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Comparison of different approaches in the Trefftz method for analysis of fluid flow between regular bundles of cylindrical fibres

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Abstract. In the paper three different approaches for the Trefftz method are compared in analysis of the fluid flow between regular bundles of cylindrical fibres. The approximate solution is a linear combination of such trial functions which fulfil exactly the governing equations. The trial functions can be defined in the Cartesian coordinate system (the first approach), in the cylindrical coordinate system (and can fulfil also some boundary conditions – the second approach) or be defined as a fundamental solutions (the third approach – the method of fundamental solutions).The average velocity and the product of the friction factor and the Reynolds number $f \cdot \mathrm{Re}$ are compared for selected parameters of a considered region.

1. Introduction

The Trefftz method (TM) was proposed in 1926 by Erich Trefftz [1]. In this method an approximate solution is a linear combination of trial functions which fulfil exactly the governing equation of the considered problem. The trial functions in the method are called the Trefftz functions or the T-functions. However real development of this method came only with development of computer techniques. One of the version of the Trefftz method is the boundary collocation technique. In the method the governing equation is fulfilled exactly by the trial functions while the boundary conditions are satisfied in an approximate way. Different applications of the method can be found in [2].

Sometimes the T-functions fulfil also some of the boundary conditions. In such case the method is often called the Trefftz method with the special purpose Trefftz functions (SPTF) [3]. Then the boundary collocation technique is applied only for unfulfilled boundary conditions. This approach was successfully applied for different problems in applied mechanics, e.g. conductive heat flow [4], fluid flow in conduits with polygonal cross-section [5] or elastic torsion of bars [6].

Another version of the Trefftz method is the method of fundamental solutions (MFS). The MFS was proposed in 1964 by Kupradze and Aleksidze [7]. The numerical implementation of the MFS was given by Mathon and Johnston [8]. In the MFS the approximate solution is a linear combination of fundamental solutions which are functions of distance between the points and the source points. The fundamental solutions fulfil exactly the governing equation and the unknown coefficients of the approximate solution are calculated using the boundary collocation technique [2]. The source points are located outside the considered region which has similar shape to the boundary of the domain. Some interesting review of applications of the MFS can be found in [9-11].

The purpose of the present paper is to compare these three approaches in analysis of the longitudinal fluid flow between bundles of cylindrical fibres. The filtration velocity and the product of the friction factor, f and the Reynolds number, Re are compared for a different arrangement of parallel cylindrical fibers.

2. Formulation of the problem

Consider the steady, fully developed, laminar, isothermal flow of an incompressible viscous fluid driven by a constant pressure in a system of regular parallel fibers. The flow is longitudinal with respect to fibers which are arranged in a regular hexagonal (Fig. 1a), square (Fig. 1b) and triangular (Fig. 1c) array. The radius of the fibers is equal to *a*, and the distance between the fibers is equal to 2*b*.

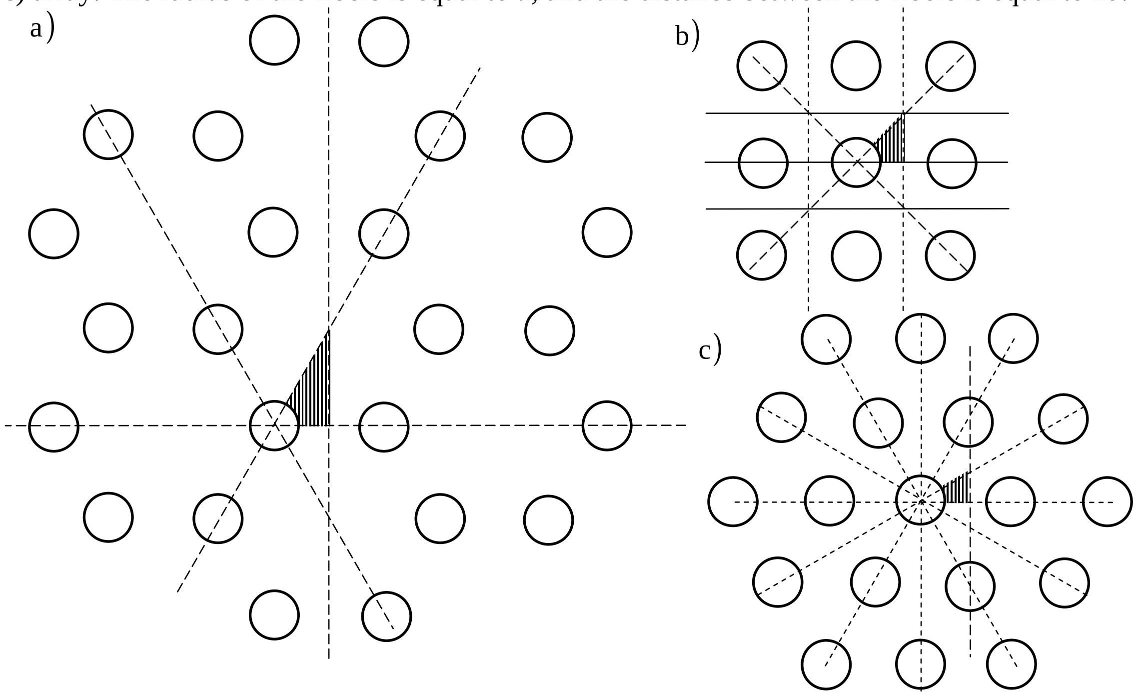

Figure 1. Three different kinds of a regular array of cylindrical rods (*a*) L = 6 hexagonal array, (*b*) L = 4 square array, and (*c*) L = 3 triangular array.

The equation of motion is reduced to a single partial differential equation in the form

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z} \quad \text{in} \quad \Omega_f \tag{1}$$

where w is the velocity component in the z direction, dp/dz is the constant pressure gradient, μ is the dynamic viscosity, and Ω_f is the repeated element of a fluid array (Fig 2a).
Equation (3) is solved with the following boundary conditions in a repeated element array (Fig. 2a)

$$w = 0 \quad \text{for} \quad \sqrt{x^2 + y^2} = a\,, \tag{2}$$

$$\frac{\partial w}{\partial y} = 0 \quad \text{for} \quad y = 0\,,\ a \le x \le b\,, \tag{3}$$

$$\frac{\partial w}{\partial x} = 0 \quad \text{for} \quad x = b\,,\ 0 \le y \le b \cdot \cot\left(\frac{\pi}{L}\right), \tag{4}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{for} \quad y = x \cdot \cot\left(\frac{\pi}{L}\right),\ a \cdot \sin\left(\frac{\pi}{L}\right) \le x \le b\,. \tag{5}$$

It is convenient to introduce the following dimensionless variables

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$$X=\frac{x}{b},\ Y=\frac{y}{b},\ E=\frac{a}{b},\ W=\frac{\mu\cdot w}{-b^2\cdot\frac{dp}{dz}}, \tag{6}$$

Now, the governing equation (3) has the dimensionless form

$$\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}=-1 \text{ in } \Omega_f, \tag{7}$$

Eq. (9) is solved with boundary conditions in a dimensionless form

$$W=0 \text{ for } \sqrt{X^2+Y^2}=E, \tag{8}$$

$$\frac{\partial W}{\partial Y}=0 \text{ for } Y=0,\ E\le X\le 1, \tag{9}$$

$$\frac{\partial W}{\partial X}=0 \text{ for } X=1,\ 0\le Y\le \cot\left(\frac{\pi}{L}\right), \tag{10}$$

$$\frac{\partial W}{\partial n}=0 \text{ for } Y=X\cdot\cot\left(\frac{\pi}{L}\right),\ E\cdot\sin\left(\frac{\pi}{L}\right)\le X\le 1. \tag{11}$$

The non-linear governing Eq. (9) with boundary conditions (10-13) yields a micro structural boundary value problem (MBVP) in a repeated element of an array of fibers.

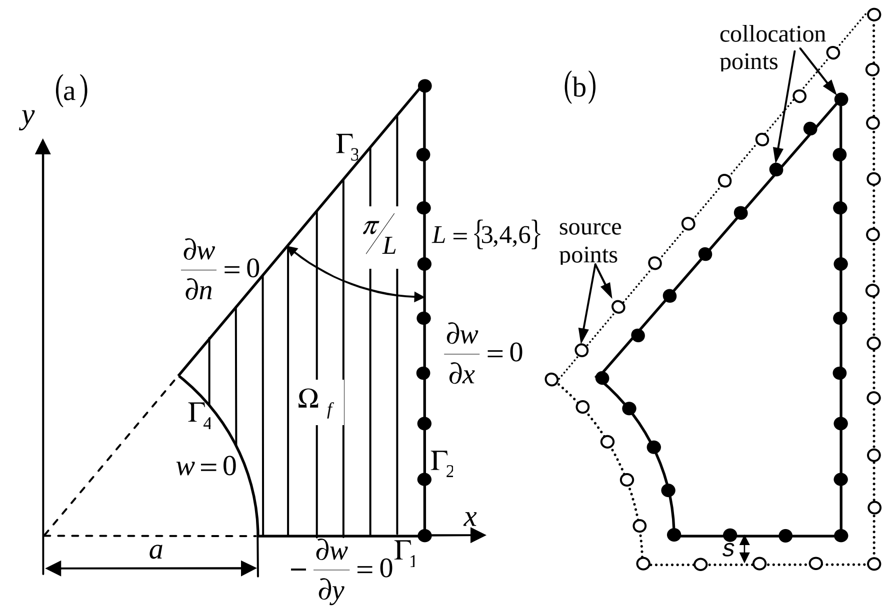

Figure 2. The repeated element a) the formulation of the boundary value problem, b) the distribution of source, collocation and interpolation points

3. Method of solution of micro structural boundary value problem

The MBVP problem can be solved using the meshless method. The approximate solutions can be expressed as a sum of general and particular solution

$$W(X,Y)=W_g(X,Y)+W_p(X,Y) \tag{12}$$

The particular solution for Eq. (9) has a form

$$W_p(X,Y) = -\frac{1}{4}\left(X^2 + Y^2\right) \tag{13}$$

The general solution $W_g(X,Y)$ fulfills the Laplace equation and can be solved using one of the three Trefftz method. The detailed differences are presented in this section.

3.1. The Trefftz method - TM

The approximate general solution using the Trefftz method can be written in the following form

$$W_g(X,Y) = \sum_{n=0}^{M} c_n F_n(X,Y) + \sum_{n=1}^{M} d_n G_n(X,Y) \tag{13}$$

where $F_n(X,Y)$ and $G_n(X,Y)$ are the trial functions defined in the Cartesian coordinate system

$$F_n(X,Y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{X^{n-2k} Y^{2k}}{(n-2k)!(2k)!}; \quad G_n(X,Y) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \frac{X^{n-2k-1} Y^{2k+1}}{(n-2k-1)!(2k+1)!}. \tag{14}$$

The above trial functions fulfill the Laplace equation in 2D. The unknown coefficients $\{c_n\}_{n=0}^{M}$ and $\{d_n\}_{n=1}^{M}$ are calculated by fulfilling the boundary conditions (8-11) in the collocation sense [2]. In this way we obtain a system of linear equations. The number of collocation points must be greater or equal to the number of unknowns coefficients ($2\cdot M$+1).

3.2. The Trefftz method with special purpose Trefftz functions - SPTF

As the general solution of the Laplace equation in 2D the following expression in the polar coordinate system can be used

$$W_g(R,\theta) = A_0 + A_1\theta + A_2\theta \ln R + A_3 \ln R + \sum_{n=1}^{M}\left(B_n R^{\lambda_n} + C_n R^{-\lambda_n}\right)\cos(\lambda_n\theta) + \sum_{n=1}^{M}\left(D_n R^{\lambda_n} + E_n R^{-\lambda_n}\right)\sin(\lambda_n\theta) \tag{15}$$

Applying it for the specific region, some of the boundary conditions can be fulfilled exactly by calculating some constants in the above solution. After that the approximate solution fulfils the governing equation and some of the boundary conditions (8, 9, 11).

$$W_g(R,\theta) = \frac{1}{4}E^2 + A_1 \ln\frac{R}{E} + \sum_{n=2}^{M} A_n\left(R^{L(n-1)_n} - \frac{E^{2L(n-1)}}{R^{L(n-1)}}\right)\cos\left(L(n-1)\theta\right). \tag{16}$$

The constants $\{A_n\}_{n=1}^{M}$ are calculated from Eq. (10) using the boundary collocation technique [2]. The number of collocation points located on Γ_2 must be greater or equal M.

3.3. The method of fundamental solutions - MFS

In the MFS the approximate solution is assumed as a linear combination of fundamental solutions. The fundamental solution fulfils exactly the governing equation and it is a function of distance between the point inside the considered region and the source point. Using the MFS the approximate solution for the Laplace equation takes the form

$$W_g(X,Y) = \sum_{j=1}^{M} c_j \ln(r_j), \tag{17}$$

where $\{c_j\}_{j=1}^{M}$ are unknown coefficients, r_j is the distance between the point (X, Y) and the j-th source point (Xs_j, Ys_j). To avoid the singularities sources points are located outside the considered domain on a fictitious boundary (see Fig. 2b). The distance between the source boundary and the real boundary is equal to *s*. Fulfilling the boundary conditions (8-11) in collocation points (Xc_i, Yc_i) we obtain a system of linear equations

$$\begin{aligned}
&\sum_{j=1}^{M} c_j \ln(r_{ij}) = W_b + \frac{1}{4}\left(Xc_i^2 + Yc_i^2\right), \quad i = 1,\ldots,N_1 \\
&\sum_{j=1}^{M} c_j \left(\frac{\partial \ln(r_{ij})}{\partial X} nx + \frac{\partial \ln(r_{ij})}{\partial Y} ny \right) = \frac{\partial W_b}{\partial n} + \frac{1}{2}\left(nx \cdot Xc_i + ny \cdot Yc_i\right), \quad i = N_1 + 1,\ldots,N
\end{aligned} \tag{18}$$

where $r_{ij}^2 = \left(Xc_i - Xs_j\right)^2 + \left(Yc_i - Ys_j\right)^2$, $\mathbf{n} = [nx, ny]$ is a unit normal vector to the boundary. The number of collocation points *N* must be greater or equal to the number of unknowns coefficients *M*.

After determined the velocity field, the filtration velocity can be calculated as

$$W_b(X,Y) = \frac{1}{\Omega} \int_{\Omega_f} W \, d\Omega_f \,, \tag{19}$$

where $\Omega = \frac{1}{2}\mathrm{ctg}\left(\frac{\pi}{\mathrm{L}}\right)$ is a total dimensionless area of the element, $\Omega_f = \frac{1}{2}\left(\mathrm{ctg}\left(\frac{\pi}{\mathrm{L}}\right) - \pi \frac{L-2}{2L} E^2\right)$ is a flow area.

Used the Darcy-Weisbach equation

$$\frac{\Delta p}{L} = f \, \frac{1}{4} \frac{\rho}{a} w_b \approx -\frac{dp}{dz} \,, \tag{20}$$

the Reynolds number definition

$$\mathrm{Re} = \frac{2a\rho w_b}{\mu} \,, \tag{21}$$

and the definition of the non-dimensional variables (6), the product of friction factor, *f* and Reynolds number, Re can be express by a dimensionless form

$$f \cdot \mathrm{Re} = \frac{8E^2}{W_b} \,. \tag{22}$$

For the fibrous porous medium the fiber volume fraction is defined as

$$\varphi = \pi E^2 \left(\frac{L-2}{2L}\right) \tan\left(\frac{\pi}{L}\right). \tag{23}$$

where *E* is the non-dimensional radius of the fiber and $L = \{3, 4, 6\}$.

4. Numerical experiments

In numerical experiments, the non-dimensional field of the velocity for longitudinal flow are calculated. The cylindrical fibers are arranged regularly in a triangular $L = 3$, square $L = 4$ and hexagonal $L = 6$ arrays.

At first we test the influence of the number of trial function *N* for a value of the error of fulfilling the boundary conditions. The calculation was performed for a square $L = 4$ arrays of fibers, $E = \{0.2, 0.4, 0.6, 0.8\}$ and results are presented on Figure 3. The smallest values of the error were obtain for the SPTF. Figure 4 shows the value of the condition number of the collocation matrix, *CN*. The condition number was calculated as a relation of the maximal to the minimal singular value of the matrix. The lowest value of *CN* was obtained for the TM and the greatest for the MFS.

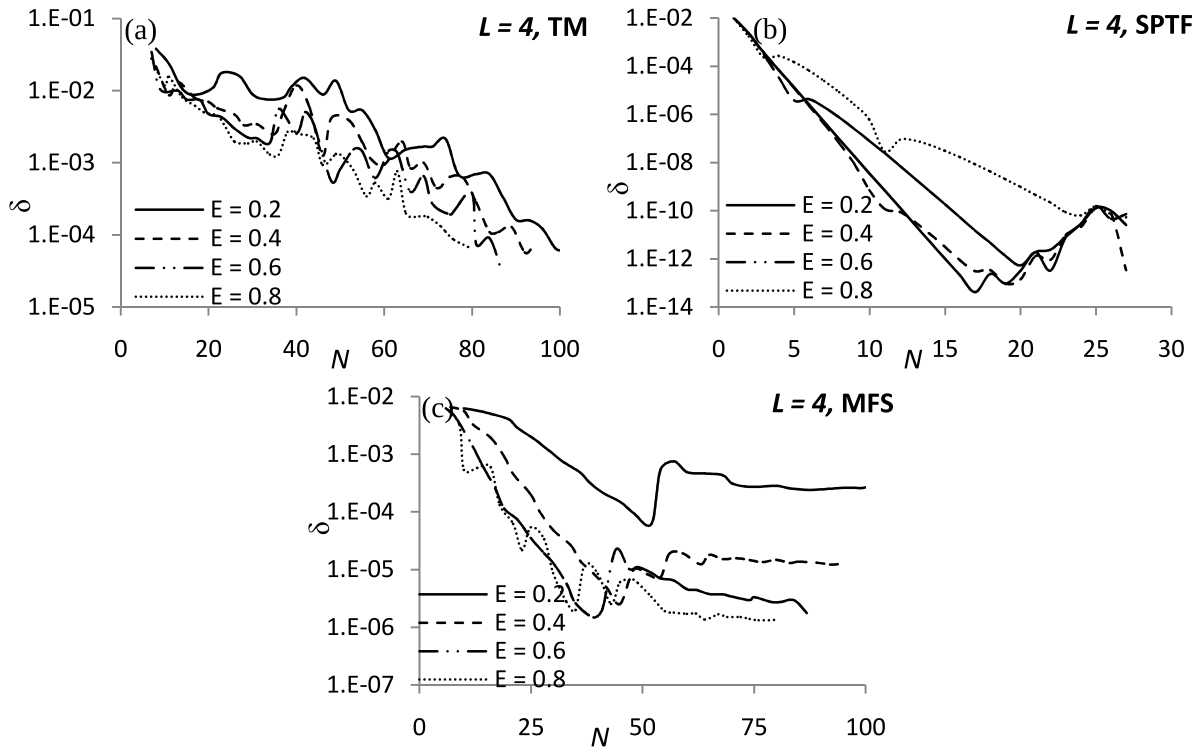

Figure 3. The error of fulfilling the boundary conditions for the square array of fibrous $L = 4$ and for three methods: TM, SPTF and MFS as a function of N

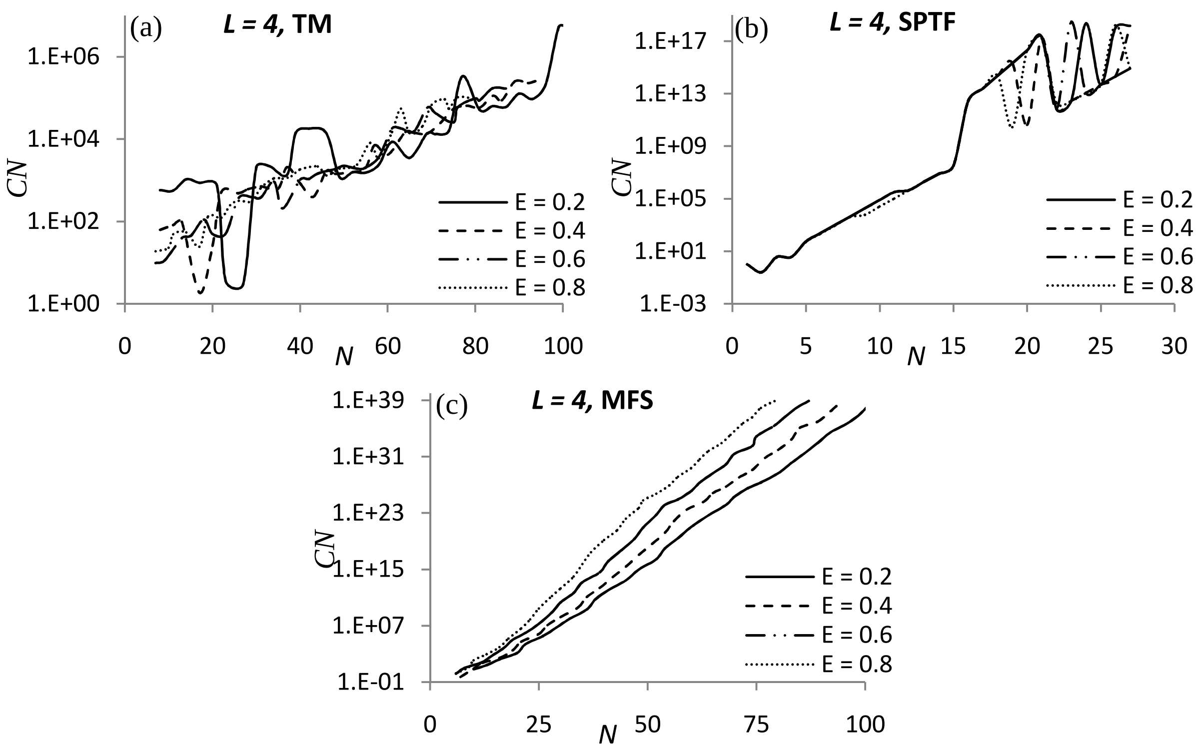

Figure 4. The condition number of collocation matrix for the square array of fibrous $L = 4$ and for three methods: TM, SPTF and MFS as a function of N

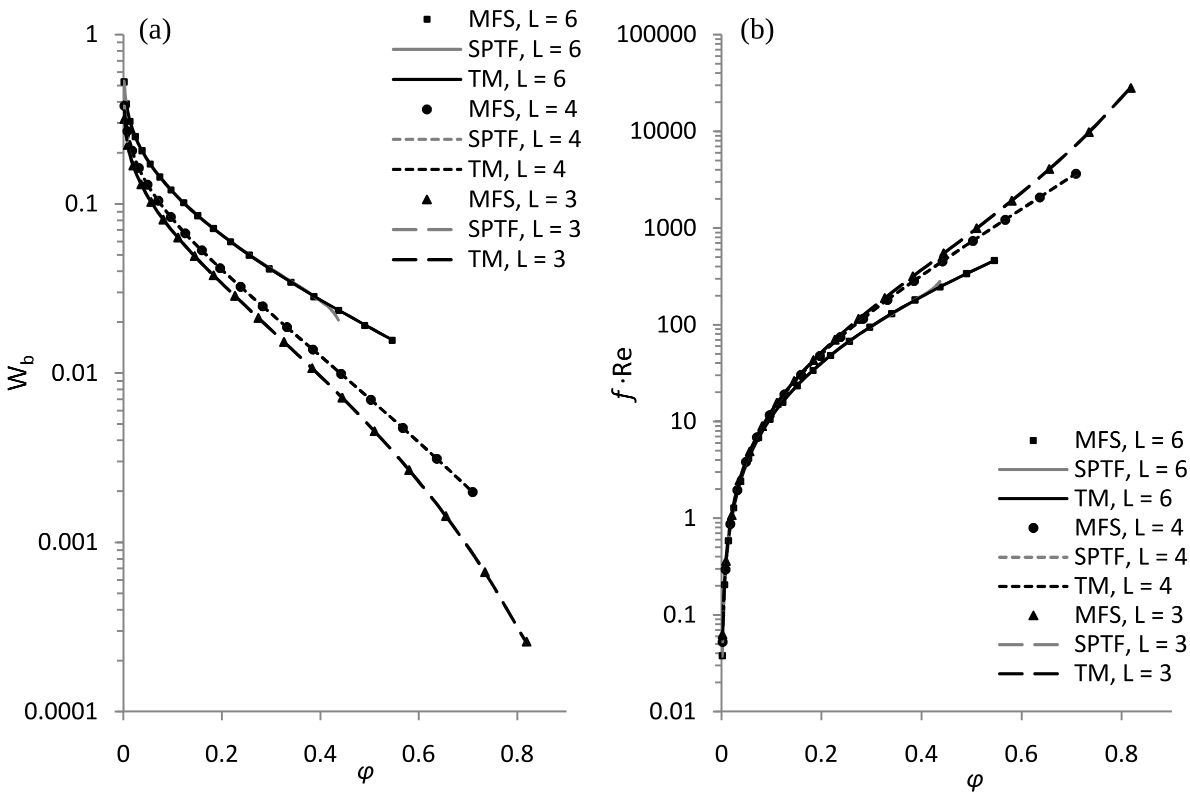

Figure 5. The filtration velocity W_b and the product $f \cdot$ Re for three different kind of fibrous array L = {3, 4, 6} and three methods: TM, SPTF and MFS as a function of fibrous volume fraction φ.

Figure 5 shows the value of the filtration velocity, W_b and the product, $f \cdot$ Re as a function of the fiber volume fraction, φ. The result obtained by the TM, the SPTF and the MFS were shown for hexagonal (L = 6), square (L = 4), and triangular array (L = 3) of fibers. For all methods the results are the same. For the Trefftz method we used 100 collocation points and M = 16. For the SPTF we used the M = 15, and the number of collocation points was also equal to 15. For the MFS the number of collocation point was equal to 86 while the number sources points was equal to 58. The distance between source contour and the boundary was equal s = 0.15.

5. Conclusions

Three different approaches for the Trefftz method were compared in analysis of the longitudinal Newtonian fluid flow through the regular bundles of cylindrical fibres. The cylindrical fibres were arranged regularly in a hexagonal, square and triangular array. The numerical algorithm was based on the boundary collocation method. For a repeated element the Trefftz method, the special purpose Trefftz functions and the method of fundamental solutions were used to determined the velocity field. Presented methods are easy to implement, accurate and meshless. The filtration velocity and the product of the friction factor and the Reynolds number were compared for those meshless methods. The results of $f \cdot$ Re and the filtration velocity obtained for all methods were the same. The SPTF is more accurate and less time consuming, because some of the boundary conditions are fulfilled exactly by the trial functions. This method need less number of collocation points, hence the collocation matrix has smaller size and is better conditioned. But we should notice that not for all the boundary value problem this method is appropriated. We can use this method easy for a domain with symmetry line and with simple boundary at which the boundary condition we can fulfil exactly.

Acknowledgments

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