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Numerical solution of the complex modified Korteweg-de Vries equation by DQM

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Abstract. In this paper, a method based on the differential quadrature method with quintic B-spline has been applied to simulate the solitary wave solution of the complex modified Korteweg-de Vries equation (CMKdV). Three test problems, namely single solitary wave, interaction of two solitary waves and interaction of three solitary waves have been investigated. The efficiency and accuracy of the method have been measured by calculating maximum error norm L_∞ for single solitary waves having analytical solutions. Also, the three lowest conserved quantities and obtained numerical results have been compared with some of the published numerical results.

1. Introduction

In nature, various problems are modeled by partial differential equations. Being one of the well-known natural phenomena it is also a model for nonlinear evolution of plasma waves [1], propagation of transverse waves in a molecular chain model [2], and in a generalized elastic solid [3, 4]. Because of its importance, many researchers have dealt with the Complex Modified Korteweg-de Vries (CMKdV) equation given in the following form

$$\frac{\partial W(x,t)}{\partial t}+\alpha\frac{\partial\left(|W(x,t)|^2 W(x,t)\right)}{\partial x}+\frac{\partial^3 W(x,t)}{\partial x^3}=0,\quad -\infty<x<\infty,\quad t>0,\tag{1}$$

where W is a complex valued function of the the spatial coordinate x and the time t, α is a constant parameter.To avoid complex computation, we need to transformation of CMKdV equation (1) into a nonlinear coupled system by decomposing $W(x,t)$ into its real and imaginary parts

$$W(x,t)=U(x,t)+iV(x,t),\quad i=\sqrt{-1}$$

and obtain the real valued-modified Korteweg-de Vries equation system,

$$U_t+\alpha\left[3U^2U_x+V^2U_x+2UVV_x\right]+U_{3x} = 0,\tag{2}$$

$$V_t+\alpha\left[3V^2V_x+U^2V_x+2UVU_x\right]+V_{3x} = 0,\tag{3}$$

where $U(x,t)$ and $V(x,t)$ are real functions.

2. Quintic B-spline DQM

Bellman *et al.*[8] first introduced DQM in 1972 where partial derivative of a function with respect to a coordinate direction is expressed as a linear weighted sum of all the functional values at all mesh points along that direction[9]. Let's take the grid distribution $a = x_1 < x_2 < \cdots < x_N = b$ of a finite interval $[a, b]$ into consideration. Provided that any given function $U(x)$ is smooth enough over the domain, its derivatives with respect to x at a grid points x_i can be approximated by a linear summation of all the functional values in the domain, namely,

$$U_x^{(r)}(x_i) = \frac{d^{(r)}U}{dx^{(r)}}\Big|_{x_i} = \sum_{j=1}^{N} w_{ij}^{(r)} U(x_j), \quad i = 1, 2, ..., N, \quad r = 1, 2, ..., N-1 \tag{4}$$

where r denotes the order of derivative, $w_{ij}^{(r)}$ represent the weighting coefficients of the $r-th$ order derivative approximation, and N denotes the number of grid points in the solution domain. Here, the index j represents the fact that $w_{ij}^{(r)}$ is the corresponding weighting coefficient of the functional value $U(x_j)$. The quintic B-splines used as a base functions which are defined as given in [10].

2.1. Weighting coefficients of the first order derivative

From Eq.(4) with value of $r = 1$, and using quintic B-splines as test functions we have obtained the following equations

$$Q_k'(x_i) = \sum_{j=k-2}^{k+2} w_{i,j}^{(1)} Q_k(x_j), \tag{5}$$

For example, for the first grid point x_1 (5), we get the following equation

$$Q_k'(x_1) = \sum_{j=k-2}^{k+2} w_{1,j}^{(1)} Q_k(x_j), \tag{6}$$

By substituting the values of quintic basis functions into Eq.(6) and using four additional equations obtained from the derivative of Eq.(6) at four different B-spline Q_k $(k = -1, 0, N+1, N+2)$ and eliminating four unknown terms from the system of equations, we obtain the following system of equations

$$\begin{bmatrix} 37 & 82 & 21 & & & & & & & \\ 8 & 33 & 18 & 1 & & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & 1 & 26 & 66 & 26 & 1 & \\ & & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & & 1 & 18 & 33 & 8 \\ & & & & & & & 21 & 82 & 37 \end{bmatrix} \begin{bmatrix} w_{1,-1}^{(1)} \\ w_{1,0}^{(1)} \\ w_{1,1}^{(1)} \\ w_{1,2}^{(1)} \\ w_{1,3}^{(1)} \\ w_{1,4}^{(1)} \\ \vdots \\ w_{1,N+1}^{(1)} \\ w_{1,N+2}^{(1)} \end{bmatrix} = \begin{bmatrix} -\frac{109}{2h} \\ -\frac{29}{h} \\ 0 \\ \frac{50}{h} \\ \frac{5}{h} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

Similarly, using the value of quintic basis functions at x_i, $(2 \le i \le N)$ grid points, respectively, the equation systems is obtained which are used to determine the weighting coefficients.

So, weighting coefficients $w_{i,j}^{(1)}$ which are related to the x_i, $(i = 1, 2, ..., N)$ are found quite easily by solving the obtained equation systems with Thomas algorithm. Determining of the weighting coefficients of the third order derivatives have same process.

3. Numerical discretization

We discritize the equations (2) − (3) separately by using forward finite difference and Crank-Nicolson.

$$2U^{n+1} + \Delta t\left[U_{3x}^{n+1} + \alpha\left(3\left(U^2U_x\right)^{n+1} + \left(V^2U_x\right)^{n+1} + 2\left(UVV_x\right)^{n+1}\right)\right]$$

$$= 2U^n + \Delta t\left[-U_{3x}^n + \alpha\left(3\left(U^2U_x\right)^n + \left(V^2U_x\right)^n + 2\left(UVV_x\right)^n\right)\right] \tag{8}$$

Then, Rubin and Graves linearization technique[11] is used at the left side of the Eq. (8) to linearize the nonlinear terms so we obtained

$$2U^{n+1} + \Delta t[U_{3x}^{n+1} + 3\alpha\left(\left(U^2\right)^n U_x^{n+1} + 2U^nU_x^nU^{n+1}\right) + \alpha\left(\left(V^2\right)^n U_x^{n+1} + 2V^nU_x^nV^{n+1}\right)$$

$$+2\alpha\left(U^{n+1}V^nV_x^n + U^nV^{n+1}V_x^n + U^nV^nV_x^{n+1}\right)]$$

$$= 2U^n + \Delta t\left[-U_{3x}^n + 3\alpha\left(U^2\right)^n U_x^n + \alpha\left(V^2\right)^n U_x^n + 2\alpha U^nV^nV_x^n\right] \tag{9}$$

Let us define some terms to use in Eq. (9) as,

$$A_i^n = \sum_{j=1}^N w_{ij}^{(1)}U_j^n = U_{x_i}^n,\quad B_i^n = \sum_{j=1}^N w_{ij}^{(3)}U_j^n = U_{3x_i}^n, C_i^n = \sum_{j=1}^N w_{ij}^{(1)}V_j^n = V_{x_i}^n,\quad D_i^n = \sum_{j=1}^N w_{ij}^{(3)}V_j^n = V_{3x_i}^n \tag{10}$$

where A_i^n and B_i^n are the first and third-order derivative approximations of U functions at the n-th time level on x_i points, respectively. And C_i^n and D_i^n are the first and third order derivative approximations of V function at the $n-th$ time level on x_i points, respectively. By the substitution of definition (10) at Eq. (9) we obtained

$$2U_i^{n+1} + \Delta t[\sum_{j=1}^N w_{ij}^{(3)}U_j^{n+1} + 3\alpha\left((U_i^n)^2\sum_{j=1}^N w_{ij}^{(1)}U_j^{n+1} + 2U_i^nA_i^nU_i^{n+1}\right) \tag{11}$$

$$+\alpha\left((V_i^n)^2\sum_{j=1}^N w_{ij}^{(1)}U_j^{n+1} + 2V_i^nC_i^nU_i^{n+1}\right)+2\alpha\left(V_i^nA_i^nV_i^{n+1} + U_i^nC_i^nV_i^{n+1} + U_i^nV_i^n\sum_{j=1}^N w_{ij}^{(1)}V_j^{n+1}\right)] = f_i^n$$

where

$$f_i^n = 2U_i^n + \Delta t\left[-B_i^n + \alpha\left(3\left(U_i^n\right)^2A_i^n + \left(V_i^n\right)^2A_i^n + 2U_i^nV_i^nC_i^n\right)\right], \tag{12}$$

we reorganised Eq. (11) for each grid points as,

$$\left[2 + \Delta t\left(w_{ii}^{(3)} + \alpha\left(3\left(U_i^n\right)^2w_{ii}^{(1)} + 6U_i^nA_i^n + \left(V_i^n\right)^2w_{ii}^{(1)} + 2V_i^nC_i^n\right)\right)\right]U_i^{n+1}$$

$$+\left[\sum_{j=1,i\neq j}^N \Delta t\left(w_{ij}^{(3)} + \alpha\left(3\left(U_i^n\right)^2w_{ij}^{(1)} + \left(V_i^n\right)^2w_{ij}^{(1)}\right)\right)U_j^{n+1}\right]$$

$$+\left[2\alpha\Delta t\left(V_i^nA_i^n + U_i^nC_i^n + U_i^nV_i^nw_{ii}^{(1)}\right)\right]V_i^{n+1} + \left[\sum_{j=1,i\neq j}^N\left(2\alpha\Delta tU_i^nV_i^nw_{ij}^{(1)}\right)V_j^{n+1}\right] = f_i^n \tag{13}$$

By the same process, the Eq. (3) is discritized, linearized and organised. Then, boundary conditions have been applied to system of equations and the first and last equations are eliminated from each systems and solved by Gauss elimination method easily.

4. Numerical examples

The accuracy of the numerical method is checked by using the error norm L_∞ and three lowest invariants:

$$\begin{aligned} L_\infty &\simeq \max_j \left| U_j^{exact} - (U_N)_j \right|, \; I_1 = \int_{-\infty}^{\infty} w dx, \\ I_2 &\simeq \sum_{j=1}^{N} h_j |w_j^n|^2, \;\; I_3 \simeq \sum_{j=1}^{N} h_j \left[\frac{\alpha}{2} |w_j^n|^4 - (w_x^n)_j^2 \right] \end{aligned}$$

4.1. Single soliton

The analytically solution of complex mKdV equation is given in[5] as:

$$W(x,t) = \sqrt{\frac{2c}{\alpha}} \text{sech} \left[\sqrt{c} \left(x - x_0 - ct \right) \right] \exp \left(i\theta \right) \tag{14}$$

where soliton standing at x_0 position initially and moving to the right hand with constant c velocity and satisfies the boundary conditions $W \to 0$ as $x \to \pm\infty$. We first take $\alpha = 2$, $\theta = 0, c = 1$, $x_0 = 0$ in $[-20, 40]$ and at $t = 0$, we obtain the initial condition.

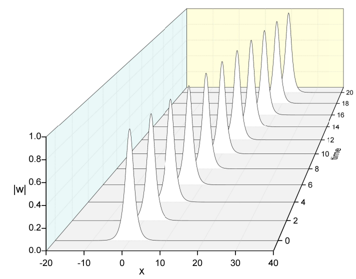

Figure 1. Single soliton

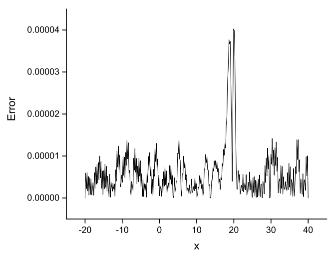

Figure 2. Absolute error

Table 1. L_∞ error norms and invariants for $\theta = 0$ and $\Delta t = 0.01$.

	Present, $N = 336$				P-G FEM [6] $N = 600$			
t	L_∞	I_1	I_2	I_3	L_∞	I_1	I_2	I_3
0	0.000000	3.141590	2.000000	0.666667	0.000000	3.141590	2.000000	0.669765
5	0.000065	3.141578	1.999999	0.666666	0.000057	3.141592	2.000000	0.669764
10	0.000070	3.141606	1.999999	0.666666	0.000108	3.141592	1.999999	0.669764
15	0.000068	3.141588	1.999999	0.666666	0.000163	3.141592	1.999999	0.669763
20	0.000066	3.141572	2.000000	0.666667	0.000218	3.141592	1.999999	0.669763

As it is seen from the Figure 1, with the time run up to the $t = 20$ the amplitude and velocity of wave do not change as a result of properties of solitons. As it is seen straightforward from Table 1 and 2 the present error norms L_∞ are smaller than earlier works [6, 7]. It is seen from Figure 2 that the maximum absolute error at time $t = 20$ is found 4.02×10^{-5} at $x = 19.94$.

4.2. Interaction of two solitary waves

The sum of two solitary waves which initial condition is given in [7] as:

$$W(x,0) = \sqrt{\frac{2c_1}{\alpha}} \text{sech} \left[\sqrt{c_1} \left(x - x_1 \right) \right] \exp \left(i\theta_1 \right) + \sqrt{\frac{2c_2}{\alpha}} \text{sech} \left[\sqrt{c_2} \left(x - x_2 \right) \right] \exp \left(i\theta_2 \right)$$

Table 2. L_∞ error norms and invariants for $\theta = \pi/4$ and $\Delta t = 0.01$.

	Present, $N = 336$			Coll. FEM [7] $N = 600$		
Time	L_∞	I_2	I_3	L_∞	I_2	I_3
0.0	0.000000	4.000000	1.333334	0.000000	4.000000	1.339529
5.0	0.000049	4.000000	1.333334	0.000078	3.999999	1.339529
10.0	0.000050	3.999999	1.333333	0.000154	3.999998	1.339528
15.0	0.000051	3.999998	1.333332	0.000231	3.999998	1.339528
20.0	0.000039	3.999994	1.333329	0.000308	3.999997	1.339526

where $x_1 = 25$ and $x_2 = 50$ are initial positions of two solitary waves, respectively in $[0, 100]$.

We investigated the interaction of two ortogonally polarized solitary waves which are interact with $y-$polarized ($\theta_1 = 0$) and $z-$polarized ($\theta_2 = \pi/2$) then the interaction of two $y-$polarized solitary waves which are interact with $y-$polarized ($\theta_1 = \theta_2 = 0$). We used fix value of $\alpha = 2, c_1 = 2, c_2 = 0.5$, and both simulations time run up to $t = 30$. As it seen clearly from Figure 3 and Figure 4 that at simulation of two ortogonally polarized solitary waves after the interaction a tail appeared behind the shorter wave and in opposition to two ortogonally polarized solitary waves there is not any tail appeared after the interaction of two $y-$polarized solitary waves. The obtained invariants given in Table 3 is acceptable good.

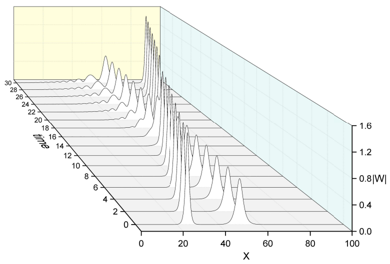

Figure 3. Two ortogonally polarized solitary waves

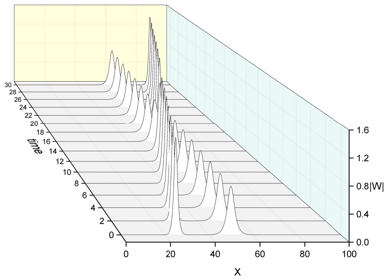

Figure 4. Two y-polarized solitary waves

Table 3. Invariants of two orthogonally and two $y-$polarized solitons for $\Delta t = 0.01$.

	two ortogonally polarized solitary waves				two y-polarized solitary waves	
t	$I_1(\mathrm{Real})$	$I_1(\mathrm{Imag})$	I_2	I_3	I_2	I_3
0	3.141593	3.141591	4.242640	2.121332	4.242640	2.121333
5	3.141630	3.141593	4.242590	2.121228	4.242587	2.121222
10	3.141667	3.141593	4.242537	2.121124	4.242529	2.121112
15	3.141534	3.141590	4.242371	2.120882	4.242330	2.120804
20	3.141566	3.141598	4.242489	2.121043	4.242473	2.121001
25	3.141812	3.142181	4.242424	2.120901	4.242431	2.120914
30	3.138532	3.135895	4.242476	2.120740	4.242377	2.120808

4.3. Interaction of three solitons

Our third test problem is interaction of three solitons which is given in [7] as follows

$$W(x,0) = \sqrt{\frac{2c_1}{\alpha}}\mathrm{sech}\left[\sqrt{c_1}\,(x-x_1)\right]\exp\left(i\theta_1\right) + \sqrt{\frac{2c_2}{\alpha}}\mathrm{sech}\left[\sqrt{c_2}\,(x-x_2)\right]\exp\left(i\theta_2\right)$$

$$+\sqrt{\frac{2c_3}{\alpha}}\text{sech}\left[\sqrt{c_3}\left(x-x_3\right)\right]\exp\left(i\theta_3\right)$$

where $x_1 = 10$, $x_2 = 30$ and $x_3 = 50$ are initial positions of three single solitons, respectively. We investigated the interaction of three solitons which are interact with $y-$polarized ($\theta_1 = \theta_2 = \theta_3 = 0$) by using $\alpha = 2, c_1 = 1, c_2 = 0.5$ and $c_3 = 0.3$ time up to $t = 80$.

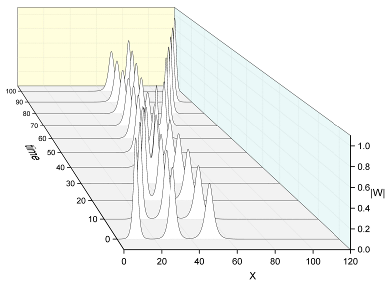

Figure 5. Three y-polarized solitons

Table 4. Invariants of three $y-$polarized solitons for $\Delta t = 0.01$.

	QBDQM $N = 501$		Coll.FEM[7] $N = 1200$	
t	I_2	I_3	I_2	I_3
0	4.510004	1.012099	4.510006	1.015897
20	4.510002	1.012098	4.510005	1.015783
40	4.509997	1.012103	4.510006	1.015031
60	4.510002	1.012097	4.510008	1.015660
80	4.510003	1.012096	4.510005	1.015752

5. Conclusion

In this work, we have implemented DQM based on quintic B-splines for numerical solution of complex mKdV equation. One of the main characteristics of the present method is to be able to obtain good results by using less number of grid points. As can be observed by the comparison between the obtained results of present method and earlier works, QBDQM results are acceptable good. The obtained results show that QBDQM can be used to produce reasonable accurate numerical solutions of the complex mKdV equation. So, QBDQM is a reliable one for getting the numerical solutions of some physically important nonlinear problems.

6. References

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