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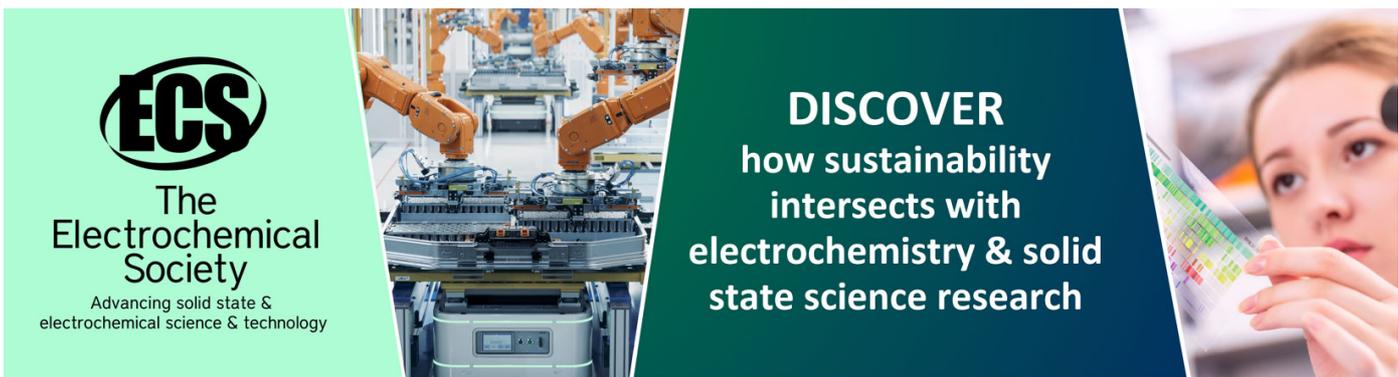
## Optimized design of thermo-mechanically loaded non-uniform bars by using a variational method

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## Optimized design of thermo-mechanically loaded non-uniform bars by using a variational method

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**Abstract.** The present paper evaluates the axial strain and stress of a thermo-mechanically loaded non-uniform bar by using a numerical method based on a variational principle. The solutions are obtained up to the elastic limit of the material based on the assumptions that material properties are independent of temperature variation and plane cross-sections remain plane maintaining axisymmetry. This approximation is carried out by Galerkin's principle, using a linear combination of sets of orthogonal co-ordinate functions which satisfy prescribed boundary conditions. The solution algorithm is implemented with the help of MATLAB<sup>®</sup> computational simulation software. Some numerical results of thermoelastic field are presented and discussed for different bar materials such as mild steel, copper, aluminium alloy 6061 (Al alloy 6061), aluminium alloy 7075 (Al alloy 7075) and diamond. The effect of geometry parameters like aspect ratio, slenderness ratio and the type of taperness is investigated and the relevant results are obtained in dimensional form. The term bar used in this paper is in generic sense and hence the formulation is applicable for all one dimensional elements, e.g., rods, pipes, truss members, etc.

### 1. Introduction

The post-elastic analyses of thermo-mechanically loaded non-uniform bars are important for designing mechanical, aerospace and civil structures. The solutions of thermal stress problem available in textbook are valid for uniform cross-section bar under uniform temperature field only. In general the solution fails because the assumption of equality of thermal and mechanical strains at every point within the domain is not true. However, an overall equilibrium of the bar system is achieved by balancing forces coming from the strain differentials, existing within the domain. Hence a detail analysis of the generalized system yields a different strain field, and subsequently a different stress field, which are often much more critical. The simple analytical method also fails when the induced thermal stress is in post-elastic region due to insufficiency in capturing the physics of material behaviour.

The solution of thermal stress problems for uniform bar subjected to a uniform temperature is found in the textbook of Timoshenko [1]. The prediction of the elasto-plastic behaviour of solid slender bars of various types of geometry as well as loading is an interesting area of work for the designers (Hill [2]). Niknam et al. [3] studied the non-linear bending of tapered functionally graded (FG) beam subjected to thermal and mechanical load with general boundary condition. They presented a closed form solution for the classical beam problem and employed Galerkin technique for the general

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case with axial force. Das et al. [4] presented a simulation study of the dynamic behaviour of non-uniform taper bars of circular and rectangular cross-section under body force loading due to gravity. The loading is controlled statically to take the bar to its post-elastic state so as to predict its dynamic behaviour in the presence of plastic deformation. Noda [5] presented an extensive review that covered a wide range of topics from thermo-elastic to thermo-inelastic problems. Grysa and Kozłowski [6] presented an approximate analytical and exact solution of the problems of heat flux and temperature determination on slabs. The interior thermal and mechanical responses are used in determining the unknown functions describing heat flux and temperature on the surface of the slab. Nayak and Saha [7] investigated the elastic limit speed of non-uniform rotating disks considering thermal effect on elasticity modulus.

The literature review in this area reveals that textbooks address analysis of thermal stress problem of uniform cross-section bar under uniform temperature field in elastic domain only and literatures explaining non-uniform bars under thermo-mechanical loading are rare. This paper attempts to address the thermo-mechanical behaviour of non-uniform bars upto the elastic limit of the material by using a numerical method based on variational principle. The effect of geometry parameters like aspect ratio, slenderness ratio and the type of taperness on the thermo-elastic performance of the bar is investigated and the relevant results are obtained in dimensional form.

## 2. Problem formulation

The present paper employs an energy approach to get the appropriate governing equations for the thermo-mechanically loaded non-uniform bars. The formulation is displacement based and the unknown displacement field is approximated by finite linear combination of admissible orthogonal functions. The mathematical model is framed on the assumption that the material of bar is homogeneous, isotropic and linear elastic. It is also assumed that the bar geometry is stub enough to exclude buckling failure from the scope of analysis. Further, the analysis is carried out based on the assumptions that material properties are independent of temperature variation and plane cross-sections remain plane maintaining axisymmetry. However, although the present analysis is carried out for an elastic material, extension of the analysis in the elasto-plastic region needs special mention.

The geometry taken for the present analysis is taper bar of solid circular cross-section with linear variation in diameter given by  $d = d_0 - \xi(d_0 - d_1)$ , where  $d_0$  is the largest diameter and  $d_1$  is the smallest diameter of the bar. The diameter of the bar ( $\phi$ ) at any axial location  $x$  is expressed in terms of the normalized axial coordinate  $\xi (= x/L)$ , where  $L$  is the total length of the bar. Furthermore, the geometry of the bar is defined by the slenderness ratio (ratio of length to radius of gyration corresponding to the minimum radius of the bar) and aspect ratio (ratio of difference in radii of two ends to length). The mathematical expression of slenderness ratio is given by  $S_R = L/k(r_1)$ , where  $k(r_1) = \sqrt{I_1/A_1}$  is the radius of gyration of the bar, corresponding to the minimum cross-section of the bar. The expression for aspect ratio is  $A_R = (r_0 - r_1)/L$ .

Displacement will occur due to thermal expansion resulting in thermal load. The magnitude of this displacement field is also governed by the boundary conditions of the bar. The solution for the displacement field is obtained from the minimization of total potential energy principle  $\delta(U + V) = 0$  where,  $U$  is the strain energy stored in the bar and  $V$  is the potential energy developed by the external forces. When the bar material is subjected to a temperature field, it experiences strain due to an expansion proportional to the temperature rise  $\Delta T$ , and the linear constitutive thermoelastic equations take the form  $\epsilon_x = \frac{\sigma_x}{E} + \alpha \Delta T$ , where  $\epsilon_x$  is the axial strain,  $\sigma_x$  is stress,  $E$  is the modulus of elasticity and  $\alpha$  is the coefficient of linear thermal expansion. The total strain energy of the bar comes from the stress and strain field, and expressed as  $\Pi = \frac{1}{2} \int_{Vol} (\sigma \epsilon) dv = \frac{1}{2} \int_0^L (\sigma_x \epsilon_x) A(x) dx$ , where  $A(x)$  is the area of

cross-section at location  $x$ . Using the linear-strain displacement relation,  $\varepsilon_x = du/dx$ , where  $u$  is the axial displacement field, we get

$$\Pi = \frac{1}{2} \int_0^L \left\{ E(u')^2 - 2E\alpha\Delta T u' \right\} A(x) dx. \quad (1)$$

In equation (1), ( )' indicate first derivative with respect to the coordinate variable. Here  $\Pi$  is the total potential energy in which one part is similar to the strain energy  $U$  stored in the bar and the other part is work function due to temperature gradient loading. So, the expression for  $U$  is given by,

$$U = \frac{1}{2} \int_0^L \left\{ E(u')^2 \right\} A(x) dx \quad (2)$$

and the expression for potential energy  $V$ , arising from the thermal loading is given by,

$$V = -\frac{1}{2} \int_0^L \left\{ 2E\alpha\Delta T u' \right\} A(x) dx. \quad (3)$$

Substituting equations (2) and (3) in the energy principle  $\delta(U+V)=0$ , the governing equilibrium equation becomes

$$\delta \left[ \frac{1}{2} \int_0^L \left\{ E(u')^2 - 2E\alpha\Delta T u' \right\} A(x) dx \right] = 0. \quad (4)$$

Equation (4) is expressed in normalized co-ordinate  $\xi$  to facilitate the numerical computation work and the governing equation takes the form

$$\frac{E}{L} \int_0^1 \left\{ u' \delta(u') \right\} A(\xi) d\xi = E\alpha\Delta T \int_0^1 \delta(u') A(\xi) d\xi. \quad (5)$$

The displacement function  $u(\xi)$  in equation (5) is approximated by a linear combination of sets of orthogonal coordinate functions as  $u(\xi) = \sum c_i \phi_i$ ,  $i=1, 2, \dots, n_f$ , where  $\phi_i$  is the set of orthogonal functions developed through Gram-Schmidt scheme,  $c_i$  is the set of unknown coefficients and  $n_f$  is number of functions. The necessary start function  $\phi_0$  is given by  $\phi_0 = \xi(1-\xi)$  which satisfies the geometric boundary conditions of the bar which are, i.e.  $u=0$  at  $\xi=0$  and  $u=0$  at  $\xi=1$ .

Now substituting the series approximation of  $u(\xi)$  in Eq. (5) and replacing the operator ' $\delta$ ' by  $\partial/\partial c_j$ ,  $j=1, 2, \dots, n$ , according to Galerkin error minimization principle, we obtain the governing differential equation in matrix form.

$$\left( \frac{E}{L} \right) \left[ \sum_{i=1}^n \sum_{j=1}^n \int_0^1 (A(\xi) \phi_i' \phi_j') d\xi \right] \{c_i\} = E\alpha\Delta T \int_0^1 (A(\xi) \phi_j') d\xi \quad (6)$$

Solution of equation (6) yields the solution vector  $\{c_i\}$ , obtained through a single step matrix inversion process. The axial displacement field ( $u$ ) can be found out for any prescribed value of  $\Delta T$  which in turn gives strain and stress fields. The numerical integration, differentiation and all other associated mathematical operations are carried out in the computational platform of MATLAB software.

### 3. Results and discussions

In this paper, a thermal stress analysis is carried out on a clamped taper bar for different materials such as mild steel, copper, Al alloy 6061, Al alloy 7075 and diamond. Mechanical properties of the bar

materials such as elasticity modulus ( $E$ ), Poisson's ratio ( $\nu$ ), coefficient of thermal expansion ( $\alpha$ ) and yield stress ( $\sigma_y$ ) are listed in table 1.

The theoretical value of temperature rise at the point of yielding ( $\Delta T_y$ ) of a taper bar of 1.2 m long, having aspect ratio  $A_R = 0.025$ , and slenderness ratio  $S_R = 20$  is given in table 2 for various bar materials. Table 2 also furnish the respective melting temperatures ( $T_m$ ) of the materials.

**Table 1.** Material properties.

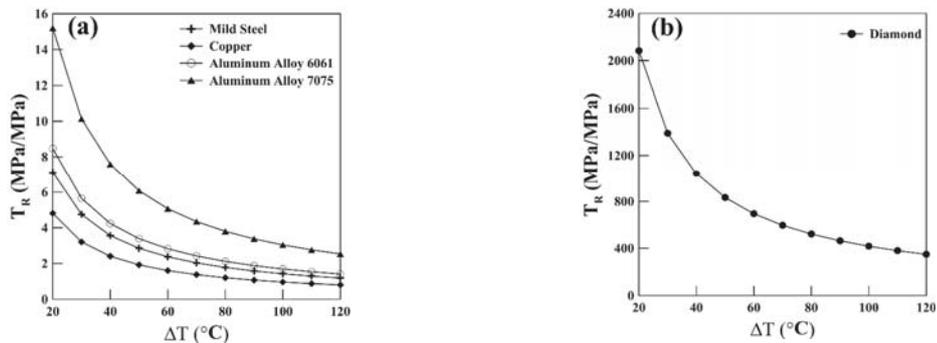
Bar material	$E$ (GPa)	$\nu$	$\alpha$ ( $10^{-6}/^{\circ}\text{C}$ )	$\sigma_y$ (MPa)
Mild steel	210	0.3	11.7	350
Copper	129	0.34	17	211
Al alloy 6061	69	0.33	23.6	276
Al alloy 7075	71	0.33	23.4	505
Diamond	1220	0.2	1.18	60000

**Table 2.** Temperature rise at yield and melting temperature.

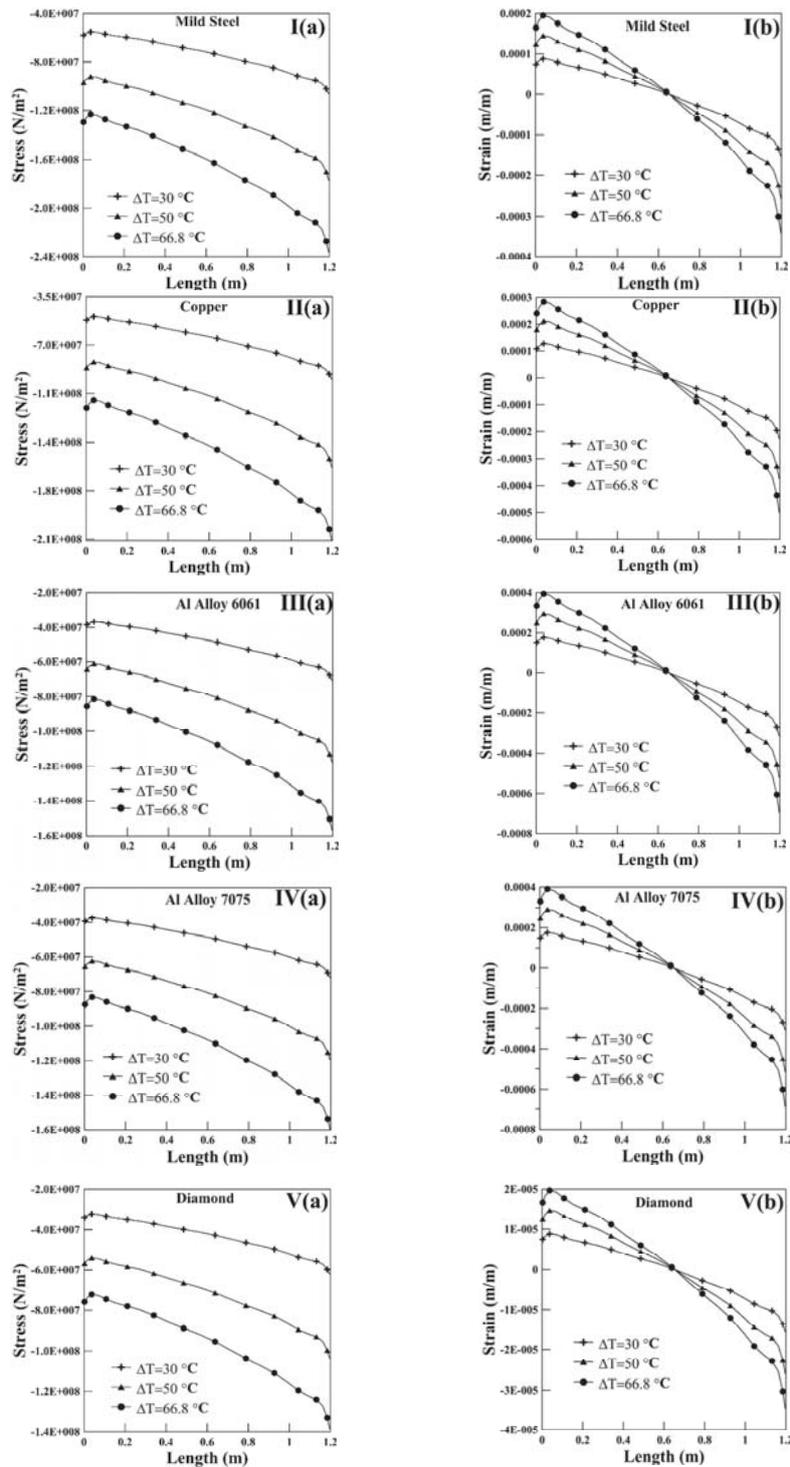
Material	Mild steel	Copper	Al alloy 6061	Al alloy 7075	Diamond
$\Delta T_y$ ( $^{\circ}\text{C}$ )	98.9	66.8	117.7	211.1	28936.9
$T_m$ ( $^{\circ}\text{C}$ )	1427	1085	582	477	-

The dimensionless parameter  $T_R = \sigma_y / \alpha E \Delta T$ , termed as thermal load resistance, is used to characterize the bar material upto the elastic limit. As the values of  $\Delta T_y$  and  $T_m$  are different for different materials as shown in table 2, an operating temperature range of (20-120)  $^{\circ}\text{C}$  has been selected. Such a selection is made because beyond this range, the plot of thermal load resistance cannot be significantly represented in a single plot for comparison. The plot of thermal load resistance ( $T_R = \sigma_y / \alpha E \Delta T$ ) with temperature rise ( $\Delta T$ ) is provided in figure 1(a) for mild steel, copper, Al alloy 6061 and Al alloy 7075 from (20-120)  $^{\circ}\text{C}$ . Since the thermal load resistance of diamond is very much higher as compared to other bar materials, so a separate figure for diamond is provided (figure 1(b)).

The stress and strain fields induced in a clamped taper bar for five different materials under three different uniform thermal loads ( $\Delta T = 30^{\circ}\text{C}$ ,  $50^{\circ}\text{C}$  and  $66.8^{\circ}\text{C}$ ) are shown in figure 2. The geometry of the bar is assumed to remain same as before ( $A_R = 0.025$ , and  $S_R = 20$ ). Since the temperature rise at yield of copper is the lowest among all materials, so for comparison purpose three temperatures upto  $66.8^{\circ}\text{C}$  is considered for all other materials in the present paper.



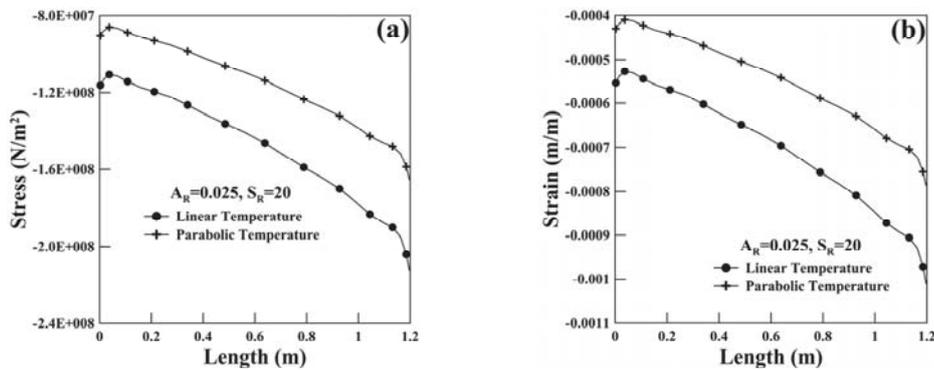
**Figure 1(a) & (b).** Variation of thermal load resistance ( $T_R$ ) with rise in temperature ( $\Delta T$ ).



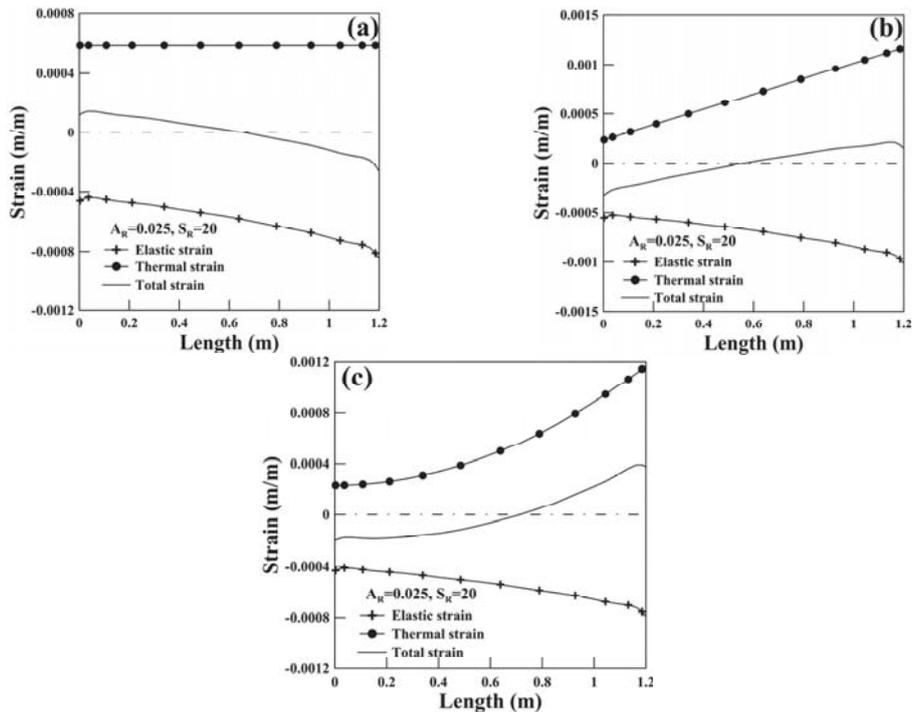
**Figure 2.** Stress and strain fields induced in different materials as indicated in plots I(a,b), II(a,b), III(a,b), IV(a,b) and V(a,b).

Although the mathematical formulation and the results of figure 2 refer to uniform temperature field, the present analysis can be carried out for various other temperature distributions also. In the present study, we consider linear and parabolic temperature distributions and the temperature boundary conditions at the largest and smallest ends of the bar are assumed as  $T(0)=20\text{ }^{\circ}\text{C}$  and  $T(L)=100\text{ }^{\circ}\text{C}$  respectively. The mathematical relations for linear and parabolic temperature distributions are taken as  $T(\xi)=T(0)+\{T(L)-T(0)\}\xi$  and  $T(\xi)=T(0)+\{T(L)-T(0)\}\xi^2$ .

The stress and strain fields induced in a clamped mild steel taper bar for linear and parabolic temperature distributions are shown in figure 3 and the plots of elastic, thermal and total strain for uniform, linear and parabolic temperature distribution is shown in figure 4.

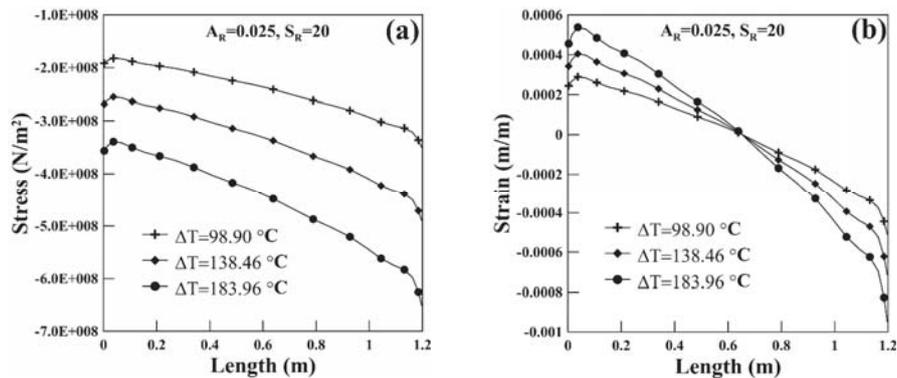


**Figure 3.** Stress and strain fields induced under linear and parabolic temperature distribution in a clamped mild steel taper bar.



**Figure 4.** Elastic, thermal and total strain induced in a clamped mild steel taper bar for (a) uniform, (b) linear and (c) parabolic temperature distribution.

The stress and strain field as mentioned here is upto the elastic limit of the bar material. However, the present method has the potential of application in structural mechanics problems involving material non-linearity. A sample result obtained from the solution of such a problem is given in figure. 5 for the mild steel bar, assuming bilinear post-elastic material behaviour. It is observed that the stress and strain field remains similar but changes in their pattern becomes more prominent when the bar material enters in post-elastic domain with increase in thermal load.



**Figure 5.** Stress and strain fields induced in a clamped mild steel taper bar at and above yield temperature.

#### 4. Conclusion

The investigation of thermoelastic stress and strain fields of clamped taper bar is formulated through a variational method for different bar materials. The effect of uniform thermal load on different bar materials has been reported. Assuming a series approximation following Galerkin's principle, the solution of the governing partial differential equation is obtained. The solution algorithm is implemented with the help of MATLAB computational simulation software. The effect of non-uniform thermal load and material non-linearity in a clamped mild steel taper bar has also been reported. Finally, the paper paves the way towards design optimization, where a uniform stress field is sought in the multi-parameter domain of the problem.

#### References

- [1] Timoshenko S 1930 *Strength of Materials: Part I Elementary Theory and Problems* (New York, D. Van Nostrand Company, Inc).
- [2] Hill R 1950 *The Mathematical Theory of Plasticity* (Oxford University Press, Oxford).
- [3] Niknam H, Fallah A and Aghdam M M 2014 Nonlinear bending of functionally graded tapered beams subjected to thermal and mechanical loading *Int. J. Nonlinear Mech.* **65** 141-147.
- [4] Das D, Sahoo P and Saha K N 2009 Dynamic analysis of non-uniform taper bars in post-elastic regime under body force loading *Appl. Math. Model.* **33** 4163-4183.
- [5] Noda N 1991 Thermal stress in materials with temperature-dependent properties *Appl. Mech. Rev.* **44** 383-397.
- [6] Grysa K and Kozłowski Z 1982 One dimensional problems of temperature and heat flux determination at the surfaces of a thermoelastic slab: Part I The analytical solutions *Nucl. Engrg. Des.* **74** 1-14.
- [7] Nayak P and Saha K N 2014 Analytical study on elastic limit speed of non-uniform rotating disks considering thermal effect on elasticity modulus *Appl. Mech. Mater.* 592-594 (DMMIR) 1001-1005.