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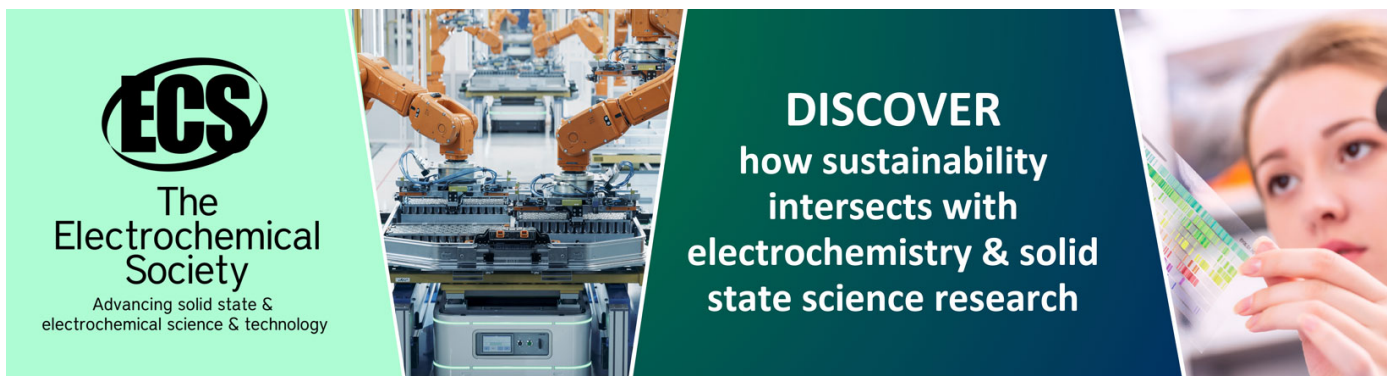
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On the Lebesgue constants of Fourier-Laplace series by Riesz Means

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Abstract. An asymptotic formula for the Lebesgue constant of the Riesz means of Fourier-Laplace series on the sphere obtained in this paper.

1. INTRODUCTION

Let us define $\sigma_n^\alpha f(x)$ the Cesaro means of order α of the partial sums of Fourier-Laplace series on unite sphere S^N as

$$\sigma_n^\alpha f(x) = \int_{S^N} \Theta^\alpha(x, y, n) f(y) d\sigma(y),$$

where the kernel

$$\Theta^\alpha(x, y, n) = \sum_{k=0}^n \frac{A_{n-k}^\alpha}{A_n^\alpha} \sum_{j=1}^{a_k} Z_k(x, y),$$

$$Z_k(x, y) = \sum_{j=1}^{a_k} Y_j^{(k)}(x) Y_j^{(k)}(y), \quad Y_j^{(k)}(y) \text{ are spherical harmonics and } A_n^\alpha = \binom{n+\alpha}{\alpha}.$$

Investigations on the behaviour of the Cesaro means $\sigma_n^\alpha f(x)$ can be found in works [4] - [5] and [11] - [16]. The different aspects of the convergence and summability can be also found in the book [18]. Since $\sigma_n^\alpha f(x)$ is an integral operator the precise estimation of its kernel $\Theta^\alpha(x, y, n)$ is essential for the study. First estimations of the kernel $\Theta^\alpha(x, y, n)$ obtained by Gronwall [6] for the case of Legendre polynomials and Kogbetliantz [8] for the Gegenbauer polynomials.

The Lebesgue constant is L_1 norm of the kernel above. Note, that estimations of the Lebesgue constants of the Cesaro means studied by Khocholava [8], Akhobadze [2] and Macharashvili [10]. The Lebesgue constants of multiple Fourier series studied in [1] and [9].

This article focuses on Lebesgue constants related to Fourier-Laplace series of the Laplace-Beltrami operator:

$$\mathcal{L}_n^\alpha = \int_{S^N} |\Theta^\alpha(x, y, n)| d\sigma(y). \quad (1.1)$$



2. MAIN RESULT.

In the present paper we consider the Riesz means instead the Cezaro means of the partial sums of Fourier-Laplace series. The Riesz means of the partial sums will also be an integral operator and its kernel can be represented by

$$\Theta^\alpha(x, y, n) = \sum_{k=0}^n \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\alpha \frac{\Gamma(k + \frac{N-1}{2}) \Gamma(k + N - 1)}{\pi^{\frac{N+1}{2}}} P_k^{\frac{N-1}{2}}(\cos \gamma),$$

where $P_k^\nu(t)$ denote the Gegenbauer polynomials as follows

$$P_k^\nu(t) = \frac{(-2)^k \Gamma(k + \nu) \Gamma(k + 2\nu)}{\Gamma(\nu) \Gamma(2(k + \nu))} (1 - t^2)^{-(\nu - \frac{1}{2})} \frac{d^k}{dt^k} \left[(1 - t^2)^{k + \nu - \frac{1}{2}} \right].$$

By this representation it is evident that $\Theta^\alpha(x, y, n)$ depends only on the spherical distance between x and y hence, allows the Riesz means of the spectral function to be written as $\Theta^\alpha(x, y, n) = \Theta_n^\alpha(\cos \gamma)$. The Riesz means of the kernel is studied in the works [3] and [17].

The main goal of the paper is to obtain the estimation of the Lebesgue constant (L_1 norm of the kernel). Let us use the same notation $\mathcal{L}_n^\alpha$ for the Lebesgue constant as in (1.1). Then following theorem is valid.

Theorem 2.1. *The Lebesgue constants for Fourier-Laplace series have the following estimations*

$$\begin{aligned} C' n^{\frac{N-1}{2}-\alpha} &< \mathcal{L}_n^\alpha < C'' n^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C' \ln n &< \mathcal{L}_n^\alpha < C'' \ln n, & \alpha = \frac{N-1}{2}, \\ 0 &< \mathcal{L}_n^\alpha < C, & \alpha > \frac{N-1}{2}. \end{aligned}$$

3. PROOF OF MAIN RESULT.

To estimate $\mathcal{L}_n^\alpha$, we first denote (1.1) as follows,

$$\mathcal{L}_n^\alpha = C \int_0^\pi |\Theta_n^\alpha(\cos \gamma)| \sin^{N-1} \gamma d\gamma \quad (3.1)$$

Let us divide the integral on the right hand side of (3.1) into three parts as follows

$$\begin{aligned} \mathcal{L}_n^\alpha &= C \int_0^{\frac{1}{n+1} \frac{\pi}{2}} |\Theta_n^\alpha(\cos \gamma)| \sin^{N-1} \gamma d\gamma + C \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} |\Theta_n^\alpha(\cos \gamma)| \sin^{N-1} \gamma d\gamma \\ &\quad + C \int_{\pi - \frac{1}{n+1} \frac{\pi}{2}}^\pi |\Theta_n^\alpha(\cos \gamma)| \sin^{N-1} \gamma d\gamma \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (3.2)$$

3.1. Estimation from above

If $0 < \gamma_0 \leq \gamma \leq \pi$ and $0 \leq \gamma \leq \pi$, the kernel $\Theta_n^\alpha(\cos \gamma)$ can be easily estimated by

$$|\Theta_n^\alpha(\cos \gamma)| \leq C_4 n^{N-1-\alpha} \quad \text{and} \quad |\Theta_n^\alpha(\cos \gamma)| \leq C_5 n^N.$$

For the Riesz means of the spectral function $\Theta_n^\alpha(\cos \gamma)$ in $0 \leq \gamma \leq \pi$ we have the following estimation

$$|\Theta_n^\alpha(\cos \gamma)| < C n^N, \quad 0 \leq \gamma \leq \pi.$$

The first integrand I_1 can be estimated as follows

$$I_1 < C n^N \int_0^{\frac{1}{n+1} \frac{\pi}{2}} \sin^{N-1} \gamma \, d\gamma \leq C n^N \int_0^{\frac{1}{n+1} \frac{\pi}{2}} \gamma^{N-1} \, d\gamma = \frac{C n^N \left(\frac{\pi}{2}\right)^N}{(n+1)^N (N)} \leq \frac{C \left(\frac{\pi}{2}\right)^N}{N} = O(1). \quad (3.3)$$

Similarly, for the third term I_3 one has

$$I_3 < C n^N \int_{\pi - \frac{1}{n+1} \frac{\pi}{2}}^{\pi} \sin^{N-1} \gamma \, d\gamma \leq C n^N \int_{\pi - \frac{1}{n+1} \frac{\pi}{2}}^{\pi} \sin^{N-1}(\pi - \gamma) \, d\gamma.$$

Applying the change of variables $\omega = \pi - \gamma$ the last estimation gives

$$I_3 < C n^N \int_0^{\frac{1}{n+1} \frac{\pi}{2}} \sin^{N-1} \omega \, d\omega = O(1). \quad (3.4)$$

To estimate I_2 we need the following estimation of the kernel $\Theta_n^\alpha(\cos \gamma)$ (refer [17])

$$\begin{aligned} \Theta_n^\alpha(\cos \gamma) &= (N-1)n^{\frac{N-1}{2}-\alpha} \frac{\sin \left[\left(n + \frac{N-1}{2} + \frac{\alpha+1}{2} \right) \gamma - \frac{\pi}{2} \left(\frac{N-1}{2} + \alpha \right) \right]}{(2 \sin \gamma)^{\frac{N-1}{2}} (2 \sin \frac{\gamma}{2})^{\alpha+1}} + \frac{\xi_n^\alpha(\gamma)(n+1)^{\frac{N-1}{2}-\alpha-1}}{(\sin \gamma)^{\frac{N+1}{2}} (\sin \frac{\gamma}{2})^{\alpha+1}} \\ &\quad + \frac{\eta_n^\alpha(\gamma)}{(n+1) (\sin \frac{\gamma}{2})^{N+1}}. \end{aligned} \quad (3.5)$$

By (3.5), the integrand I_2 is bounded by the sum of the following integrals

$$\begin{aligned} I_2 &< C \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{n^{\frac{N-1}{2}-\alpha}}{(\sin \gamma)^{\frac{N-1}{2}} (\sin \frac{\gamma}{2})^{1+\alpha}} \sin^{N-1} \gamma \, d\gamma + C \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{n^{\frac{N-3}{2}-\alpha}}{(\sin \gamma)^{\frac{N-1}{2}} (\sin \frac{\gamma}{2})^{1+\alpha}} \sin^{N-1} \gamma \, d\gamma \\ &\quad + C \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{|\varepsilon_n(\gamma)|}{(\sin \frac{\gamma}{2})^{N+1}} \sin^{N-1} \gamma \, d\gamma = I_2' + I_2'' + I_2'''. \end{aligned} \quad (3.6)$$

We first estimate I_2' :

$$\begin{aligned} I_2' &= C \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{n^{\frac{N-1}{2}-\alpha} \sin^{N-1} \gamma}{(\sin \gamma)^{\frac{N-1}{2}} (\sin \frac{\gamma}{2})^{1+\alpha}} \, d\gamma = C \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{n^{\frac{N-1}{2}-\alpha} (\sin \gamma)^{\frac{N-1}{2}}}{(\sin \frac{\gamma}{2})^{1+\alpha}} \, d\gamma \\ &= C n^{\frac{N-1}{2}-\alpha} \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} (\sin \gamma)^{\frac{N-1}{2}-\alpha-1} \, d\gamma \leq C n^{\frac{N-1}{2}-\alpha} \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \gamma^{\frac{N-1}{2}-\alpha-1} \, d\gamma. \end{aligned}$$

Consider the cases: $\alpha < \frac{N-1}{2}$, $\alpha = \frac{N-1}{2}$ and $\alpha > \frac{N-1}{2}$.

If $\alpha < \frac{N-1}{2}$, then

$$I'_2 \leq \frac{C}{\frac{N-1}{2} - \alpha} \left[\left(\pi - \frac{\pi}{2(n+1)} \right)^{\frac{N-1}{2} - \alpha} - \left(\frac{\pi}{2(n+1)} \right)^{\frac{N-1}{2} - \alpha} \right] n^{\frac{N-1}{2} - \alpha} \leq C n^{\frac{N-1}{2} - \alpha}. \quad (3.7)$$

If $\alpha = \frac{N-1}{2}$,

$$I'_2 \leq C \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{1}{\gamma} d\gamma = C \ln \left[(n+1) \left(2 - \frac{1}{n+1} \right) \right] = C \ln(2n+1) \leq C \ln n. \quad (3.8)$$

If $\alpha > \frac{N-1}{2}$,

$$I'_2 \leq C n^{\frac{N-1}{2} - \alpha} \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \gamma^{\frac{N-1}{2} - \alpha - 1} d\gamma \leq \frac{C n^{\frac{N-1}{2} - \alpha}}{(\alpha - \frac{N-1}{2})(n+1)^{\frac{N-1}{2} - \alpha}} \left(-\frac{\pi}{2} \right)^{\frac{N-1}{2} - \alpha} \leq C_\alpha. \quad (3.9)$$

Hence, from (3.7), (3.8) and (3.9) we obtain

$$I'_2 \leq \begin{cases} C n^{\frac{N-1}{2} - \alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ C_\alpha, & \alpha > \frac{N-1}{2}. \end{cases} \quad (3.10)$$

The term I''_2 is estimated by:

$$\begin{aligned} I''_2 &= C \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{(n+1)^{\frac{N-3}{2} - \alpha} |\eta_n(\gamma)|}{(\sin \gamma)^{(N-1)/2+1} \left(\sin \frac{\gamma}{2} \right)^{\alpha+1}} \sin^{N-1} \gamma d\gamma \leq C n^{\frac{N-3}{2} - \alpha} \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{(\sin \gamma)^{\frac{N-3}{2}}}{(\sin \frac{\gamma}{2})^{\alpha+1}} d\gamma \\ &\leq C n^{\frac{N-3}{2} - \alpha} \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \gamma^{\frac{N-5}{2} - \alpha} d\gamma. \end{aligned}$$

Let us consider the following cases:

If $\alpha < \frac{N-3}{2}$

$$I''_2 \leq C n^{\frac{N-3}{2} - \alpha} \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \gamma^{\frac{N-5}{2} - \alpha} d\gamma = \frac{C n^{\frac{N-3}{2} - \alpha}}{\frac{N-3}{2} - \alpha} \left[\left(\pi - \frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2} - \alpha} - \left(\frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2} - \alpha} \right] \leq C n^{\frac{N-3}{2} - \alpha}. \quad (3.11)$$

If $\alpha = \frac{N-3}{2}$

$$I''_2 \leq C n^{\frac{N-3}{2} - \alpha} \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{1}{\gamma} d\gamma \leq C \ln n. \quad (3.12)$$

If $\alpha > \frac{N-3}{2}$

$$\begin{aligned} I_2'' &\leq C n^{\frac{N-3}{2}-\alpha} \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \gamma^{\frac{N-5}{2}-\alpha} d\gamma = \frac{C n^{\frac{N-3}{2}-\alpha}}{\alpha - \frac{N+1}{2}} \left[\left(\frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2}-\alpha} - \left(\pi - \frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2}-\alpha} \right] \\ &\leq \frac{C}{\alpha - \frac{N+1}{2}} \left(\frac{\pi}{2(n+1)} \right)^{\frac{N-1}{2}-\alpha} n^{\frac{N-3}{2}-\alpha} \leq C. \end{aligned} \quad (3.13)$$

By (3.11), (3.12) and (3.13), I_2' is estimated by

$$I_2' \leq \begin{cases} C n^{\frac{N-3}{2}-\alpha}, & \alpha < \frac{N-3}{2}, \\ C \ln n, & \alpha = \frac{N-3}{2}, \\ C, & \alpha > \frac{N-3}{2}. \end{cases} \quad (3.14)$$

Finally, to estimate I_2 from above, I_2''' is estimated as follows:

$$I_2''' = \frac{C}{n+1} \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{|\varepsilon_n(\gamma)|}{(\sin \frac{\gamma}{2})^{N+1}} \sin^{N-1} \gamma d\gamma \leq \frac{C}{n+1} \int_{\frac{1}{n+1} \frac{\pi}{2}}^{\pi - \frac{1}{n+1} \frac{\pi}{2}} \frac{1}{\gamma^2} d\gamma \leq \frac{C}{n+1} \frac{\pi(n+1)}{2} \leq C, \quad (3.15)$$

Then from (3.6) taking into the note (3.10), (3.14) and (3.15) we obtain

$$I_2 \leq \begin{cases} C n^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ C, & \alpha > \frac{N-1}{2}. \end{cases} \quad (3.16)$$

Consequently, using (3.3), (3.4) and (3.16) the Lebesgue constant $\mathcal{L}_n^\alpha$ (see (3.2)) is estimated as follows

$$\mathcal{L}_n^\alpha \leq \begin{cases} C n^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ C, & \alpha > \frac{N-1}{2}. \end{cases} \quad (3.17)$$

3.2. Estimation from below

The next step is to obtain the lower bound of the Lebesgue constant. We proceed by first estimating I'_2 from below:

$$\begin{aligned}
 I'_2 &= C \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi - \frac{1}{n+1}\frac{\pi}{2}} n^{\frac{N-1}{2}-\alpha} (N-1) \frac{|\sin[(n + \frac{N-1}{2} + \frac{\alpha+1}{2})\gamma - \frac{\pi}{2}(\frac{N-1}{2} + \alpha)]|}{(2\sin\gamma)^{\frac{N-1}{2}} (2\sin\frac{\gamma}{2})^{1+\alpha}} \sin^{N-1}\gamma \, d\gamma \\
 &= \frac{C}{2^\alpha} \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi - \frac{1}{n+1}\frac{\pi}{2}} n^{\frac{N-1}{2}-\alpha} (N-1) \frac{|\sin[(n + \frac{N-1}{2} + \frac{\alpha+1}{2})\gamma - \frac{\pi}{2}(\frac{N-1}{2} + \alpha)]|}{(\sin\frac{\gamma}{2})^{\alpha - \frac{N-3}{2}}} \left(\cos\frac{\gamma}{2}\right)^{\frac{N-1}{2}} d\gamma \\
 &\geq C n^{\frac{N-1}{2}-\alpha} \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\frac{3}{4}\pi} n^{\frac{N-1}{2}-\alpha} (N-1) \frac{|\sin[(n + \frac{N-1}{2} + \frac{\alpha+1}{2})\gamma - \frac{\pi}{2}(\frac{N-1}{2} + \alpha)]|}{\gamma^{\alpha - \frac{N-3}{2}}} d\gamma.
 \end{aligned}$$

A new variable $\omega = (n + \frac{N}{2} + \frac{\alpha}{2})\gamma - \frac{\pi(N-1+2\alpha)}{4}$ is introduced and substituted into the chain of inequalities:

$$\begin{aligned}
 I'_2 &\geq C n^{\frac{N-1}{2}-\alpha} \int_{(n + \frac{N-1}{2} + \frac{\alpha+1}{2})\frac{\pi}{2(n+1)} - \frac{\pi(N-1+2\alpha)}{4}}^{(n + \frac{N-1}{2} + \frac{\alpha+1}{2})\frac{3\pi}{2} - \frac{\pi(N-1+2\alpha)}{4}} \frac{|\sin\omega|}{\left(\frac{\omega + \frac{\pi(N-1+2\alpha)}{4}}{n + \frac{N-1}{2} + \frac{\alpha+1}{2}}\right)^{\alpha - \frac{N-3}{2}}} \frac{d\omega}{n + \frac{N-1}{2} + \frac{\alpha+1}{2}} \\
 &\geq C n^{\frac{N-1}{2}-\alpha} \left(n + \frac{N-1}{2} + \frac{\alpha+1}{2}\right)^{\alpha - \frac{N-1}{2}} \int_{\pi}^{\frac{n\pi}{2}} \frac{|\sin\omega| \, d\omega}{\left(\omega + \frac{\pi(N-1+2\alpha)}{4}\right)^{\alpha - \frac{N-3}{2}}}.
 \end{aligned}$$

Since

$$\left(n + \frac{N-1}{2} + \frac{\alpha+1}{2}\right) \frac{\pi}{2(n+1)} - \frac{\pi(N-1+2\alpha)}{4} \leq \pi,$$

and

$$\left(n + \frac{N-1}{2} + \frac{\alpha+1}{2}\right) \frac{3\pi}{4} - \frac{\pi(N-1+2\alpha)}{4} \geq \frac{n\pi}{2},$$

we can now estimate I'_2 as

$$I'_2 \geq C \int_{\pi}^{\left[\frac{n}{2}\right]\pi} \frac{|\sin\omega| \, d\omega}{\left(\omega + \frac{N-1}{2} + \alpha\right)^{\alpha - \frac{N-3}{2}}} = C \sum_{\tau=1}^{\left[\frac{n}{2}\right]-1} \int_{\tau\pi}^{(\tau+1)\pi} \frac{|\sin\omega| \, d\omega}{\left(\omega + \frac{N-1}{2} + \alpha\right)^{\alpha - \frac{N-3}{2}}}.$$

Applying the change of variable $\beta = t + \tau\pi$, we obtain

$$\begin{aligned}
 I'_2 &\geq C \sum_{\tau=1}^{\left[\frac{n}{2}\right]-1} \int_0^{\pi} \frac{|\sin(t + \tau\pi)| \, dt}{\left(t + \tau\pi + \frac{\pi(N-1+2\alpha)}{4}\right)^{\alpha - \frac{N-3}{2}}} \geq C \sum_{\tau=1}^{\left[\frac{n}{2}\right]-1} \int_0^{\pi} \frac{\sin t \, dt}{\tau^{\alpha - \frac{N-3}{2}}} = C \sum_{\tau=1}^{\left[\frac{n}{2}\right]-1} \frac{1}{\tau^{\alpha - \frac{N-3}{2}}} \\
 &\geq C \int_1^{\left[\frac{n}{2}\right]-1} \frac{dt}{t^{\alpha - \frac{N-3}{2}}}.
 \end{aligned}$$

Once more, the following 3 cases are considered:

If $\alpha < \frac{N-1}{2}$ then

$$I_2' \geq C \int_1^{\left[\frac{n}{2}\right]-1} t^{\frac{N-1}{2}-\alpha-1} dt = \frac{C}{\frac{N-1}{2}-\alpha} \left\{ \left(\left[\frac{n}{2}\right] - 1 \right)^{\frac{N-1}{2}-\alpha} - 1 \right\} \geq C n^{\frac{N-1}{2}-\alpha}. \quad (3.18)$$

If $\alpha = \frac{N-1}{2}$ then

$$I_2' \geq C \int_1^{\left[\frac{n}{2}\right]-1} \frac{dt}{t} = C \ln \left(\left[\frac{n}{2}\right] - 1 \right) \geq C \ln n. \quad (3.19)$$

If $\alpha > \frac{N-1}{2}$ then

$$I_2' \geq C \int_1^{\left[\frac{n}{2}\right]-1} t^{\frac{N-1}{2}-\alpha-1} dt = \frac{C}{\alpha - \frac{N-1}{2}} \left\{ 1 - \left(\left[\frac{n}{2}\right] - 1 \right)^{\frac{N-1}{2}-\alpha} \right\} \geq C. \quad (3.20)$$

From (3.18), (3.19) and (3.20), we have

$$I_2' \geq \begin{cases} C n^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ C, & \alpha > \frac{N-1}{2}. \end{cases} \quad (3.21)$$

Applying the reverse triangle inequality, $|a - b| \geq |a| - |b|$ with the Riesz mean of the spectral function from (3.5), gives the following

$$\begin{aligned} I_2 &= C \int_{\frac{\pi}{2(n+1)}}^{\pi - \frac{\pi}{2(n+1)}} |\Theta_1^\alpha(\cos \gamma) + \Theta_2^\alpha(\cos \gamma) + \Theta_3^\alpha(\cos \gamma)| \sin^{N-1} \gamma \, d\gamma \\ &\geq C \int_{\frac{\pi}{2(n+1)}}^{\pi - \frac{\pi}{2(n+1)}} |\Theta_1^\alpha(\cos \gamma)| \sin^{N-1} \gamma \, d\gamma - C \int_{\frac{\pi}{2(n+1)}}^{\pi - \frac{\pi}{2(n+1)}} |\Theta_2^\alpha(\cos \gamma)| \sin^{N-1} \gamma \, d\gamma \\ &\quad - C \int_{\frac{\pi}{2(n+1)}}^{\pi - \frac{\pi}{2(n+1)}} |\Theta_3^\alpha(\cos \gamma)| \sin^{N-1} \gamma \, d\gamma \\ &= I_2' - I_2'' - I_2'''. \end{aligned} \quad (3.22)$$

Given the inequalities (3.21), (3.14) and (3.15) of (3.22), I_2 is bounded as follows

$$I_2 \geq \begin{cases} C n^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ 0, & \alpha > \frac{N-1}{2}, \end{cases} \quad (3.23)$$

and by manner of (3.2), $\mathcal{L}_n^\alpha \geq I_2$. Consequently, gives the following estimates

$$\mathcal{L}_n^\alpha \geq \begin{cases} Cn^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ 0, & \alpha > \frac{N-1}{2}. \end{cases} \quad (3.24)$$

Combination of the estimated upper bound of the Lebesgue constant in (3.17) and lower bound in (3.24), provides estimate of the Lebesgue constant, $\mathcal{L}_n^\alpha$ in form of

$$\begin{aligned} C' n^{\frac{N-1}{2}-\alpha} &< \mathcal{L}_n^\alpha < C'' n^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C' \ln n &< \mathcal{L}_n^\alpha < C'' \ln n, & \alpha = \frac{N-1}{2}, \\ 0 &< \mathcal{L}_n^\alpha < C, & \alpha > \frac{N-1}{2}. \end{aligned}$$

Theorem 2.1 is proved.

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