (100°), 7075 (100°) from [17]. The S-N curves and a-N curves are displayed on the **Fig.3** for aluminium alloy AlZnMgCu0,5, titanium alloy Ti6Al4V, and steel X35NiCrMo31 [18].

7. Conclusion

The paper introduces the closed form analytical expression for crack length over number of cycles. Two new fractional-differential functional dependences express the damage growth per cycle are introduced. These functions allow the unification of different fatigue laws in a single expression. The unified fatigue law provides closed form analytical solutions for crack length upon the mean value and range of cyclic variation of stress intensity factor. The solution expresses the number of cycles to failure as the function of the initial size of the crack and eliminates the solution of nonlinear ordinary differential equation of the first order. The explicit formulas for stress against the number of cycles to failure are delivered for both proposed unified fatigue laws.

The different common expressions, which account the influence of the stress ratio, are immediately applicable. For the proposed unified propagation functions, the ranges for the stress load factor are extended using the introduction of effective stress intensity. The solution leads to the effective stress intensity factor range, effective mean value of stress intensity factor and effective stress ratio.

The proposed methods are applicable for the modeling of tribological effects on fatigue. Fretting fatigue is an outcome involving different influences. The major factors that effect fretting fatigue are slip amplitude, coefficient of friction, shear load and loading phase difference [21-24]. In the recent paper [25] study, three numerical models were introduced for simulation the effect of both in phase and out of phase loading on contact stresses and damage initiation locations. In the paper [25] was pointed out that phase difference affects the shear traction and tensile stress profiles at the contact interface, whereas no significant effect is observed on convergence efficiency. The methods, which proposed in the current article, could be applied for the simulation of the fretting fatigue processes.

The proposed method is also appropriate for investigation of fatigue behaviour compression springs at a very high number of cycles [26]. The reported variation of the slope in S-N diagram could be explained by the transition to a fractal fracture regime [27,28]. The corresponding models of fracture under static load conditions were proposed in [29]

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