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Scalar field cosmology via non-local integrals of motion

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Abstract. In re-parametrization invariant systems, such as mini-superspace Lagrangians, the existence of constraints can lead to the emergence of additional non-local integrals of motion defined in phase space. In the case of a FLRW flat/non-flat space-time minimally coupled to an arbitrary scalar field, we manage to use such conserved quantities to completely integrate the system of equations of motion. This is achieved without constraining the potential in any way. Thus, obtaining the most general solution that encompasses all possible cosmological scenarios which can be based on the existence of a scalar field.

1. Introduction and general theory

Symmetries are widely used in the study of cosmological systems, especially within the minisuperspace approximation [1]-[8]. The study of Noether symmetries of both regular [9] and singular Lagrangians [10], [11] that are used in this process, provides the means to constrain quantities involved in these systems so as to yield them integrable ([12], [13]). In this work we are going to exploit certain non-local symmetries that appear in singular systems to fully integrate a scalar field with an arbitrary potential, minimally coupled to gravity under the ansatz of an FLRW space-time (spatially flat/non-flat). A work where an integration of this type, without imposing conditions on the potential, has been previously done in homogeneous cosmology for Bianchi types I and V [14].

It was proven in [10], that for singular systems described by Lagrangians of the form

$$\bar{L} = \frac{1}{2N(t)} \bar{G}_{\mu\nu}(q) \dot{q}^{\mu}(t) \dot{q}^{\nu}(t) - N(t) U(q)$$
(1.1)

all conformal Killing fields of $\overline{G}_{\mu\nu}$ can be used to write down integrals of motion. To observe that, let us for simplicity perform a re-parametrization of the form $N \rightarrow n = NV$. Then, it can be seen that for the equivalent system

$$L = \frac{1}{2n} G_{\mu\nu}(q) \dot{q}^{\mu} \dot{q}^{\nu} - n, \qquad (1.2)$$

where $G_{\mu\nu} = U \bar{G}_{\mu\nu}$, if there exist vector fields $\xi^{\alpha}(q)$ defined on the configuration space for which the relation $\pounds_{\xi} G_{\mu\nu} = \omega(q) G_{\mu\nu}$ holds, then the quantity

$$Q = \xi^{\alpha} p_{\alpha} + \int n(t)\omega(q(t))dt, \qquad (1.3)$$

with $p_{\alpha} = \frac{\partial L}{\partial q^{\alpha}}$, defines integrals of motion on the phase space due to the existence of the Hamiltonian constraint

$$\mathcal{H} = \frac{1}{2} G^{\mu\nu} p_{\mu} p_{\nu} + 1 \approx 0 \tag{1.4}$$

It can be easily verified that, due to ξ being a conformal Killing vector with conformal factor ω , Q is a constant of motion

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \{Q, \mathcal{H}\} = \omega \mathcal{H} \approx 0.$$

For a general description of constrained systems and their properties, see ([15],[16])

In the special case were $\omega = 0$, i.e. the ξ 's are Killing vectors of the scaled mini-supermetric $G_{\mu\nu}$, relation (1.3) leads to autonomous integrals of motion that do not exhibit any explicit time dependence. On the other hand, when $\omega \neq 0$, non local integrals of motion emerge that are actually rheonomic, due to the explicit time dependence owed to the integral in (1.3).

Our starting point, for the derivation of the mini-superspace Lagrangian, is the action

$$S = \int d^4x \left(\sqrt{-g}R + \epsilon \,\phi_{,\mu} \phi^{,\mu} + 2V(\phi) \right) \tag{1.5}$$

where R is the Ricci scalar and g the determinant of the spacetime metric $g_{\mu\nu}$. We also allow for the existence of a phantom field by introducing a pure sign constant $\epsilon = \pm 1$. As is known, variation with respect to $g^{\mu\nu}$ yields the equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}$$
(1.6)

with $T_{\mu\nu} = \epsilon \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} (\epsilon \phi^{,\kappa}\phi_{,\kappa} - 2V(\phi)) g_{\mu\nu}$, being the energy-momentum tensor for the matter part. Variation of (1.5) with respect to the scalar field leads to the Klein-Gordon equation

$$\epsilon \Box \phi - V'(\phi) = 0, \qquad (1.7)$$

where the prime denotes differentiation over ϕ .

It can be easily shown that, under the ansatz

$$ds^{2} = -N(t)^{2}dt^{2} + a(t)^{2} \left(\frac{1}{1-kr^{2}}dr^{2} + r^{2}d\theta^{2} + d\varphi^{2}\right),$$
(1.8)

the demand for the field to be spatially homogeneous $\phi = \phi(t)$, and a re-parametrization of the lapse function

$$N = \frac{n}{2a \left(a^2 V(\phi) - 3k\right)}.$$
(1.9)

the set of equations (1.6) and (1.7) is equivalent to the Euler-Lagrange equations derived by a Lagrangian of the form (1.2) with $q^{\mu} = (a, \phi)$ and the mini-supermetric being

$$G_{\mu\nu} = 4a^2 \left(a^2 V(\phi) - 3k \right) \begin{pmatrix} -6 & 0\\ 0 & \epsilon \, a^2. \end{pmatrix}$$
(1.10)

Any of the conformal Killing fields of $G_{\mu\nu}$ can now be used to define integrals of motion for the corresponding system.

2. The spatially flat case

In the k = 0 case, the mini-supermetric (1.10) exhibits a homothetic vector that is independent of the scalar field potential $V(\phi)$, namely $\xi = \frac{a}{6} \frac{\partial}{\partial a}$, that satisfies $\pounds_{\xi} G_{\mu\nu} = G_{\mu\nu}$. This results in the existence of a conserved quantity in phase space that is written as

$$Q = \frac{a}{6}p_a + \int n(t)dt = \frac{a}{6}\frac{\partial L}{\partial \dot{a}} + \int n(t)dt = -\frac{4a^5\dot{a}V(\phi)}{n} + \int n(t)dt = \kappa$$
(2.1)

with κ being a constant.

We proceed by fixing the gauge through the choice $\phi(t) = t$. At the same time we express the scaled lapse function n(t) with the help of a new non-constant function h(t) and we reparameterize the potential - that now can be considered as a function of time V(t) - with respect to a new (again non-constant) function A(t):

$$n(t) = \dot{h}(t), \qquad V(t) = \frac{(h(t) - \kappa)\dot{h}(t)}{4\dot{A}(t)}.$$
 (2.2)

As a result, (2.1) reduces to a local expression that can be easily integrated to give

$$a(t) = \pm 6^{1/6} \left(A(t) + c_1 \right)^{1/6}.$$
(2.3)

Substitution of this solution into the Euler-Lagrange equation for a(t) leads to a first order differential equation for A(t) that is solved by

$$A(t) = \frac{3m^4}{2} \exp\left(-\int \frac{\dot{h} \pm \left(\dot{h}^2 + \epsilon \,(\kappa - h)^2\right)^{1/2}}{\kappa - h} \,dt\right) - c_1,\tag{2.4}$$

with m being a non zero constant. In these solutions function h(t) remains free, reflecting the arbitrariness of the potential V(t).

It can be seen by (2.3) and (2.4), that the constant c_1 is not important for the solution, so we might as well consider it to be zero. The same is true for κ , since one need only define a new parametrization as $h(t) = \kappa + \exp\left(\frac{\omega}{2} - 3\int \frac{\epsilon}{\omega}dt\right)$. It can be verified that with this reparametrization of the lapse function N(t) as given by (1.9), together with the scale factor a(t)and the potential the line element can be eventually written (for $\dot{\omega} > 0$, if $\dot{\omega} < 0$ one needs only implement another parametrization and bring it in the same final form):

$$ds^{2} = -m^{4} \dot{\omega}^{2} e^{6 \int (\epsilon/\dot{\omega}) dt} dt^{2} + e^{\omega/3} \left(dr^{2} + r^{2} d\theta^{2} + r^{2} \sin^{2} \theta d\varphi^{2} \right), \qquad (2.5)$$

while the respective scalar field potential for each $\omega(t)$ is $V(t) = \frac{(\dot{\omega}^2 - 6 \epsilon)e^{-6 \int (\epsilon/\dot{\omega})dt}}{12m^4 \dot{\omega}^2}.$

Solution (2.5) can be further simplified by performing a change in the time variable from t to ω . Since, ω is an arbitrary function of t, we choose to invert the relation $\omega(t)$ by the use of an arbitrary function $F(\omega)$ defined as follows:

$$\phi(t) = t = \int \sqrt{\epsilon \frac{F'(\omega)}{6}} d\omega = \phi(\omega), \qquad (2.6)$$

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the prime denoting differentiation with respect to the argument ω . The line element (2.5) under a slight redefinition of the $F(\omega)$ function $(F(\omega) \to F(\omega) - \log m^4)$ that does not alter (2.6) - can be written

$$ds^{2} = -e^{F(\omega)}d\omega^{2} + e^{\omega/3}\left(dr^{2} + r^{2}d\theta^{2} + r^{2}\sin^{2}\theta d\varphi^{2}\right)$$
(2.7)

while the potential transforms to

$$V(\omega) = \frac{1}{12} e^{-F(\omega)} \left(1 - F'(\omega) \right).$$
 (2.8)

The set (2.6) - (2.8) satisfies the Einstein plus Klein-Gordon equations in the new time variable ω , with $F(\omega)$ remaining of course an arbitrary function due to the fact that we have not adopted a particular form for the potential.

3. The $k \neq 0$ case

In this case there exists no homothetic vector of the mini-superspace metric $G_{\mu\nu}$. However, we shall make use of the conformal vector $\xi = \frac{\partial}{\partial \phi}$, with corresponding factor $\frac{a^2 V'(\phi)}{a^2 V(\phi) - 3k}$. Subsequently, the following non-local integral of motion can be defined

$$Q = p_{\phi} + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k} dt = \frac{\partial L}{\partial \dot{\phi}} + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k} dt$$

$$= \frac{4 \epsilon a^4 \dot{\phi} \left(a^2 V(\phi) - 3k\right)}{n} + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k} dt.$$
(3.1)

It can be straightforwardly checked that, equation $Q = \kappa$ is the first integral of the Klein-Gordon equation (1.7) (or equivalently, of the Euler-Lagrange equation with respect to ϕ).

Again, we choose to adopt the gauge $\phi(t) = t$, and allow for the following reparameterizations

$$n(t) = \frac{2\dot{h}\left(a^2V - 3k\right)}{a^2\dot{V}}, \qquad V(t) = \int \frac{\dot{w}}{a^6}dt$$
(3.2)

with w(t) and h(t) being non constant functions of time. Now, the corresponding equation (2.1) reduces to a local expression that can be integrated to yield

$$h(t) = \frac{1}{2} \left(\kappa \pm \sqrt{4c_1 + \kappa^2 - 8\epsilon w} \right).$$
(3.3)

By choosing to perform another re-parametrization $w(t) = \frac{a\dot{v}}{\dot{a}} + \frac{1}{8\epsilon} \left(4c_1 + \kappa^2\right) - 6v$, in which we introduce a new function of time v(t), together with a substitution of (3.2) and (3.3) into the quadratic constraint $\frac{\partial L}{\partial n} = 0$, we get a differential equation for v(t), which is solved by

$$v(t) = \exp\left(\int \left(\frac{6\dot{a}}{a} - \frac{\epsilon a}{\dot{a}}\right) dt\right) \left(\int \frac{ka^5 \exp\left(-\int \left(\frac{6\dot{a}}{a} - \frac{\epsilon a}{\dot{a}}\right) dt\right)}{2\dot{a}} dt + c_2\right).$$
 (3.4)

where c_2 is an integration constant. If we set $a(t) = e^{\omega/6}$ and adopt a suitable time change

$$\phi(t) = t = \int \left[\frac{1}{6\epsilon} \left(\frac{S''(\omega)}{S'(\omega)} + \frac{1}{3}\right)\right]^{1/2} d\omega = \phi(\omega)$$
(3.5)

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where $S(\omega) = \exp\left(12k\int e^{F(\omega)-\omega/3}d\omega\right) - \frac{6c_2}{k}$, the result can be significantly simplified. The ensuing line element is

$$ds^{2} = -e^{F(\omega)}d\omega^{2} + e^{\omega/3}\left(\frac{1}{1-kr^{2}}dr^{2} + r^{2}d\theta^{2} + r^{2}\sin\theta d\varphi^{2}\right),$$
(3.6)

while the corresponding potential becomes

$$V(\omega) = \frac{1}{12} e^{-F(\omega)} \left(1 - F'(\omega)\right) + 2k e^{-\omega/3}.$$
(3.7)

As a result, the set of relations (3.5) - (3.7) satisfies Einstein's equation plus scalar field for the case $k \neq 0$, with $F(\omega)$ remaining again arbitrary. It can be seen that if one considers k = 0, then this solution becomes exactly the one given in the previous section. However, it has to be noted that in the process of deriving the above relations the assumption $k \neq 0$ has been taken into account.

4. Discussion

In this paper we derived the general solution of a general scalar field minimally coupled to gravity for a FLRW space-time, without having to impose any conditions over the potential. This result is not only important in the context of general relativity but also may be used for the derivation of solutions in other theories of gravity, which under certain transformations may be mapped to a minimally coupled scalar field action (such as f(R) or scalar tensor theories of gravity).

A key element for the derivation of the result has been the re-parametrization invariance of the singular action (1.2). The choice of gauge in which the scalar field is time allowed us to "merge" and parameterize the potential with other functions of time. Finally, the complete integration of the system was made possible with the use of one of the infinite non-local symmetries that can be defined in phase space with the help of conformal Killing vectors of the two dimensional mini-supermetric.

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References

- [1] Vakili B and Khazaie F 2012 Clas. Quant. Grav. 738 035015
- [2] Capozziello S, De Laurentis M, Odintsov S D 2012 Eur. Phys. J. C72 2068
- [3] Terzis P A, Dimakis N and Christodoulakis T 2014 Phys. Rev. D 90 123543
- [4] Vakili B 2014 Phys. Lett. B 738 488-492
- [5] Zampeli A 2015 J. Phys. Conf. Ser. 633 012019
- [6] Zampeli A, Pailas T, Terzis P A and Christodoulakis T 2015 Preprint gr-qc/1511.08382
- [7] Paliathanasis A and Capozziello S 2016 Preprint gr-qc/1602.08914
- [8] Belinchón J A, Harko T and Mak M K 2016 Astrophys. Space Sci. 361 52
- [9] Tsamparlis M and Paliathanasis A 2011 Gen. Relativ. Grav. 43 1861
- [10] Christodoulakis T, Dimakis N and Terzis P A 2014 J. Phys. A: Math. Theor. 47 095292
- [11] Terzis P A, Dimakis N, Christodoulakis T, Paliathanasis A and Tsamparlis M 2016 J. Geom. Phys. 101 52-64
- [12] Paliathanasis A, Tsamparlis M and Basilakos S 2011 Phys. Rev. D 84 123514
- [13] Dimakis N, Christodoulakis T, Terzis P A 2014 J. Geom. Phys. 77 97-112
- [14] Christodoulakis T, Grammenos T, Helias C, Kevrekidis P G and Spanou A 2006 J. Math. Phys. 47 042505
- [15] Dirac P A M 1964 Lectures on Quantum Mechanics (Yeshiva University, Academic Press, New York)
- [16] Sundermeyer K 1982 Constrained Dynamics (Springer-Verlag)