PAPER • OPEN ACCESS

An iterative method to include spatial dispersion for waves in nonuniform plasmas using wavelet decomposition

To cite this article: Pablo Vallejos et al 2016 J. Phys.: Conf. Ser. 775 012016

View the article online for updates and enhancements.

You may also like

- <u>Dispersion relation and instability for an</u> <u>anisotropic nonuniform flowing plasma</u> Min Uk Lee, Gunsu S Yun and Jeong-Young Ji
- Particle dynamics, black hole shadow and weak gravitational lensing in the f (Q) theory of gravity
 Allah Ditta, Xia Tiecheng, Farruh Atamurotov et al.
- Propagation of microwave radiation through an inhomogeneous plasma layer in a magnetic field
 B A Balakirev, V A Bityurin, A N Bocharov et al.





DISCOVER how sustainability intersects with electrochemistry & solid state science research



This content was downloaded from IP address 18.119.28.237 on 16/05/2024 at 15:20

An iterative method to include spatial dispersion for waves in nonuniform plasmas using wavelet decomposition

Pablo Vallejos, Thomas Johnson, Torbjörn Hellsten

Fusion Plasma Physics, EES, KTH, SE-10044 Stockholm, Sweden

E-mail: pablova@kth.se

Abstract. A novel method for solving wave equations with spatial dispersion is presented, suitable for applications to ion cyclotron resonance heating. The method splits the wave operator into a dispersive and a non-dispersive part. The latter can be inverted with e.g. finite element methods. The spatial dispersion is evaluated using a wavelet representation of the dielectric kernel and added by means of iteration. The method has been successfully tested on a low frequency kinetic Alfvén wave with second order Larmor radius effects in a nonuniform plasma slab.

1. Introduction

Plasma waves with perpendicular wavelengths comparable to the ion Larmor radius will experience a non-local response; an acceleration at one point along the gyro orbit will induce a current along the whole orbit. Consequently, the dielectric response is an integral operator, integrating the acceleration along the gyro orbit, which will depend on the wavelength, i.e. it is spatially dispersive.

Numerical methods, such as finite element (FE), finite difference (FD), and Fourier spectral methods, are efficient for solving non-dispersive electromagnetic problems. These methods can be used in certain limits for spatially dispersive problems, e.g. for calculating fast-wave propagation in fusion plasmas during ICRF with negligible spatial dispersive effects [1]. In general, the numerical modelling of waves in spatially dispersive media tends to be significantly more complicated than similar non-dispersive electromagnetic problems, due to its integral character. The inclusion of spatial dispersion effects to all orders in finite Larmor radius was first solved by Sauter et al [2], who derived a set of integro-differential equations for the wave fields in a plane slab which was solved with FE discretisation. Spatial dispersion can also be handled by Fourier spectral methods (see e.g. [3]), but have the disadvantage of producing large and dense matrices. Such matrices are time consuming to invert and memory expensive. Recently, an alternative technique has been proposed for plasma waves with spatial dispersion that uses either FE or FD methods and iterate on the induced current [4].

In this study we extend the method proposed in [5] by generalising the operator splitting. We propose to identify non-dispersive parts by evaluating the dispersive response at an approximate wave vector. The operator splitting is performed between the spatially dispersive and non-dispersive parts of the wave operator. The dispersive part is considered as an inhomogeneous

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI. Published under licence by IOP Publishing Ltd 1

Joint Varenna-Lausanne International	Workshop on the Theory	of Fusion Plasmas 2016	IOP Publishing
Journal of Physics: Conference Series	s 775 (2016) 012016	doi:10.1088/	1742-6596/775/1/012016

term in the wave equation, which is solved by means of iteration with Anderson acceleration [6]. The evaluation of the dispersive response is performed using a Morlet wavelet representation.

The paper is organised as follows: In section 2, the iterative procedure is formulated and the relation to kinetic Alfvén waves is described. In section 3, the spatially dispersive response is evaluated using wavelets. In section 4, a numerical example is presented, showing that the iteration procedure works. Conclusions are drawn in section 5.

2. Electromagnetic wave equations with spatial dispersion

The problem we are aiming to solve is a wave equation with a spatially dispersive response

$$\mathcal{L}[\mathbf{E}](\mathbf{r}) = i\omega\mu_0 \mathbf{J}_{ant}(\mathbf{r}), \qquad (1)$$

where

$$\mathcal{L}[\mathbf{E}](\mathbf{r}) \equiv \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - \frac{\omega^2}{c^2} \mathcal{K}[\mathbf{E}](\mathbf{r}), \qquad (2)$$

$$\mathcal{K}[\mathbf{E}](\mathbf{r}) \equiv \int \frac{d\mathbf{k}}{(2\pi)^3} \int d\mathbf{r}' K(\mathbf{r}, \mathbf{k}) \mathbf{E}(\mathbf{r}') \exp\left[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')\right].$$
(3)

The dielectric kernel K(x, k) is described in Ref. [7, 8].

When studying the propagation of a particular wave (e.g. the fast wave during ICRF), a solution to Eq. (1) can be obtained by assuming that the wave-vector is given by an approximate dispersion relation, $k_D(\mathbf{r})$. For example, when modelling ion cyclotron heating a large part of the wave is usually well described by a fast-wave dispersion relation [9]. The wave equation can then be written on a form where the dielectric response is no longer spatially dispersive

$$\mathcal{L}_0[\mathbf{E}](\mathbf{r}) \equiv \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - \frac{\omega^2}{c^2} K(\mathbf{r}, k_D(\mathbf{r})) \mathbf{E}(\mathbf{r}) = i\omega\mu_0 \mathbf{J}_{ant}(\mathbf{r}) \,. \tag{4}$$

This equation can be solved using standard FE or FD methods [1].

In this report we propose to solve Eq. (1) by first splitting the wave operator $\mathcal{L} = \mathcal{L}_0 + (\mathcal{L} - \mathcal{L}_0)$ and formally rewriting the equation on the form

$$\mathbf{E}(\mathbf{r}) = -\mathcal{L}_0^{-1} \left[(\mathcal{L} - \mathcal{L}_0) [\mathbf{E}](\mathbf{r}) + i\omega\mu_0 \mathbf{J}_{ant} \right] (\mathbf{r}) , \qquad (5)$$

where \mathcal{L}_0^{-1} can be generated using FE or FD methods. Eq. (5) can be solved using a fixed-point iteration scheme. This formulation is most effective when the spatial dispersion is weak, such that \mathcal{L} and \mathcal{L}_0 have similar solutions. However, the formulation is not restricted to this limit. In fact, using a fixed point iteration with Anderson acceleration [6, 4], as used in this report, a wide range of inhomogeneous problems with strong spatial dispersion can be solved.

2.1. Second order ODE describing Kinetic Alfvén waves

To study the properties of the proposed scheme, Eq. (5), we will study the solutions to an ODE of second order. This equation can be derived from Eq. (1) when considering kinetic Alfvén waves in a plasma that is homogeneous along straight field lines, while assuming that the ratio of the ion Larmor radius over the wavelength is small. When aligning the coordinates such that the magnetic field is in the z direction and the perpendicular wave number is in the x direction, the wave equation may be written as [9]

$$n_{\parallel}^{2} E(x) - \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx' \exp\left[ik(x-x')\right] K_{xx}(x,k) E(x') = 0, \qquad (6)$$

where the K_{xx} is the dielectric tensor component in the (x, x)-direction, n_{\parallel} is the parallel refractive index and E(x) is the x-directed electric field Fourier transformed in the z-direction. By expanding the dielectric tensor in the perpendicular wave number and neglecting the weak dependence on the parallel wave number, $K_{xx}(x,k) \approx K_0(x) - K_1(x)k_{\perp}^2$, an ODE is obtained

$$\left(\frac{\partial^2}{\partial x^2} + f^2(x)\right) E(x) = 0, \qquad (7)$$

where $f^2(x) = (K_0(x) - n_{\parallel}^2)/K_1(x)$.

3. Wavelet representation of the wave equation

Dielectric responses that include finite Larmor radius effects form an integral operator, or alternatively a differential operator of infinite order. To evaluate such operators, the basis for describing the electric field should ideally have inifinite number of derivatives. In inhomogeneous media, a spatially localised basis is preferable for computational efficiency. We therefore propose the use of a Morlet basis and continuous wavelet transform. The Morlet wavelet not only satisfies the conditions above, but also has a narrow Fourier spectrum such that harmonic functions have a narrow wavelet spectra.

3.1. Continuous wavelet transform

The wavelet transform is performed using the basis

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \,, \tag{8}$$

where

$$\psi(x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \left(e^{i\sigma x} - \kappa \right)$$
(9)

is the complex Morlet wavelet (see Fig. 1) with $\kappa = e^{-\sigma^2/2}$ and $\sigma = 6$. This choice of basis is localised in both real space around x = b and in wave number around $k = \sigma/a$ (the Fourier transform is a Gaussian with width 1/a, see Fig. 1). The wavelet transform, **WT**, is defined as [10]

$$E_{a,b} = \mathbf{WT}[E(x)](a,b) = \int_{-\infty}^{\infty} E(x)\psi_{a,b}^{*}(x)dx, \qquad (10)$$

with the inverse transform

$$E(x) = \mathbf{W}\mathbf{T}^{-1}\left[E_{a,b}\right](x) = \frac{1}{C_{\psi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_{a,b}\psi_{a,b}(x)\frac{dadb}{a^2},$$
(11)

$$C_{\psi} = 2\pi \int_{-\infty}^{\infty} |\hat{\psi}_{a,b}(k)|^2 \frac{dk}{k} \,, \tag{12}$$

where $\hat{\psi}_{a,b}(k)$ is the Fourier transform of $\psi_{a,b}(x)$.

3.2. Wavelet representations of the dielectric kernel

The spatially dielectric response in the wave equation, Eq. (1), can be expressed using a Morlet representation of the electric field (for simplicity, this derivation will be performed assuming an electric field that only depends on a single coordinate x)

$$\mathcal{K}[\mathbf{E}](x) = \frac{1}{C_{\psi}} \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} \frac{dk}{2\pi} K(x,k) \mathbf{E}_{a,b} \hat{\psi}_{a,b}(k) e^{ikx} \,. \tag{13}$$



Figure 1. Wavelet basis defined in Eq. (8). *Left*: Solid black and dashed red lines represent the real and imaginary part of $\psi_{1,0}(x)$, respectively. *Right*: Fourier transformed complex Morlet wavelets, $\hat{\psi}_{1,0}(k)$ (solid black) and $\hat{\psi}_{2,0}(k)$ (dashed red).

Since $\hat{\psi}_{a,b}(k)$ is localised around $k \sim \sigma/a$ (the wave number of the eikonal factor in the Morlet basis) one can make the expansion

$$K(x,k) = K\left(x,\frac{\sigma}{a}\right) + \left(k - \frac{\sigma}{a}\right) \frac{\partial K(x,k)}{\partial k}\Big|_{k=\frac{\sigma}{a}} + \frac{1}{2}\left(k - \frac{\sigma}{a}\right)^2 \frac{\partial^2 K(x,k)}{\partial k^2}\Big|_{k=\frac{\sigma}{a}} + \dots$$
(14)

Note that when K is a second order polynomial in k, such as for the kinetic Alfvén wave in Eq. (7), this expansion is exact. The dielectric response is then given by an inverse wavelet transform

$$\mathcal{K}[\mathbf{E}](x) = \mathbf{W}\mathbf{T}^{-1}\left[K_{a,b}\left(x,\frac{\sigma}{a}\right)\mathbf{E}_{a,b}\right](x)$$
(15)

$$K_{a,b}(x,k) = K(x,k) + i\left(\frac{x-b}{a^2}\right)\frac{\partial K(x,k)}{\partial k} + \frac{1}{2}\left(\frac{1}{a^2} - \left(\frac{x-b}{a^2}\right)^2\right)\frac{\partial^2 K(x,k)}{\partial k^2} + \dots$$
(16)

The coefficients in this expansion are Hermite polynomials. In the derivation of Eq. (16), the terms proportional to κ have been neglected, since they give a negligible contribution.

3.3. Wavelet representations of the kinetic Alfvén wave equation

The equation for the kinetic Alfvén wave, Eq. (6), can be expressed using the dielectric response in Eq. (16)

$$f^{2}(x)E(x) = \mathbf{W}\mathbf{T}^{-1}\left[\left\{k^{2} + i2\left(\frac{x-b}{a}\right)k - \left(\frac{1}{a^{2}} - \left(\frac{x-b}{a^{2}}\right)^{2}\right) + \dots\right\}\Big|_{k=\sigma/a}E_{a,b}\right](x).$$
 (17)

The same equation can be derived by inserting the Morlet representation into Eq. (7)

$$f^{2}(x)E(x) = -\frac{\partial^{2}}{\partial x^{2}} \left\{ \frac{1}{C_{\psi}} \int_{-\infty}^{\infty} \frac{da}{a^{2}} \int_{-\infty}^{\infty} db E_{a,b} \psi_{a,b}(x) \right\}.$$
 (18)



Figure 2. Solution to Eq. (17) and comparisons with the WKB solution of Eq. (18). *Left*: The real and imaginary parts of the electric field in blue, with the WKB solution is indicated by crosses. *Right*: Wavelet representation $E_{a,b}$ of the electric field with $k = \sigma/a$.

4. Results

The ODE in Eq. (17) has been solved on an interval (0, 120) for $f(x) = \frac{3}{2} + \frac{1}{2\pi} \operatorname{atan}([x - 60]/5)$, such that the solution to the local dispersion relation have wavelength between π and 2π .

Morlet wavelets are defined on an infinite interval. Applying them to a finite interval means the wavelet transform is no longer invertable near the boundaries. To ensure that the wavelet transform can be inverted inside our domain, the transform has been performed in an extended domain (-30, 150). In the extended layers, here called the "matched layers", a harmonic solution has been imposed that matches the dispersion relation at the boundary. The matched layers also provides the boundary conditions to the differential equation; in the layer to the left a right propagating wave is imposed with unit amplitude, $E_x = \exp(ikx)$, while at the right boundary a matching procedure is introduced to identify the complex amplitudes of the right and left propagating waves (although in the problems studied the left propagating waves can be neglected).

Numerical solutions of Eq. (17) are illustrated in Fig. 2. The solution in the figure has been compared with a WKB solution $(E(x) \sim \exp(\int^x k(x')dx')/\sqrt{k(x)})$, showing good agreement. More specifically, the solution represents correctly both the amplitude and the phase of the wave in an inhomogeneous domain. The wavelet spectrum, $E_{a,b}$ in Fig. 2, illustrates how the wavelet representation is localised near wavenumbers satisfying the dispersion relation at x = b. The spectrum is calculated on the finite domain (0, 120), excluding the matched layers, causing pollution (a numerical widening of the spectrum) in the wavelet-spectrum near the boundaries b = 0 and b = 120. This illustrates the importance of the matched layers to provide a clean transform.

While Eq. (13) provides an exact response, it is computationally more expensive than the expanded formulation in Eq. (16). To understand the type of error generated by expanding the dielectric response in Eq. (14), we have compared solutions with different order expansions. The results are shown in Fig. 3. In figure a) and b), both the second and third terms in Eq. (16) are neglected (thus approximating $K(x, k) = K(x, \sigma/a)$) and the solutions exhibit strong oscillations and a non-negligable offset in both the frequency and amplitude. In figure c) and d), the first order term in $(k - \sigma/a)$ has been added (second term in Eq. (16)), which reduced both the



Figure 3. Solutions of Eq. (17) represented in terms of the wave amplitude, |E(x)|, and the wave vector, $k(x) = \partial_x Im[\ln(E(x))]$. Figure a) and b) are evaluated when neglecting terms of order σ and σ^2 . Figure c) and d) are evaluated when neglecting terms of order σ . Figure c) and d) are evaluated when neglecting terms of order σ . Figure c) and d) are evaluated with all terms.

oscillations and the offset. Finally, in figures e) and f), all terms in Eq. (16) are kept, yielding good agreement with WKB solution.

5. Discussion and conclusions

A novel iterative technique for solving the spatially dispersive wave equation, Eq. (5), has been proposed. The technique has the potential of including spatial dispersive effects in a simple manner. Using fixed-point Picard iterations, this equation tends to be unstable. However,

Joint Varenna-Lausanne International Workshop on the Theory of F	Jusion Plasmas 2016	IOP Publishing
Journal of Physics: Conference Series 775 (2016) 012016	doi:10.1088/1742	-6596/775/1/012016

solutions can be found using Anderson acceleration. For Picard-unstable problems a large number of iterations is required to find a solution using Anderson acceleration; e.g. the solutions presented in section 4 were found after about 100 iterations. Initial studies indicate that the number of iterations depend mainly on the complexity of the solution, while the dependence on the initial guess and the grid resolution is weak. The present method is rather slow, however, there are several possibilities for optimization.

The numerical complexity of the wavelet representation can be simplified by expanding the dielectric response function for wave numbers near the fundamental wave number of the Morlet wavelet, σ/a . For dielectric responses of finite order in k, the expansion can be made to exactly represent the operator. However, for response tensors with all-order FLR effect a truncated expansion is of interest. We have shown that truncations neglecting second order may cause an oscillation of the wave amplitude, while keeping only zeroth order terms may give rise to an error in both the wave number and the amplitude of the solution. The higher order terms in this expansion describe the spectral width of Morlet wavelet. The nature of these oscillations are still being investigated, however, our conclusion is that the spectral width of the Morlet wavelet has to be taken into account to obtain a converged solution.

The Morlet wavelet has several attractive features, such as differentiability, localisation in space and wave number. However, the Morlet wavelet basis has redundancy in the representation of continuous functions, such that there is more than one way to represent the same signal. While it is still possible to generate an inverse transform, the wavelet representation tends to be computationally inefficient. The wavelet representation redundancy can be reduced by smart reduction of the grid parameters in wavelet space, i.e. $\{a, b\}$, when performing the inverse transform.

References

- [1] Villard L, Appert K, Gruber R and Vaclavik J 1986 Computer Physics Reports 4 95
- [2] Sauter O and Vaclavik J 1994 Computer Physics Communications 84 226
- [3] Jaeger E F, Berry L A, D'Azevedo E, Batchelor D B and Carter M D 2001 Physics of Plasmas 8 1573
- [4] Green D L and Berry L A 2014 Comput. Phys. Commun. 185 736
- [5] Hellsten T, Johnson T and Vallejos P 2014 Journal of Physics: Conference Series 561 012010
- [6] Walker H F and Ni P 2011 SIAM J Numer. Anal. 49 1715
- [7] Smithe D 1989 Plasma Phys. Controlled Fusion 31 1105
- [8] Dumont R J, Phillips C K and Smithe D N 2005 Phys. of Plasmas 12 042508
- [9] Stix T H 1992 Waves in Plasmas (Springer-Verlag New York)
- [10] Daubechies I 1992 Ten lectures on wavelets (Philadelphia, PA, USA: Society for Industial and Applied Mathematics)