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On Equiconvergence of Fourier Series and Fourier Integral

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Abstract. In this paper we prove a precise equiconvergence relation between index of the Bochner-Riesz means of the expansions and power of the singularity of the distributions with compact support.

1. Introduction

The effect of the singularity of the distribution on the regular points is essential. For example, partial sums of Fourier trigonometric series of the Delta function diverges at a regular point. However its regularization converges.

As a rule the study of summability of Fourier integral much easier than the Fourier series. Ease of the study of summability of Fourier integral is caused by computability of the kernel of the integral and its regularizations. Therefore, it is important to compare the Fourier integral and series.

The urgency of the study of these questions is promoted by the problems of the quantum mechanics which deal mostly with the singular data. The solutions of the problems in the area require a more deeper insight the summability of the expansions into the series and/or integral.

In fact, the behavior of the Fourier series and integral in terms of convergence and/or summability are identical for the some spaces of the functions. This type of behaviour of the series and integrals is known as an equiconvergence. The uncontested importance of this fact is that it allows to reduce the study to the convenient one. However there are some cases when the behaviour of these expansions are different.

Recall that $\mathcal{E}(T^N)$ is the space of infinite time differentiable on $T^N = (-\pi, \pi)^N$ functions, with the topology of uniform convergence of the sequence of functions and their partial derivatives in compact subsets of T^N [5]. Elements of the space $\mathcal{E}(T^N)$ we consider as the "test" functions. The space of linear continuous (with the respect to the topology of the space $\mathcal{E}(T^N)$) functionals is denoted by $\mathcal{E}'(T^N)$. Elements of $\mathcal{E}'(T^N)$ are distributions with the compact support on $T^N = (-\pi, \pi)^N$. The topology in $\mathcal{E}'(T^N)$ is called weak topology. Any distribution with the compact support can be expanded into Fourier series in the weak topology. For example, in this topology the Dirac delta function can be represented as

$$\delta(x) = (2\pi)^{-N} \sum_{n\in Z^N} \exp(inx). \tag{1.1}$$

The support of the Delta function is $\{0\}$. But series (1.1) diverges at the points far from the zero. For example, it diverges at $x = \frac{\pi}{2}$. In fact, the singularity of the Delta function at zero has significant affects at the points far from this point.

In $N = 1$ the arithmetic means of the series (1.1) convergence to zero at any non zero point. This statement is not valid if $N \geq 2$. Therefore, the convergence in the case $N \geq 2$ requires higher order regularization.

For any distribution $f \in \mathcal{E}'(T^N)$ and any real number $s \geq 0$ the Riesz means of order s of the spherical partial sums of series (1.1) is defined by

$$\sigma_\lambda^s f(x) = (2\pi)^{\frac{-N}{2}} \sum_{|n|<\lambda} \Big(1 - \frac{|n|^2}{\lambda^2}\Big)^s f_n \ \exp(inx). \tag{1.2}$$

where f_n is value of the functional f on a "test" function $(2\pi)^{-\frac{N}{2}} \exp(-iy\xi)$. If $s = 0$ and $f = \delta$, then from (1.2) we obtain (1.1).

Let us extend distribution f to $\mathcal{R}^\mathcal{N}$ as follows

$$F = \begin{cases} \ \ f \quad \text{in } T^N \ , \\ \ 0, \ \text{in } \mathcal{R}^\mathcal{N} \setminus \mathcal{T}^\mathcal{N}. \end{cases} \tag{1.3}$$

Note that the distribution F belongs to the space $\mathcal{E}'(\mathcal{R}^\mathcal{N})$. By $\hat{F}$ we denote its Fourier transformation. For instance, the Delta functions Fourier transformation has the following form $\hat{\delta}(x) = 1$. Then the Bochner-Riesz means of order s of the Fourier integral of the Delta function is

$$\Theta_\lambda^s(x) = (2\pi)^{\frac{-N}{2}} \int\limits_{|y|<\lambda} \Big(1 - \frac{|y|^2}{\lambda^2}\Big)^s \exp(iy\cdot x)dy. \tag{1.4}$$

Then we can define (1.4) for any distribution $F \in \mathcal{E}'(\mathcal{R}^\mathcal{N})$ as follows

$$R_\lambda^s(x) =< F, \Theta_\lambda^s(x-y) > . \tag{1.5}$$

where F acts to $\Theta_\lambda^s(x-y)$ with the respect to the variable y.

At the critical index $s = \frac{N-1}{2}$, Bochner [4] proved that the localization for (1.5) holds, and at the same time fails for the partial sum (1.2) in the class L_1. He also proved that the localization in the critical index is valid for both expansions in L_2. This result for the expansions in eigenfunctions of the Laplace operator proved by Levitan [7]. Below the critical index $\frac{N-1}{2}$ the problem studied by Il'in [6].

We note that the summability of the spectral expansions of distributions is studied in [1]. These questions for the Fourier series are studied in [8] and for the Forier integral is studied in [9].

2. Main Result

Let ℓ is a real number and $p \geq 1$. By $L_p^\ell(T^N)$ we denote the periodic Sobolev space of distributions (in case of integer ℓ called periodic Lebesque space)

$$L_p^\ell(T^N) = \Big\{ f \in \mathcal{E}' : \sum_{n\in Z^N} (1+|n|^2)^{\ell/2} f_n \exp(inx) \in \ \ L_p(T^N) \Big\}.$$

Theorem 1. *Let $\ell > 0$ and $s = \frac{N-1}{2} + \ell$. Then for any $f \in L_p^{-\ell}(T^N)$ with $1 < p \leq 2$ and $suppf \subset \Omega \subset\subset T^N$ one has*

$$\sigma_\lambda^s f(x) = R_\lambda^s F(x) + \ O(1)\|f\|_{-\ell,p}$$

where $x \in T^N \setminus \overline{\Omega}$ and $\|\cdot\|_{-\ell,p}$ is a norm in $L_p^{-\ell}(T^N)$ defined by

$$\|f\|_{-\ell,p} = (2\pi)^{-\frac{N}{2}}\|(1+|n|^2)^{\ell/2} f_n \exp(inx)\|_p.$$

We point out that nn case $s < \frac{N-1}{2} + \ell$ the statement of the theorem is not valid for any distribution [10]. In case $p = 2$ Theorem 1 is proved in [10].

3. Preliminary statments

Recall that the Riesz means of the partial sums of the series (1.1) is

$$D_\lambda^s(x) = \ (2\pi)^{-N} \sum_{|n|<\lambda} \left(1 - \frac{|n|^2}{\lambda^2}\right)^s \exp(inx). \tag{3.1}$$

Then (1.2) can be written as follows

$$\sigma_\lambda^s f(x) =< f, D_\lambda^s(x-y) >, \tag{3.2}$$

where f is acting to $D_\lambda^s(x-y)$ by y.

Note, that the integral (1.4) can be directly evaluated as [2]

$$\Theta_\lambda^s(x) = (2\pi)^{-N/2} 2^s \Gamma(s+1) \frac{\lambda^{\frac{N}{2}-s} J_{\frac{N}{2}+s}(\lambda|x|)}{|x|^{\frac{N}{2}+s}}, \tag{3.3}$$

where $J_\nu(r)$ is the Bessel function.

Using the estimation for the Bessel function

$$|J_\nu(r)| \leq const \frac{1}{\sqrt{r}}, \quad r > 1.$$

from (3.3) we obtain

$$|\Theta_\lambda^s(x)| \leq const(1+|x|)^{-N-\ell}. \tag{3.4}$$

Note that the Fourier transformation of the function (3.3) is

$$\hat{\Theta}_\lambda^s(\xi) = \begin{cases} \left(1 - \frac{|\xi|^2}{\lambda^2}\right)^s, & \text{if } |\xi| \leq \lambda, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

Thus, we can write a similar estimation for (3.4)

$$|\hat{\Theta}_\lambda^s(\xi)| \leq \ const(1+|\xi|)^{-N-\ell} \tag{3.6}$$

where $\ell > 0$, $s = \ \frac{N-1}{2} + \ell$.

From the inequalities (3.4) - (3.6) and the Poisson formula of summation [2] it follows that

$$\sum_{n \in Z^N} \Theta_\lambda^s(x + 2\pi n) = (2\pi)^{-\frac{N}{2}} \sum_{n \in Z^N} \hat{\Theta}_\lambda^s(n) \exp inx. \tag{3.7}$$

Obviously from (3.5) it follows that the left side of (3.7) is equal to (3.1). Hence, from (3.7) we obtain

$$D_\lambda^s(x) = \sum_{n \in Z^N} \Theta_\lambda^s(x + 2\pi n). \tag{3.8}$$

Rewrite formula (3.8) as

$$D_\lambda^s(x) = \Theta_\lambda^s(x) \ + \Theta_{*,\lambda}^s(x), \tag{3.9}$$

where $\Theta_{*,\lambda}^s(x)$ is a function

$$\Theta_{*,\lambda}^s(x) \ = \sum_{n \in Z^N, n \neq 0} \Theta_\lambda^s(x + 2\pi n).$$

From (3.9) and estimations (3.4) and (3.6) one finds

Lemma 2. *Let $\ell > 0$, $s = \frac{N-1}{2} + \ell$. Then one has*

$$|\Theta_{*,\lambda}^s(x)| = O(\lambda^{-\frac{\ell}{4}}),$$

uniformly in any compact set $K \subset T^N$

4. Proof of Main result

Let $f \in L_p^{-\ell}(T^N)$ with compact support. Then from (3.9) one gets

$$< f, D_\lambda^s(x-y) >=< f, \Theta_\lambda^s(x-y) > + < f, \Theta_{*,\lambda}^s(x-y) >, \tag{4.1}$$

where the distribution f acts with respect to variable y .

Comparing the left side of (4.1) with (3.2) and the first term on the rights side of (4.1) with (1.5) we obtain

$$\sigma_\lambda^s f(x) - R_\lambda^s F(x) =< f, \Theta_{*,\lambda}^s(x-y) > .$$

Lemma 3. *Let $s = \frac{N-1}{2} + \ell$, $\ell > 0$, $f \in L_p^{-\ell}(T^N) \cap \mathcal{E}'(T^N)$, $1 < p \leq 2$ and let $suppf \subset \Omega \subset\subset T^N$. Then one has*

$$< f, \Theta_{*,\lambda}^s(x-y) >= \ \mathcal{O}(1)\|f\|_{-\ell,p}$$

uniformly in any compact set $K \subset T^N \setminus \overline{\Omega}$

Proof. Let Ω_0 such that $\Omega_0 \subset\subset \Omega$ and *suppf* $\subset \Omega_0$. Then taking into account that $f \in L_p^{-\ell}(T^N)$ we obtain the following inequality

$$| < f, \Theta_{*,\lambda}^s(x-y)| \leq \|f\|_{-\ell,p} \|\Theta_{*,\lambda}^s(x-y)\|_{\ell,p,0}$$

where $\|\Theta_{*,\lambda}^s(x-y) > \|_{\ell,p,0}$ is a norm of $\Theta_{*,\lambda}^s(x-y)$ in $L_p^\ell(\Omega_0)$ via $y \in \Omega_0$.

There exists a constant $c > 0$ such that $|x-y| > c$, for all $x \in K$ and $y \in \Omega_0$. Thus we have [3]

$$\|\Theta_{*,\lambda}^s(x-y)\|_0 = O(\lambda^{\frac{-\ell}{4}}), \tag{4.2}$$

where $\|\Theta_{*,\lambda}^s(x-y)\|_0$ is a norm of a function $\Theta_{*,\lambda}^s(x-y)$ in the space $L_2(\Omega_0)$.

Then the statement of Lemma 3 immediately follows from (4.2) and

$$\|\Theta_{*,\lambda}^s(x-y)\|_{\ell,0} = O(\lambda^{\frac{\ell}{4}}) \quad \|\Theta_{*,\lambda}^s(x-y)\|_0,$$

Hence, Theorem 1 follows from Lemma 3.

5. Conclusions

The behaviour of the regularized Fourier integral and Fourier series are the same in the considered spaces of distributions under certain conditions for the index of summability of the expansions and power of the singularity of the distributions. The results of the present paper can be extended to the more general operators.

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