Efficient operation of anisotropic synchronous machines for wind energy systems

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Efficient operation of anisotropic synchronous machines for wind energy systems

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Abstract. This paper presents an analytical solution for the Maximum-Torque-per-Ampere (MTPA) operation of synchronous machines (SM) with anisotropy and magnetic cross-coupling for the application in wind turbine systems and airborne wind energy systems. For a given reference torque, the analytical MTPA solution provides the optimal stator current references which produce the desired torque while minimizing the stator copper losses. From an implementation point of view, the proposed analytical method is appealing in terms of its fast online computation (compared to classical numerical methods) and its efficiency enhancement of the electrical drive system. The efficiency of the analytical MTPA operation, with and without consideration of cross-coupling, is compared to the conventional method with zero direct current.

Nomenclature
N, R, C are the sets of natural, real, and complex numbers. x ∈ R or x ∈ C is a real or complex scalar. x ∈ R^n (bold) is a real valued vector with n ∈ N. X ∈ R^{n×m} (capital bold) is a real valued matrix with n ∈ N rows and m ∈ N columns. x_s^k = (x_s^d, x_s^q)^T ∈ R^{n×m} is a stator space vector expressed in the synchronously rotating k-coordinate system (with orthogonal axes known as the direct (d) and quadrature (q) axes) and may represent voltage, current or flux linkage.

1. Introduction
For wind turbine systems (WTS), the optimal choice for the generator (electrical machine) adopted is still controversially discussed [1]. Doubly-fed induction machines (DFIMs) were considered one of the favourable choices for WTS, owing to their accompanied cost effective power converters and simple control of active/reactive power [2]. However, the depletion of DFIMs with respect to synchronous machines (SMs) became more pronounced due to three reasons: (i) partially rated power converters of DFIMs offer a rather poor voltage ride through capability [3, 4], (ii) SMs do not have slip rings (i.e. lower maintenance), eliminate the need for a gearbox (or allow for a drastic reduction of the gear ratio), have a simpler terminal wiring, and offer better grid support capabilities, and (iii) the increased calls for energy-efficient drives discriminates between SMs and DFIMs [5]. Particular interest has been set to permanent magnet SMs (PMSMs) and PM-excited reluctance SMs (PME-RSMs) [6], since they possess lower electrical losses than DFIMs due to the absence of the copper losses in the rotor. In general, for all WTS topologies such as large-scale or small-scale WTSs or airborne wind energy systems (AWESs), the selected SMs must be robust, cheap, and, most importantly, efficient. For pumping-mode AWESs [7, Ch. 2 & 3], the SM will operate in generator and also motor...
mode. Albeit PM machines can be controlled to operate at high power factor [5], this does not necessarily imply that the electrical losses within the adopted SM are minimised [8]. Thus, an optimal control strategy for the SMs is vital to fully realize its intrinsic efficiency. For PMSMs or PME-RSMs, the quadrature-axis stator current \( i^q_s \) is proportional to the produced electromagnetic torque \( m_m \) if the direct-axis stator current \( i^d_s \) is set to zero. However, in most cases, the electromagnetic torque is a nonlinear function of both stator currents [6, 9–11].

Therefore, for a given reference torque \( m_{m,\text{ref}} \), optimal stator current references \( i^d_{\text{s,ref}} \) and \( i^q_{\text{s,ref}} \) must be computed (see Fig. 1). These optimal current references allow for reduced stator copper losses and higher efficiencies than conventional methods with \( i^d_{\text{s,ref}} = 0 \) [10, 11]. Typically, (i) magnetic cross-coupling between the \( d \)- and \( q \)-axes is neglected (even though it has a direct effect on the machine torque; see (4) or [9, 10]) and (ii) numerical solutions are employed to compute the optimal references [12].

In [12], optimal feedforward torque control—known as Maximum-Torque-per-Ampere (MTPA)—was suggested by formulating an optimisation problem where a reference current vector with minimal magnitude was computed numerically, which in turn due to the anisotropy gives a noticeable efficiency increment compared to the conventional \( i^d_{\text{s,ref}} = 0 \) approach [13]. For high speeds, further efficiency enhancements can be achieved by considering the iron losses in the machine as well [9, 14]. However, modelling of iron losses is not straight forward (e.g. requires finite element analyses [9]) and, due to the rather low mechanical speeds in WTS, iron losses can be neglected [10]. Nevertheless, solving the MTPA optimisation problem requires nonlinear solvers such as the bisection method (known for its numerical stability but its poor convergence rate [12]) or the Newton-Raphson method (known for its fast convergence but its rather weak numerical stability depending on the initial condition [12]). Moreover, those solvers impose a rather high computational burden on the fixed-point processors adopted for such drives. Only very few results exist which compute an analytical solution to the MTPA optimization problem. In [8] and [15], an analytical expressions for the optimal \( d \)-axis reference current (the \( q \)-axis reference current comes from the outer speed control loop) has been presented, but magnetic cross-coupling and its impact on the electromagnetic torque were neglected.

In this paper, an analytical solution for the MTPA optimization problem considering magnetic cross-coupling (i.e. a non-zero mutual inductance \( L_m \neq 0 \)) is presented which allows to compute the optimal \( d \)- and \( q \)-axis current references

\[
i^k_{\text{s,ref}} = \begin{pmatrix} i^d_{\text{s,ref}} \\ i^q_{\text{s,ref}} \end{pmatrix} = \text{MTPA}(m_{m,\text{ref}}, L_m, \ldots, \text{other machine parameters})
\]  

(1)

\( i^k_{\text{s,ref}} \) based on the machine parameters and a given (feasible\(^1\)) reference torque \( m_{m,\text{ref}} \). The presented method, derived from Lagrangian optimization, is applicable to PMSMs, PME-RSMs or electrically-excited SMs (EESMs) with non-negligible anisotropy and magnetic cross-coupling. Due to space limitations, in the remainder of the paper, only the derivation for permanent-magnet synchronous machines (PMSMs) is shown and discussed.

2. MTPA with analytical solution for PMSMs with magnetic cross-coupling

In this section, the main result of the paper is derived based on the dynamical model of anisotropic PMSMs with magnetic cross-coupling. The result emanates from a Lagrangian formalization of the problem and invoking Ferrari’s method to solve quartic polynomials [16].

\(^1\) It is assumed that the currents can actually produce the desired reference torque within the electrical drive system; e.g. voltage constraints/maximum-torque-per-volt (MPTV) are not considered.
2.1. Dynamical model of anisotropic PMSMs with magnetic cross-coupling

The generic model of an anisotropic PMSM in the $k = (d, q)$-reference frame is given by [17]

\[
\begin{bmatrix}
\dot{u}_s^d(t) \\
\dot{u}_s^q(t)
\end{bmatrix} =
R_s \begin{bmatrix}
\psi_s^d(t) \\
\psi_s^q(t)
\end{bmatrix} + n_p \omega_m(t) \begin{bmatrix}
0 \\
1
\end{bmatrix} \begin{bmatrix}
\psi_s^d(t) \\
\psi_s^q(t)
\end{bmatrix} + \frac{d}{dt} \psi_s^k(t),
\]

\[
\begin{bmatrix}
\psi_s^k(0) \\
\phi_m(0)
\end{bmatrix} \in \mathbb{R}^2
\]

\[
\begin{bmatrix}
\psi_s^d(t) \\
\psi_s^q(t)
\end{bmatrix} = \begin{bmatrix}
L_s^d & L_m \\
L_m & L_s^q
\end{bmatrix} \begin{bmatrix}
i_s^d \\
i_s^q
\end{bmatrix} + \begin{bmatrix}
\psi_{pm} \\
0
\end{bmatrix}
\]

\[
\begin{cases}
\sigma = L_s^k = (L_s^k)^\top > 0, \\
\psi_m = \psi_{pm}
\end{cases}
\]

with stator voltages $u_s^k = (u_s^d, u_s^q)^\top$ (in V), stator currents $i_s^k = (i_s^d, i_s^q)^\top$ (in A), flux linkages $\psi_s^k = (\psi_s^d, \psi_s^q)^\top$ (in Wb), stator resistance $R_s$ (in Ω), number of pole pairs $n_p$, mechanical angle $\phi_m$ (in rad) and speed $\omega_m$ (in rad/s), inertia $\Theta$ (in kg m²), machine torque $m_{eq}$ and turbine torque $m_t$ (both in N m), gear ratio $g_r$ and viscous friction coefficient $\nu_f$ (in \(\frac{N m}{rad}\)). The stator flux linkage vector

\[
\psi_s(t) = \begin{bmatrix}
L_s^d & L_m \\
L_m & L_s^q
\end{bmatrix} \begin{bmatrix}
i_s^d \\
i_s^q
\end{bmatrix} + \begin{bmatrix}
\psi_{pm} \\
0
\end{bmatrix}
\]

is assumed to be an affine² function of the stator current vector $i_s^k$ and the permanent-magnet flux linkage vector $\psi_{pm}^k$ (in Wb). The inductance matrix $L_s^k$ with stator inductances $L_s^d > 0$, $L_s^q > 0$ and mutual (cross-coupling) inductance $L_m \neq 0$ is symmetric and positive-definite (i.e. $L_s^d L_s^q - L_m^2 > 0$ [17]). If the anti-diagonal elements $L_m$ of the inductance matrix $L_s^k$ are non-zero, a change in the $d$-axis current $i_s^d$ imposes a change in the $q$-axis flux linkage $\psi_s^q$ and vice versa. This effect is referred to as magnetic cross-coupling. Moreover, if the main-diagonal elements $L_s^d$ and $L_s^q$ have different values, i.e. $L_s^d - L_s^q \neq 0$ H, the (three-phase) stator flux linkage depends on the actual rotor position/angle, and the machine is said to be anisotropic. Fig. 2 illustrates the third quadrant (generator mode) of the flux linkage (3) of the simulated PMSM (see Sec. 3). Due to magnetic cross-coupling, the flux maps are slightly tilted. The electro-mechanical torque for such an anisotropic machine is given by³

\[
m_{eq}(i_s^k) = \frac{3}{2} n_p (i_s^k)^\top J_s i_s^k = \frac{3}{2} n_p \psi_{pm}^k i_s^k + \frac{3}{2} n_p (L_s^d - L_s^q) i_s^d i_s^q + \frac{3}{2} n_p L_m ((i_s^q)^2 - (i_s^d)^2),
\]

and can be split into three components: (i) permanent magnet torque, (ii) reluctance torque, and (iii) torque due to magnetic cross coupling (which is usually neglected [10]). Clearly, there exist infinitely many current vectors $i_s^k$ to produce the same torque (see (4) and/or Fig. 3).

2.2. Problem formulation

The MTPA optimization problem can be formulated as: Find the minimum current magnitude $\|i_s^k\|^2 = (i_s^k)^\top i_s^k$ for a given reference torque $m_{eq,ref}$ (see e.g. $m_{eq,ref}$ as in (17) in regime II), i.e.

\[
e_{s,ref} = \text{MTPA}(m_{eq,ref}, L_s^d, L_s^q, L_m, \psi_{pm}, \ldots) := \arg \min_{i_s^k} \|i_s^k\|^2 \text{ such that } m_{eq}(i_s^k) = m_{eq,ref}.
\]

² Note that a constant matrix $L_s^k$ is considered. Most publications deal with constant inductances $L_s^d$ and $L_s^q$ while the cross-coupling inductance $L_m$ is neglected [9, 10, 12]. Future work will consider nonlinear flux linkages.

³ For the sake of readability, the time-dependency will be omitted in the following. The factor 3/2 is due to an amplitude-correct Clarke transformation [18].
Due to the equality constraint, the optimization problem can be rewritten as Lagrangian
\[
\max_{t_s, \kappa} \mathcal{L}(t_s^k, \kappa) := -(t_s^k)^\top t_s^k + \kappa \left[ m_m(t_s^k) - m_{m,ref} \right] \tag{5}
\]
with Lagrangian multiplier \(\kappa \in \mathbb{C} \setminus \{0\}\). Clearly, \((t_s^k)^\top J L_s^k t_s^k = (t_s^k)^\top \left[ \frac{1}{2} (J L_s^k + L_s^k J^\top) \right] t_s^k\) and \((t_s^k)^\top J \psi_{pm} = (\psi_{pm}^k)^\top J^\top t_s^k\). Then, by defining \(I_2 := \begin{bmatrix} 1 & 0 \end{bmatrix}\),
\[
T := \frac{3}{2} n_p (J L_s^k + L_s^k J^\top) = \frac{3}{2} n_p \left[ \frac{L_m^d - L_m^q}{2} \right] = T^\top \quad \text{and} \quad t := \frac{3}{4} n_p J \psi_{pm} \left( \frac{0}{3} \right). \tag{6}
\]
the machine torque can be expressed as \(m_m(t_s^k) = (t_s^k)^\top T t_s^k + 2 (t_s^k)^\top t_s^k\) and the Lagrangian in (5) can be rewritten as quadratic as follows
\[
\mathcal{L}(t_s^k, \kappa) = -(t_s^k)^\top I_2 t_s^k + \kappa \left[ \frac{3}{2} n_p (t_s^k)^\top J \psi_{pm}^k - m_{m,ref} \right]
= (t_s^k)^\top \left[ -I_2 + \kappa \frac{3}{2} n_p (J L_s^k + L_s^k J^\top) \right] t_s^k + \kappa \left[ \frac{3}{2} n_p (\psi_{pm}^k)^\top J^\top t_s^k - \kappa m_{m,ref} \right]
= (t_s^k)^\top \left[ -I_2 + \kappa T \right] t_s^k + \kappa 2 t^\top t_s^k - \kappa m_{m,ref}. \tag{7}
\]

2.3. Analytical solution for the current references
To find the maximum of (5), it is sufficient to (i) set the gradient of (7) to zero and find its (optimal) solutions in \(t_s^k, \kappa\) and (ii) check negative definiteness of the Hessian of (7). The gradient and the Hessian of (7) are given by
\[
g_\mathcal{L}(t_s^k, \kappa) := \left( \frac{d \mathcal{L}(t_s^k, \kappa)}{d t_s^k} \right)^\top = \left( \frac{d}{d t_s^k} \mathcal{L}(t_s^k, \kappa) \right)^\top = \left( 2 \left[ -I_2 + \kappa T \right] t_s^k + \kappa 2 t^\top t_s^k - m_{m,ref} \right)^\top \in \mathbb{R}^3 \tag{8}
\]
and
\[
H_\mathcal{L}(t_s^k, \kappa) := \left( \frac{d^2 \mathcal{L}(t_s^k, \kappa)}{d t_s^k} \right)^\top = \left[ -I_2 + \kappa T, T t_s^k + t, \left( t_s^k \right)^\top T^\top + t^\top, 0 \right] = H_\mathcal{L}(t_s^k, \kappa)^\top \in \mathbb{R}^{3 \times 3}. \tag{9}
\]
respectively.
(i) Setting the gradient (8) to zero, i.e. \( \mathbf{g}_e(i_s^{k*}, \kappa) = \mathbf{0}_3 \), and solving the first two rows for \( i_s^{k*} = i_s^k \) yields

\[
\kappa (\kappa + 1) \kappa (\kappa - 1) = - \begin{bmatrix} - I_2 + \kappa \mathbf{T} \end{bmatrix}^{-1} \kappa \mathbf{T} = - \begin{bmatrix} -1 - \frac{3}{2} \kappa L_m n_p \kappa \frac{3}{2} n_p (L_s^d - L_s^q) \\ \kappa \frac{3}{2} n_p (L_s^d - L_s^q) - 1 + \frac{3}{2} \kappa L_m n_p \end{bmatrix}^{-1} \kappa \left( \frac{3}{4} n_p \psi_{pm} \right).
\]

Inserting \( i_s^{k*}(\kappa) \) into the last row of the gradient (8) gives a fourth-order polynomial

\[
a_4 \kappa^4 + a_3 \kappa^3 + a_2 \kappa^2 + a_1 \kappa + a_0 = 0
\]

in \( \kappa \) with five coefficients \( a_0, a_1, \ldots, a_4 \in \mathbb{R} \). The coefficients of (11) for the considered anisotropic PMSM model (2) depend on the machine parameters and are given by (details are omitted)

\[
a_0 = -16 m_{\text{m,ref}}^2, \quad a_1 = -18 n_p^2 (\psi_{pm})^2,
\]

\[
a_2 = m_{\text{m,ref}} \left[ 36 L_m^2 n_p^2 + \left( 3 n_p (L_s^d - L_s^q) \right)^2 + \frac{81}{2} L_m n_p^2 (\psi_{pm})^2 \right], \quad a_3 = 0, \quad \text{and}
\]

\[
a_4 = -\frac{81}{32} n_p^4 \left[ 2 m_{\text{m,ref}} \left( 4 L_m^2 + (L_s^d - L_s^q)^2 \right)^2 + 3 n_p (\psi_{pm})^2 \left( 4 L_m^2 + (L_s^d - L_s^q)^2 \right) \right].
\]

Note that the resulting quartic is a depressed quartic (since \( a_3 = 0 \)), which simplifies the calculation of its four roots \( \kappa_1^*, \ldots, \kappa_4^* \). The four roots depend on the coefficients \( a_i, i = 0, \ldots, 4 \) only (and, hence, on machine parameters and reference torque only) and can be computed analytically e.g. by invoking Ferrari’s method [16].

(ii) To obtain the optimal solution \( (i_s^{k*}(\kappa^*), \kappa^*) \) of the optimization (maximization) problem (5), the Hessian matrix (9) must be negative definite. The root \( \kappa^* \in \{ \kappa_1^*, \ldots, \kappa_4^* \} \), which renders the Hessian \( \mathbf{H}_e(i_s^{k*}(\kappa^*), \kappa^*) \) negative definite, is the optimal Lagrangian multiplier. Therefore, the Hessian matrix must be evaluated for all \( \kappa_i^*, i = 1, \ldots, 4 \), and checked for negative definiteness. This can be done by employing Sylvester’s criterion [19, Prop. 8.2.8]: All leading principle minors of (9) must be negative, which is the case if (for details see the Appendix)

\[
\begin{align*}
(a) \quad \kappa_i^* & > - \frac{1}{3 n_p L_m} \quad \implies -1 - \frac{3}{2} n_p L_m \kappa_i^* < 0 \\
(b) \quad |\kappa_i^*| & > \frac{1}{2 n_p \sqrt{ \frac{1}{4} (L_s^d - L_s^q)^2 + L_m^2 } } \quad \implies \det \left[ - I_2 + \kappa_i^* \mathbf{T} \right] < 0.
\end{align*}
\]

Both conditions (a) and (b) must be satisfied simultaneously, hence the following must hold

\[
\exists \kappa_i^* \in \{ \kappa_1^*, \ldots, \kappa_4^* \}: \quad - \frac{1}{3 n_p L_m} < \kappa^* := \kappa_i^* < - \frac{1}{2 n_p \sqrt{ \frac{1}{4} (L_s^d - L_s^q)^2 + L_m^2 } } < 0.
\]

For generator (or motor mode) only one \( \kappa^* = \kappa_i^* \) solves the optimization problem; which, finally, allows to compute the unique reference currents \( i_s^{k,\text{ref}} := i_s^{k*}(\kappa^*) = \text{MTPA}(m_{\text{m,ref}}, L_s^d, L_s^q, L_m, \psi_{pm}, \ldots) \) with \( i_s^{k*} \) as in (10) and \( \kappa^* \) as in (13).

Remark 1: As an alternative to evaluating the Hessian matrix, (10) can be evaluated for all four roots \( \kappa_1^*, \ldots, \kappa_4^* \). The vector \( i_s^{k*}(\kappa_i^*) \) with the smallest magnitude is the optimal solution.
2.4. Torque and MTPA hyperbolas (implicit and explicit expressions)

To illustrate the effect of anisotropy and magnetic cross-coupling on the current reference computation, the plots of the torque hyperbola and the MTPA hyperbola in the current loci are helpful (see Fig. 3). For reference torque $m_{m, \text{ref}}$, the torque hyperbola is implicitly given by

$$
\text{TRQ}(m_{m, \text{ref}}) := \left\{ \begin{bmatrix} i_s^k \end{bmatrix} \in \mathbb{R}^2 \bigg| \begin{bmatrix} i_s^k \end{bmatrix}^T T i_s^k + 2 t^T i_s^k = 0 \right\}
$$

where $T$ and $t$ are as in (6). Solving this quadric for $i_s^d$ yields the explicit formula

$$
\text{TRQ}(i_s^d, m_{m, \text{ref}}) = \frac{- (L_a - L_b^q) i_s^d + \psi_{pm}}{2L_m} + \frac{\sqrt{[(L_a - L_b^q)^2 + 4L_m^2(i_s^d)^2 + 2\psi_{pm}(L_a - L_b^q)i_s^d + 4L_m^{2m, \text{ref}} + \psi_{pm}^2]}}{2L_m}, \quad L_m \neq 0
$$

$$
\frac{\frac{3}{2} n_p \psi_{pm} + (L_a - L_b^q) i_s^d}{L_m}, \quad L_m = 0
$$

as function of $i_s^d$ which holds for all $i_s^d \neq \psi_{pm}/(L_a - L_b^q)$. Exemplary torque hyperbolas are plotted in Fig. 3 (see green lines). Note that, for $L_m = 0$ and $L_b^q = L_b^q$ (isotropic case), the torque hyperbola becomes a horizontal line (see Fig. 3a).

To derive an expression for the MTPA hyperbola, in the first two rows of the gradient (8) (i.e. $2 \left[ -I_2 + \kappa T \right] i_s^k + \kappa 2 t = 0_2$), the Lagrangian multiplier $\kappa$ can be eliminated. Solving this for $\kappa$ and rearranging gives the implicit expression of the MTPA hyperbola as follows

$$
\text{MTPA} := \left\{ \begin{bmatrix} i_s^k \end{bmatrix} \in \mathbb{R}^2 \bigg| \begin{bmatrix} i_s^k \end{bmatrix}^T \left[ \frac{3}{4} n_p (L_a - L_b^q) \begin{bmatrix} 2L_m i_s^k - 3 \frac{n_p L_m}{2} - 3 \frac{n_p (L_a - L_b^q)}{2} \end{bmatrix} \right] i_s^k + \left( \frac{3}{4} n_p \psi_{pm}, 0 \right) i_s^k = 0 \right\}.
$$

Solving this quadric for $i_s^d$ yields the explicit expression of the MTPA hyperbola

$$
\text{MTPA}(i_s^d) = \begin{cases} 2L_m i_s^d + \sqrt{[(L_a - L_b^q)^2 + 4L_m^2(i_s^d)^2 + 2\psi_{pm}(L_a - L_b^q)i_s^d + 4L_m^{2m, \text{ref}} + \psi_{pm}^2]} \quad & L_m \neq L_b^q \\ (i_s^d = 0) \quad & L_m = 0 \land L_b^q = L_b^q. \end{cases}
$$

In Fig. 3b, it can be clearly seen that, for a machine with $L_m \neq 0$, neglecting the magnetic cross-coupling will have a significant effect on the shape of the MTPA hyperbola and, hence, on the current reference computation. The computed reference will not have a minimal magnitude.

3. Simulation results

For the upcoming simulations, a small-scale wind turbine system (SS-WTS) with a 17.7 kW PMSM is considered. The PMSM has rated torque $m_{m, \text{rated}} = 49.3 \text{ Nm}$, rated speed $\omega_{m, \text{rated}} = 360 \text{ rad/s}$ and the following electrical parameters

$$
L_a^q = 3.5 \text{ mH}, \quad L_b^q = 1.5 L_b^d, \quad L_m = 0.15 L_b^d, \quad \psi_{pm} = 0.2 \text{ Wb}, \quad n_p = 3 \quad \text{and} \quad R_s = 0.12 \Omega. \quad (16)
$$

Three different MTPA strategies are implemented:

- the proposed MTPA considering anisotropy and cross-coupling (i.e. $L_a^d \neq L_b^q$ and $L_m \neq 0$, see solid blue line (-----) in Fig. 4 and 5),
- the standard MTPA($L_m = 0$) considering anisotropy but neglecting cross-coupling (i.e. $L_a^d \neq L_b^q$ and $L_m = 0$, see dashed blue line (-----) in Fig. 4 and 5) [10], and
Iron losses are not considered. For WTS, the angular speed will (in most cases) not exceed the rated speed.

Simulations have been conducted in Matlab/Simulink (R2016a) to compare the performance and the effects of the MTPA strategies on the efficiency of the electrical drive system and the overall small-scale wind turbine system. Moreover, the proposed MTPA considering anisotropy and cross-coupling was implemented by using the analytical solution and a conventional numerical solution (Newton-Raphson). Both solutions match with high accuracy (deviation $< 10^{-26}$), but the analytical solution was obtained 20 times faster than the numerical one. In the remainder, only the analytical solutions for the three MTPA strategies are implemented and discussed.

**Remark 2:** Note that the proposed algorithm is applicable to any PMSM, PME-RSM or EESM with non-negligible anisotropy and magnetic cross-coupling independently of the machine’s power rating. Small-scale machines tend to have a higher ratio of reluctance torque to permanent magnet torque (represented by (i) and (ii) in (4), respectively) and, therefore, the application to small-scale wind turbine systems is discussed, since it illustrates better the efficiency gain of the proposed analytical MTPA strategy.

### 3.1. Current loci and efficiency improvement of the electrical drive system

Fig. 3 shows the current loci (third quadrant / generator mode) of an isotropic PMSM (see Fig. 3a) and an anisotropic PMSM with parameters as in (16) (see Fig. 3b). Anisotropy and magnetic cross-coupling are not negligible and should be considered in the MTPA strategy. Neglecting $L_m$ leads to an incorrect computation of the reference currents (see Fig. 3b) and, hence, higher losses (see Fig. 4b). For copper losses $P_{\text{Cu}} = \frac{3}{2} R_s \| i_s \|^2$, friction losses $P_{\text{fric}} = \nu \omega_m^2$ and mechanical power $P_{\text{mech}} = m_i (i_s) \omega_m$, the efficiencies

$$\eta = \frac{P_{\text{mech}} - P_{\text{Cu}} - P_{\text{fric}}}{P_{\text{mech}}} = 1 - \frac{P_{\text{Cu}} + P_{\text{fric}}}{P_{\text{mech}}}$$

are compared in generator mode for different speeds $\omega_m$ using the simple MTPA$(i_s^d = 0)$, the conventional MTPA$(L_m = 0)$ and the proposed MTPA (considering cross-coupling). As illustrated in Fig. 4, considering anisotropy and cross-coupling allows for a significant efficiency improvement differing for different machine speeds. The efficiency increase compared to the simple MTPA$(i_s^d = 0)$ is measured by the relative efficiency gain (see Fig. 4c)

$$\Delta \eta := \eta(X) - \eta(\text{MTPA}(i_s^d = 0)) = \frac{P_{\text{Cu}}(\text{MTPA}(i_s^d = 0)) - P_{\text{Cu}}(X)}{P_{\text{mech}}} \text{ with } X \in \{\text{MTPA}, \text{MTPA}(L_m = 0)\}.$$
3.2. Overall small-scale wind turbine system (SS-WTS) with anisotropic PMSM

Now, three identical SS-WTSs with anisotropic PMSM (2) parametrized as in (16), gear ratio $g_t = 14.7$, inertia $\Theta = 0.199 \frac{kg}{m}$ and viscous friction coefficient $\nu_t = 5 \cdot 10^{-3} \frac{N \cdot m}{rad}$ are implemented in Matlab/Simulink. The SS-WTS has turbine power $P_t = c_p(\lambda)P_w$ (in W) and turbine torque $m_t = P_t \omega_{w}$ \cite{18} where $P_w = \frac{1}{2} \pi \varrho r_t^2 v_w^3$ (in W), $\varrho = 1.293 \frac{kg}{m}$, $r_t = 3.38 m$ and $v_w$ (in $\frac{m}{s}$) are wind power, air density, turbine radius and wind speed, respectively. The power coefficient $c_p(\cdot)$ from \cite[Example III.2]{20} with maximum $c_p^* := c_p(\lambda^*) = \max_{\lambda} c_p(\lambda) = 0.441$ and optimal tip speed ratio $\lambda^* = 6.91$ is used. For maximum power point tracking (MPPT) in region II (i.e. $v_w \in [5, 12] \frac{m}{s}$ for the simulations), the nonlinear turbine speed controller \cite{18, 21}

$$m_{m, \text{ref}} = -k_p^* \omega_m^2 + \nu_m \omega_m - \frac{\Theta_m - \Theta_e}{\omega_e} \xi$$

where $k_p^* := \frac{g_t^5 \pi c_p(\lambda^*)}{2 g_t r_t}$ and $\xi(s) = \left[1 - \frac{1}{1 + s T_m}\right] \omega_m(s)$ (17)

with inertia compensation\cite{21} (p. 482) and (viscous) friction feedforward compensation is implemented for all three SS-WTS. The underlying current control loops \cite{17} are identical for all three SS-WTS. The only difference between the three SS-WTS is that the reference torque $m_{m, \text{ref}}$ (controller output) is fed to one of the three introduced MTPA strategies (see above).

The simulation results are depicted in Fig. 5 (line colors are as introduced above). The wind speed $v_w(\cdot)$ varies within the interval $[7, 12] \frac{m}{s}$. Only the SS-WTS with the proposed MTPA (considering cross-coupling) achieves maximum power point tracking (MPPT) at steady state, since $\lambda \rightarrow \lambda^*$ for all wind speeds (see second subplot in Fig. 5). For the SS-WTS with MTPA($d = 0$) and MTPA($L_m = 0$), the speed controller (17) does not guarantee MPPT; not even at steady state. Due to the incorrect current reference computation, the produced machine torque $m_m \neq m_{m, \text{ref}}$ differs from the reference torque (17) (see also \cite{22}) and $\lambda \rightarrow \lambda^*$ cannot be established. Hence, the maximum turbine power $c_p^* P_w$ can not be extracted, i.e. $P_t = c_p(\lambda) P_w < c_p^* P_w$. The relative extracted power $\Delta P_{\text{ext}}$ (normalized with respect to the maximum turbine power $c_p^* P_w$) and the relative torque deviation $\Delta m_m$ (normalized with respect to the reference torque (17)) of each SS-WTS can be computed by

$$\Delta P_{\text{ext}} := \frac{(c_p(\lambda) P_w - P_{\text{Cu}} - P_{\text{fric}} - c_p^* P_w)}{c_p^* P_w} \quad \text{and} \quad \Delta m_m := \frac{m_m - m_{m, \text{ref}}}{m_{m, \text{ref}}}$$

respectively. $\Delta m_m$ and $\Delta P_{\text{ext}}$ are shown in the third and fourth subplot of Fig. 5, respectively. For example, for the SS-WTS with MTPA($d = 0$) at $v_w = 12 \frac{m}{s}$, the relative torque deviation can reach more than 15% with a relative loss in extracted power of almost 8.5%.

Note that $\xi \approx T_e \frac{d}{dt} \omega_m$ approximates the time derivative of the machine speed and $\Theta_e < \Theta$ (in kg m$^2$) is the desired reduced (virtual) inertia of the WTS. For the simulations, $\Theta_e = \Theta/4$ and $T_e = 0.2$ s were chosen.
Figure 5: Simulation results for a SS-WTS with anisotropic PMSM (parametrized as in (16)) operated in region II (maximum power point tracking) under (step-like) varying wind speeds $v_w$ ($\in [7, 12]$ m/s).

4. Conclusion
In this paper, an analytical solution for the Maximum-Torque-per-Ampere (MTPA) optimization problem has been derived for permanent-magnet synchronous machines with anisotropy and magnetic cross-coupling. The proposed MTPA strategy has been applied to a small-scale wind turbine system. The main contributions of this paper have been (i) the derivation of the analytical solution considering magnetic cross-coupling (which has not been done in literature before) and (ii) performance and efficiency comparisons with both, the conventional method (with zero direct current) and the state of the art MTPA strategy (neglecting magnetic cross-coupling). It has been shown that including the magnetic cross-coupling into the optimization problem can significantly increase the efficiency of the system (copper losses are reduced). Moreover, the analytical solution allows for an almost instantaneous computation of the optimal reference currents. Compared to the numerical Newton-Raphson method, the analytical solution converged to the correct solution 20 times faster. The proposed analytical MTPA strategy and its reference current computation can be adopted within already existing small-scale or large-scale wind turbine systems (WTSs) without any further modifications. Independently of the machine’s power rating, the proposed solution can be applied to any permanent-magnet synchronous machine (PMSM), permanent-magnet-excited reluctance synchronous machine (PME-RSM) or electrically-excited synchronous machine (EESM) with non-negligible anisotropy and magnetic cross-coupling (see also Remark 2). Future work will focus on the extension of the analytical solution to nonlinear flux maps (i.e., when the inductances are functions of the stator currents).

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Appendix

Note that, by defining

\[ M(i_s^k) := \begin{bmatrix} I_2, & [ - I_2 + \kappa T ]^{-1}(T_i^k + t) \end{bmatrix} \in \mathbb{R}^{3 \times 3} \quad \text{with} \quad \det M(i_s^k) = 1 \]

and by invoking [19, Fact 2.16.2], the Hessian matrix

\[ \boldsymbol{H}_2(i_s^k, \kappa) = 2M(i_s^k)^\top \begin{bmatrix} - I_2 + \kappa T, & 0_2, & -((i_s^k)^\top T + t)^\top [ - I_2 + \kappa T ]^{-1}(T_i^k + t) \end{bmatrix} M(i_s^k) \]

can be written as product of three matrices. Hence,

\[ \det [ \boldsymbol{H}_2(i_s^k, \kappa) ] = -2((-i_s^k)^\top T + t)^\top [ - I_2 + \kappa T ]^{-1}(T_i^k + t) \cdot \det [ - I_2 + \kappa T ], \]

which, with \( \det M(i_s^k) = \det M(i_s^k)^\top = 1 \) and \( \det [ - I_2 + \kappa T ] < 0 \) (12), implies that \( \alpha = ||T_i^k + t||_Q^2 < 0 \) (a weighted norm with negative definite \( Q := [- I_2 + \kappa T] < 0 \)) is negative for all non-zero vectors \( T_i^k + t \neq 0_2 \). Inserting \( i_s^k = i_s^{k,*}(\kappa) \) with \( i_s^{k,*}(\kappa) \) as in (10) yields

\[ T_i^{k,*}(\kappa) + t = \begin{cases} \kappa T[- I_2 + \kappa T]^{-1}t + t & \text{for all} \quad \kappa = \kappa^* \quad \text{as in (12)} \end{cases} \]

References