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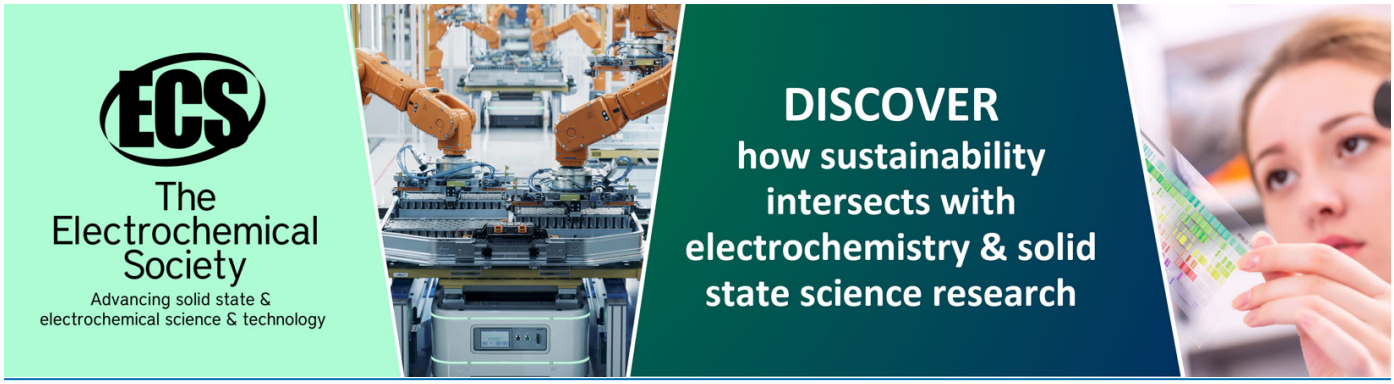

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On the pertinence to Physics of random walks induced by random dynamical systems: a survey

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Abstract.

Let $\mathbb{X}$ be an abstract space and $\mathbb{A}$ a denumerable (finite or infinite) alphabet. Suppose that $(p_a)_{a\in\mathbb{A}}$ is a family of functions $p_a : \mathbb{X} \to \mathbb{R}_+$ such that for all $x \in \mathbb{X}$ we have $\sum_{a\in\mathbb{A}} p_a(x) = 1$ and $(S_a)_{a\in\mathbb{A}}$ a family of transformations $S_a : \mathbb{X} \to \mathbb{X}$. The pair $((S_a)_a, (p_a)_a)$ is termed an *iterated function system with place dependent probabilities*. Such systems can be thought as generalisations of random dynamical systems. As a matter of fact, suppose we start from a given $x \in \mathbb{X}$; we pick then randomly, with probability $p_a(x)$, the transformation S_a and evolve to $S_a(x)$. We are interested in the behaviour of the system when the iteration continues indefinitely.

Random walks of the above type are omnipresent in both classical and quantum Physics. To give a small sample of occurrences we mention: random walks on the affine group, random walks on Penrose lattices, random walks on partially directed lattices, evolution of density matrices induced by repeated quantum measurements, quantum channels, quantum random walks, etc.

In this article, we review some basic properties of such systems and provide with a pathfinder in the extensive bibliography (both on mathematical and physical sides) where the main results have been originally published.

1. Introduction and motivation

The (discrete) time evolution of several phenomena in (classical and quantum) physics, biology, ecology, economics, etc. are modelled by *dynamical systems.* Mathematically, a dynamical system is a map $S : \mathbb{X} \to \mathbb{X}$ from some measurable space $(\mathbb{X}, \mathcal{X})$ into itself. The system starts at time $n = 0$ at some initial state $x_0 \in \mathbb{X}$; the next moment $n = 1$, the system gets at state $x_1 = S(x_0)$ and recursively for all (integer) instants $n \in \mathbb{N}$, we have $x_{n+1} = S(x_n)$. At this level, we don't specify further either the nature of the space $\mathbb{X}$ beyond its measurability properties (very often it is a topological or metric space) or the nature of the transformation S (very often it exhibits some non-linearity). Nevertheless, we mention that even very innocent looking examples on gentle spaces — like iteration of the logistic map $S(x) = 4x(1-x)$ on $\mathbb{X} = [0,1]$ —- can exhibit a tremendously complicated behaviour (see [1] for instance, for a user-friendly introduction to the topic).

When dealing with realistic systems, it is natural to suppose that the evolution is influenced by external parameters, collectively denoted by some symbol $a \in \mathbb{A}$, i.e. instead of having a single evolution transformation S, we have a collection $(S_a)_{a\in\mathbb{A}}$, parametrised by $\mathbb{A}$. Now, on a realistic system the control of external parameters can never be complete; to model such imperfect knowledge, we introduce a randomness on the parameters a. Thus, the evolution

is described by a random dynamical system, when the system at moment n is in state x, the applied transformation S_a is chosen among the possible ones with probability $p_a(x)$; the evolution is described by the Markovian kernel

$$P(x,B) := \mathbb{P}(X_{n+1} \in B | X_n = x) = \sum_{a \in \mathbb{A}} p_a(x) \delta_{S_a(x)}(B), \quad \text{for } B \in \mathcal{X}. \tag{1}$$

Such systems, in their simplest form with place independent probabilities (i.e. with p_a constants), have been introduced — and termed iterated function systems — in [18] where the conditions of the existence of an attracting set supporting their invariant measure have been given. The full-fledged place-dependent variant has been introduced in [2] when $\mathbb{X}$ is a metric space; the fractal dimension of the attractor has been obtained under the condition that the place-dependent probabilities are Lipschitz-continuous. Conditions of asymptotic stability of such systems are obtained in [19] (extending previous results in [24]).

Although these systems are very versatile, can model numerous phenomena, and exhibit very rich a structure in numerical simulations, rigorous mathematical results are lacking when the place-dependent probabilities are not continuous. (A very partial case is treated in [20]).

It is worth noting that the joint process $(A_n, X_n)_{n \in \mathbb{N}}$ taking values in $\mathbb{A} \times \mathbb{X}$ is also a Markov chain with transition probability kernel $Q((a,x),(b,B)) := \mathbb{P}((A_{n+1} = b, X_{n+1} \in B | A_n = a, X_n = x) = p_b(x)\delta_{S_b(x)}(B)$ for $a, b \in \mathbb{A}$ and $B \in \mathcal{X}$; the kernel P, introduced in equation (1), is obtained from the kernel Q by marginalisation when the $\mathbb{A}$-component is not observed. On the contrary, the marginalisation of Q where the $\mathbb{X}$-component is not observed does not — in general — give rise to a Markov chain but only to what is called a *hidden Markov chain*. A hidden Markov chain can become an ordinary Markov chain under some specific circumstances (for instance when the p_a's are constant). Let us mention finally that the restriction of $\mathbb{A}$ being denumerable can be removed is some cases.

In the sequel, we present some models fitting this formalism.

2. Random walks on lattices

2.1. Simple random walks on directed sublattices of $\mathbb{Z}^d$

Let $\mathbb{X} = \mathbb{Z}^d$, $\mathbb{A} = \{\pm e_1, \ldots, \pm e_d\} \subset \mathbb{X}$, and for any $a \in \mathbb{A}$ and any $x \in \mathbb{X}$, define $S_a(x) = x + a$. If the functions $p_a(x) = (2d)^{-1}$ for all a and x, the process (X_n) governed by (1) is equivalent to the simple symmetric random walk on $\mathbb{Z}^d$, introduced, almost a century ago, in the celebrated paper of Pólya [33]. A lot of properties of simple random walks can be subsumed under their recurrence/transience behaviour; already in [33] it has been established that the simple symmetric random walk on $\mathbb{Z}^d$ returns almost surely infinitely often (it is recurrent) to its starting point in $d \leq 2$ but almost surely only a finite number of times (it is transient) in $d \geq 3$. It is worth noting that interesting connections of random walks on $\mathbb{Z}^d$ with computational trajectories of finite automata or with properties of (Abelian) groups can be made. Let us mention also that simple random walks serve in modelling a plethora of physical phenomena: every time a physical phenomenon is described by a differential equation involving a Laplacian, one can coin a random walk representation of the phenomenon.

It is quite surprising that although random walks on $\mathbb{Z}^d$ have a centennial history, random walks on directed sublattices of $\mathbb{Z}^2$ had not been mathematically studied until 2003. These walks are more realistic if one wishes to model information flow on internet or vehicle traffic in urban networks for instance; they have been considered by hydrologists [28] and a related model has been heuristically studied by computer simulations in [34] but no mathematical result was known before [6, 7]. They are obtained with $\mathbb{A}$ and $\mathbb{X}$ as in the simple random walk on $\mathbb{Z}^d$ case but with different choices of place-dependent p_a's. Recall that $\mathbb{X} = \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$, hence every $x \in \mathbb{Z}^2$ can be written as a pair $\mathbf{x} = (x_1, x_2) \in \mathbb{Z} \times \mathbb{Z}$. Three different models

of partially directed lattice, depicted in the figure 1, have been studied in [6]. These directed lattices arise as sublattices of $\mathbb{Z}^2$ by rendering their horizontal lines unidirectional.

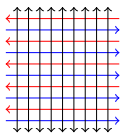
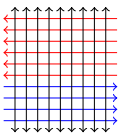
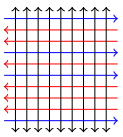
Alternate lattice	Half-plane one-way	Randomly horizontally directed
		
$p_a(\mathbf{x}) = \begin{cases} 1/3 & \text{if } a = \pm e_2 \\ 1/3 & \text{if } a = e_1 \text{ and } x_2 \in 2\mathbb{Z} \\ 1/3 & \text{if } a = -e_1 \text{ and } x_2 \in 2\mathbb{Z}+1 \\ 0 & \text{otherwise.} \end{cases}$	$p_a(\mathbf{x}) = \begin{cases} 1/3 & \text{if } a = \pm e_2 \\ 1/3 & \text{if } a = e_1 \text{ and } x_2 < 0 \\ 1/3 & \text{if } a = -e_1 \text{ and } x_2 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$	$p_a(\mathbf{x}) = \begin{cases} 1/3 & \text{if } a = \pm e_2 \\ 1/3 & \text{if } a = \sigma_{x_2} e_1 \\ 0 & \text{otherwise.} \end{cases}$

Figure 1: Three different lattices having one-way horizontal lines and the corresponding probability functions p_a. (i) In the alternate lattice the horizontal lines are going eastward or westward according to the parity of the ordinate. (ii) In the half-plane lattice, all upper half-plane lines are westward and all lower half-plane eastward. (iii) In the random horizontal lattice a honest coin is flipped to decide the way each line is going and this choice is encoded into $\{-1, 1\}$-valued random variable σ_{x_2}.

We proved that the random walk on the alternate lattice is recurrent, on the half-plane one-way is transient, and on the randomly directed lattice almost surely transient. It is worth noting that all the above lattice have zero net drift; the dramatic change of behaviour in the recurrence/transience properties is due solely to the directedness of the lattice.

As expected, several new results have been triggered by this work (see [14, 15, 30, 32, 11, 13, 8] for instance). It is worth noting that from the above models only (i) can be obtained by the computational trajectories of a finite automaton; model (ii) requires a push-down automaton while model (iii) a genuine Turing machine with bi-infinite external tape. Additionally, the set of all possible trajectories looses the natural group structure to remain merely a semi-groupoid.

2.2. Random walks on Penrose tilings

Another non-trivial model that fits the aforementioned description is the random walk on the complexes obtained by the edges of Penrose tilings. Introduced first as a mathematical recreational problem, such tilings proved essential in the crystallographic description of some alloys (like Al-Mn, Ho-Mg-Zn) that exhibit 5-fold and icosahedral symmetries — forbidden by classical crystallography — in their X-ray diffraction patterns[1]. Recall that Penrose tiling of $\mathbb{R}^d$ (see figure 2) is an aperiodic covering of the space by finitely many types of prototiles. Although the initial construction of Penrose [31] for $d = 2$ was made by use of local matching rules, a cut-and-project method giving a systematic construction for every d has been introduced in [21, 29].

The idea of this cut-and-project method is to decompose $\mathbb{R}^N = E \oplus E'$ into a direct sum of mutually orthogonal Euclidean spaces E and E' with $N > d$, $\dim E = d$ and $\dim E' = N - d$. When the unit cube $C = \{y \in \mathbb{R}^N : y = \sum_{i=1}^N y_i e_i, y_i \in [0,1]\}$ is translated parallel to E it forms the "strip" $\mathcal{S} = C + E$. All points of the integer lattice $\mathbb{Z}^N$ within the strip $\mathcal{S}$ are then orthogonally projected on E. When the principal directions of E are incommensurate with $\mathbb{Z}^N$, the points in the strip are in bijection with their orthogonal projections on E or E'. When projected on E, they produce an aperiodic tiling of E by finitely many types of prototiles. To illustrate this method we give in the figure 2 below this construction for the aperiodic tiling

[1] This experimental observation [35] was essential for awarding the 2011 Nobel prize in Chemistry to Dan Shechtman.

of the line by short and long segements, obtained by projecting a strip in $\mathbb{Z}^2$ on a line with irrational slope and the tiling of the plane by projecting the strip in $\mathbb{Z}^5$ with respect to a given irrationally oriented plane.

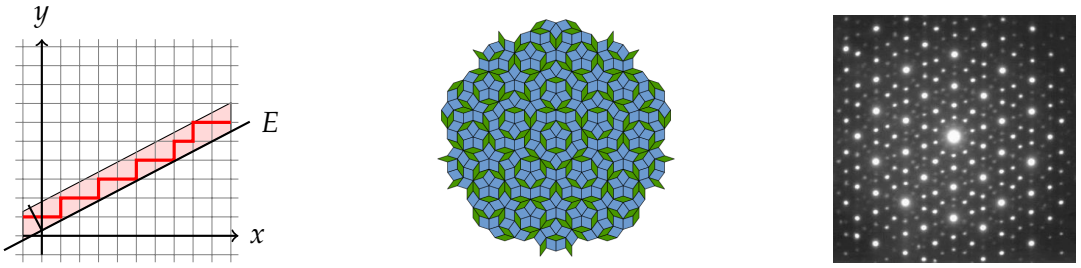


Figure 2: *Left:* The aperiodic tiling of the one-dimensional space E by short and long segments. The shaded region within the two parallel oblique lines is the strip $\mathcal{S}$. The tiling is obtained by projecting on E the horizontal and vertical segments of the line zigzagging inside the strip. *Centre:* The aperiodic tiling of $\mathbb{R}^2$ by 2 different types of lozenges, each coming into 5 different orientations to give rise in 10 types of prototiles arranged in a pattern having locally a 5-fold symmetry (Public domain image, source: Wikipedia) *Right:* X-ray experimental diffraction pattern on a Zn-Mg-Ho quasicrystal showing a clear local 5-fold symmetry. (Image under Creative Commons BY-SA 3.0 licence, source: Wikipedia).

The projected figure constitutes the Penrose tiling of the plane. The corners and the edges of the lozenges define a lattice in $\mathbb{R}^2$. The simple random walk on a given vertex at time n, jumps with equal probability to a nearest-neighbour (in the graph distance defined by the edges) at moment $n+1$. It is worth noting that in the cut-and-project method, the parameter N can be chosen arbitrarily large (but finite): some vertices of the lattice can have thus arbitrarily large coordination number. In spite of that, in [37], results on the heat kernel of the simple random walk on a Penrose lattice have been obtained; in [10] the random walk is shown to be generically recurrent for $d \leq 2$ and generically transient for $d \geq 3$.

Remark also that the trajectories of the random walk can be shown to be in bijection with the computational trajectories of Turing machine; additionally, they have a natural groupoid structure but not that of a group. The random walk on the Penrose lattice is isomorphic to a random walk fitting the iterated function systems with place-dependent probabilities formalism by choosing $\mathbb{A} = \{\pm e_1, \ldots, \pm e_N\}$, $\mathbb{X} = \mathbb{Z}^N$ and

$$p_a(x) = \begin{cases} \frac{1}{d(x)} & \text{if } x + a \in \mathcal{S} \\ 0 & \text{otherwise,} \end{cases}$$

where $d(x) := \sharp\{b \in \mathbb{A} : x + b \in \mathcal{S}\}$, for all $x \in \mathcal{S}$. Since the integer points in the strip are in bijection with the corners of tiles in E, any trajectory of the random walk in the strip is bijectively projected to a trajectory on edges of the tiling.

3. Random walks on (semi-)groups

3.1. The "$ax + b$" group

In [36, 23] problems stemming from random walks in random envrionment have been studied. It appeared then that a random process $(X_n)_{n\in\mathbb{N}}$ on the set $\mathbb{X}$ of d-dimensional vectors occurred naturally. If $\mathbb{A}$ is a fixed set of $d \times d$ matrices and of d-dimensional vectors, the evolution of the process consists in choosing randomly a pair $a = (m, v) \in \mathbb{A}$ (i.e. a pair of a matrix m and a vector v and define $X_{n+1} = S_a(X_n)$ where $S_a = mx + v$. Obviously, this evolution fits the general formalism given in the introduction. In case the matrices m and vectors v have positive

components, the asymptotic behaviour of X_n can be obtained in a more straightforward manner [22]. Connection of this process with a one-dimensional *disordered model of statistical mechanics* has been established in [12], while in [9], the asymptotic behaviour of the model has been shown be connected with the tail behaviour of the invariant measure of the process (X_n) by using Mellin transform. Same ideas have been developed in [16, 5], while in [17], a highly non trivial generalisation to the case of not necessarily positive matrices has been made.

3.2. Random dynamical systems on $\mathbb{R}_+$

In [3], we studied systems with $\mathbb{A} \simeq \mathbb{Z}$, $\mathbb{X} = \mathbb{R}_+$ and $S_a(x) = (a + x \pm x^\gamma)_+$ where γ is some fixed power in $(0,1)$. When the random variable $A \in \mathbb{A}$ has moments, nothing very interesting occurs since the asymptotic behaviour of random dynamical system is governed by the systematic drift term x^γ. An interesting situation occurs when the random variable A is distributed according to a law having heavy tails, i.e. the distribution of A does not decay sufficiently fast for large values of $|A|$ for it to have expectation. More precisely, we suppose that $\mathbb{P}(|A| > y) \asymp \frac{C}{y^\alpha}$, with $\alpha \in (0,1)$. A complete characterisation of the type diagram is obtained exhibiting a competition between the heavy-tail behaviour and the systematic drift, i.e. we obtain a non trivial type diagram in the (α, γ) plane.

4. Evolution of density matrices under repeated unsharp quantum measurements

The state of a quantum system is described by a *density operator*, i.e. a positive, trace-class and unit-trace operator x, acting on a separable Hilbert space $\mathbb{H}$; this set is denoted $\mathfrak{D}(\mathbb{H})$ and plays the *rle* of $\mathbb{X}$ in this quantum setting. When the system is isolated, a standard result in quantum mechanics states that the evolution is through a unitary operator U acting on x by $U^* x U$. When a sharp measurement of a quantity Q (represented by a self-adjoint operator acting on $\mathbb{H}$ and supposed — for simplicity — with discrete spectrum in this survey) is performed, the quantum mechanics postulates establish that the value a occurs with probability $\mathrm{tr}(xE[a])$ where $E[a]$ denotes the spectral measure of Q at a; the family of $(E[a])_{a\in\mathbb{A}}$ is a family of orthogonal orthoprojections. After having obtained a value $a \in \mathbb{A}$, the state of the system evolves irreversibly to a new density operator $x' = S_a(x) = \frac{E[a]xE[a]}{\mathrm{tr}(xE[a])}$. In this setting, the set $\mathbb{A}$ can be chosen to be the spectrum of Q and the probability of occurrence of every a is $p_a(x) = \mathrm{tr}(xE[a]) = \mathrm{tr}(E[a]xE[a]) = \mathrm{tr}(E[a]^* xE[a])$.

Unsharp quantum measurements are families of positive operators $(E[a])_{a\in\mathbb{A}}$, verifying $\sum_{a\in\mathbb{A}} E[a] = I$ but the members $E[a]$ of the family are not orthogonal orthoprojections. Since they are positive however, there exists a family of operators $(Z[a])$ such that $E[a] = Z[a]^* Z[a]$ (they are called *Kraus operators* in the literature). The probability of getting the value a in this unsharp measurement is $p_a(x) = \mathrm{tr}(xE[a]) = \mathrm{tr}(xE[a]) = \mathrm{tr}(Z[a]xZ[a]^*)$ while the state of the system evolves into the new state $x' = S_a(x) = \frac{Z[a]xZ[a]^*}{\mathrm{tr}(Z[a]xZ[a]^*)}$. Therefore, this problem also fits the general formalism introduced above (see [4, 26] for instance) with $\mathbb{X} = \mathfrak{D}(\mathbb{H})$. It is remarkable that repeated quantum measurements induce a classical Markov chain on the space $\mathbb{X} = \mathfrak{D}(\mathbb{H})$.

In [27], the asymptotic behaviour of the thus induced Markov chain has been obtained for the case where $\mathbb{A}$ is finite while in [25] these results have been extended to the case of denumerably infinite $\mathbb{A}$.

5. Conclusion and open problems

We have demonstrated that there exists a unified formalism encompassing very diverse models that have been previously introduced in various areas of physics (statistical mechanics, disordered media, quantum mechanics, etc.). This formalism puts together mathematical methods developed somehow separately in the context of dynamical systems and/or Markov

processes and as such has a very rich mathematical structure. In spite of the vast applicability of the formalism, very little is known about existence and uniqueness of the invariant measure for the random dynamical system described by the Markov kernel of equation (1) when the probabilities $(p_a)_{a\in\mathbb{A}}$ are genuinely place-dependent and this dependence is not continuous. This constitutes a challenge in the theoretical understanding of these systems. On the physical side, the existence of an underlying unified framework can be used to transfer methods developed ad-hoc in each area to the other. Within every model presented above, only some partial aspects are understood. We expect that the transfer of methods from one area to the other will help answering some of the remaining questions.

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