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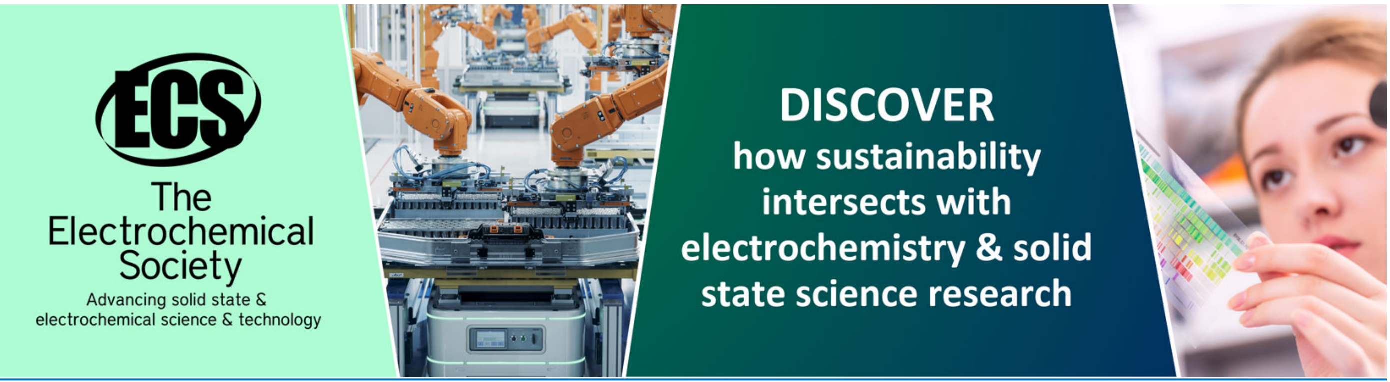

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Uniqueness and numerical methods in inverse obstacle scattering

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Abstract. The inverse problem we consider in this tutorial is to determine the shape of an obstacle from the knowledge of the far field pattern for scattering of time-harmonic plane waves. In the first part we will concentrate on the issue of uniqueness, i.e., we will investigate under what conditions an obstacle and its boundary condition can be identified from a knowledge of its far field pattern for incident plane waves. We will review some classical and some recent results and draw attention to open problems. In the second part we will survey on numerical methods for solving inverse obstacle scattering problems. Roughly speaking, these methods can be classified into three groups. Iterative methods interpret the inverse obstacle scattering problem as a nonlinear ill-posed operator equation and apply iterative schemes such as regularized Newton methods, Landweber iterations or conjugate gradient methods for its solution. Decomposition methods, in principle, separate the inverse scattering problem into an ill-posed linear problem to reconstruct the scattered wave from its far field and the subsequent determination of the boundary of the scatterer from the boundary condition. Finally, the third group consists of the more recently developed sampling methods. These are based on the numerical evaluation of criteria in terms of indicator functions that decide whether a point lies inside or outside the scatterer. The tutorial will give a survey by describing one or two representatives of each group including a discussion on the various advantages and disadvantages.

1. Introduction

The propagation of acoustic waves in a homogeneous isotropic medium with constant speed of sound c is governed by the wave equation

$$\Delta U = \frac{1}{c^2}\,\frac{\partial^2 U}{\partial t^2}$$

for the velocity potential U. For time-harmonic waves with frequency ω the time dependence is factored out in the form

$$U(x,t) = \mathrm{Re}\left\{u(x)e^{-i\omega t}\right\}$$

leading to the Helmholtz equation

$$\Delta u + k^2 u = 0$$

for the space dependent part u with positive wave number $k = \omega/c$. The scattering of an incident wave u^i by an obstacle D, that is, a bounded domain $D \subset \mathbb{R}^3$ with a connected complement, is modelled by an exterior boundary value problem

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{1.1}$$

for the scattered wave subject to a boundary condition

$$B(u^i + u^s) = 0 \quad \text{on } \partial D \tag{1.2}$$

and the Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial r} - iku^s = o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \tag{1.3}$$

uniformly for all directions. The total wave u is obtained via superposition $u = u^i + u^s$ and, for most of this tutorial we assume the incident wave to be a plane wave, that is,

$$u^i(x, d) = e^{ik\, x\cdot d}$$

where the unit vector d is the direction of propagation. The most frequently occurring boundary conditions are the Dirichlet boundary condition

$$B(u) := u$$

for a sound-soft scatterer and the impedance boundary condition

$$B(u) := \frac{\partial u}{\partial \nu} + ik\lambda u$$

with the exterior unit normal vector ν to ∂D and some impedance function $\lambda \geq 0$ on ∂D. Note that the Neumann boundary condition for sound-hard scatterers is included as the case where $\lambda = 0$. For simplicity, throughout the tutorial we assume that the boundary ∂D of the scatterer D is C^2 smooth.

The Sommerfeld radiation condition characterizes outgoing waves and ensures uniqueness for the obstacle scattering problem for both of the above boundary conditions (see [7]). For brevity, solutions u^s to the Helmholtz equation that satisfy the Sommerfeld radiation condition are called radiating solutions. They can be shown to have an asymptotic behavior of the form

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \quad \hat{x} := \frac{x}{|x|}, \tag{1.4}$$

uniformly with respect to all directions. The function u_∞ is known as the *far field pattern* of the scattered wave and is an analytic function of $\hat{x}$ on the unit sphere $\Omega := \{x \in \mathbb{R}^3 : |x| = 1\}$. As one of the most important tools in scattering theory, Rellich's lemma (see Theorem 2.13 in [7]) provides a one-to-one correspondence between a radiating solution u^s to the Helmholtz equation and its far field pattern u_∞ in the sense that $u_\infty = 0$ on Ω (or on an open subset of Ω) implies that $u^s = 0$ in its domain of definition.

The *inverse scattering problem* that we are concerned with is to determine the shape and location of the scatterer D from a knowledge of the far field pattern u_∞ for one or several incident plane waves. We note that this inverse problem is nonlinear in the sense that the scattered wave depends nonlinearly on the scatterer D. More importantly, it is ill-posed since the determination of D does not depend continuously on the far field pattern in any reasonable norm. This issue of ill-posedness will be handled using standard regularization techniques, e.g., Tikhonov regularization (see [7]).

We illustrate the nonlinearity and ill-posedness of the inverse obstacle scattering problem by looking at a simple example. For this we consider as incident field the entire solution v^i to the Helmholtz equation given by

$$v^i(x) = \frac{\sin k|x|}{|x|}, \quad x \in \mathbb{R}^3. \tag{1.5}$$

Because of

$$\frac{\sin k|x|}{|x|} = \frac{k}{4\pi}\int_\Omega e^{ik\,x\cdot d}\,ds(d), \quad x \in \mathbb{R}^3,$$

the field v^i is a Herglotz wave function (see [7]), i.e., a superposition of plane waves. For D a sound-soft ball of radius R centered at the origin the scattered wave is given by

$$v^s(x) = -\frac{\sin kR}{e^{ikR}}\,\frac{e^{ik|x|}}{|x|}\,, \quad |x| \geq R. \tag{1.6}$$

This leads to the total field

$$v(x) = \frac{1}{|x|e^{ikR}}\,\sin k(|x| - R), \quad |x| \geq R, \tag{1.7}$$

and the far field pattern

$$v_\infty(\hat{x}) = -\frac{\sin kR}{e^{ikR}}\,, \quad \hat{x} \in \Omega. \tag{1.8}$$

Therefore, given the a priori information that the scatterer is a ball centered at the origin, (1.8) provides a nonlinear equation for determining the radius R.

Concerning the ill-posedness we consider a perturbed far field pattern

$$v_\infty^\delta(\hat{x}) = -\frac{\sin kR}{e^{ikR}} + \delta Y_n(\hat{x})$$

with some $\delta \in \mathbb{R}$ and a spherical harmonic Y_n of degree n. Then, in view of the asymptotic behavior of the spherical Hankel functions for large argument, the corresponding total field is given in terms of an outgoing spherical wave function

$$v^\delta(x) = \frac{\sin k(|x| - R)}{e^{ikR}\,|x|} + \delta\,k\,i^{n+1}\,h_n^{(1)}(k|x|)Y_n\left(\frac{x}{|x|}\right)$$

with the spherical Hankel function $h_n^{(1)}$ of order n and of the first kind (see Section 2.4 in [7]). This implies

$$v^\delta(x) = \delta\,k\,i^{n+1}\,h_n^{(1)}(kR)Y_n\left(\frac{x}{|x|}\right), \quad |x| = R,$$

and consequently, by the asymptotics of the spherical Hankel functions for larger order, it follows that

$$|v^\delta(x)| \approx \delta\,k\left(\frac{2n}{ekR}\right)^n Y_n\left(\frac{x}{|x|}\right), \quad |x| = R.$$

This illustrates that small changes in the data v_∞ can cause large errors in the solution of the inverse problem, or a solution even may not exist anymore since v^δ may fail to have a closed surface as zero level surface.

The above inverse problem serves as a model problem for analyzing inverse scattering techniques in nondestructive evaluation such as radar, sonar, ultrasound imaging, seismic imaging etc. However, we should note that in practical applications the inverse scattering problem will never occur in the above idealized form. In particular, the far field pattern or some other measured quantity of the scattered wave will be available only for observation directions within a limited aperture either in the near or in the far field region. In addition, as it is the case for example in applications of inverse scattering techniques in land mine detection, the background might not homogeneous and then must be modelled as a layered medium.

In this tutorial our main concern is with the issues of uniqueness and (stabilized) reconstruction algorithms. In the subsequent section 2 we will address the issue of uniqueness. After settling the uniqueness issue one might be tempted to ask for existence of solutions to the inverse scattering problem. However, for inverse problems, in general, this is the wrong question to ask. For inverse scattering problems, positive answers would need to characterize far field patterns for which the corresponding total field vanishes on a closed surface and this problem is beyond the capability of analysis. This is also reflected through the above example for the ill-posedness of the inverse obstacle scattering problem. Therefore, after settling uniqueness, the main task in inverse obstacle scattering is to design methods for the approximate and stable solution under the assumption of a correct or a perturbed far field pattern for a scatterer D. The remaining sections 3–5 will introduce the main ideas of iterative methods, decomposition methods and sampling methods for approximately solving the inverse obstacle scattering problem. Although most of our analysis in sections 3–5 can be extended to the impedance and/or the Neumann boundary condition, we confine our presentation of reconstruction methods to the case of the Dirichlet boundary condition.

For more detailed presentations of the current state of research in inverse obstacle scattering we refer to the monographs [3, 7, 37] and the surveys [5, 9, 26, 39, 40].

2. Uniqueness

Since by Rellich's lemma the far field pattern uniquely determines the scattered wave and consequently the total wave in the exterior of the scatterer, the question of uniqueness for the inverse problem is equivalent to the question whether the total wave can satisfy the boundary condition (1.2) for two different domains D_1 and D_2. We immediately can exclude the case where the two scatterers are disjoint, i.e., $\overline{D_1} \cap \overline{D_2} = \emptyset$. In this situation, the scattered wave u^s is well defined in all of $\mathbb{R}^3$, since it is defined in the exterior of both D_1 and D_2. Consequently, the scattered wave u^s is an entire solution to the Helmholtz equation satisfying the radiation condition and therefore it must be identically zero. However, then the total wave coincides with the incident field and this leads to a contradiction, because the plane wave by itself cannot satisfy the boundary condition. For the Dirichlet and Neumann condition this is obvious, since the plane wave is given by an exponential function. For the impedance boundary condition, $Bu^i = 0$ on ∂D would imply that $\nu \cdot d + \lambda = 0$ on ∂D. This, with the aid of $\lambda \geq 0$ and $\lambda \neq 0$, leads to a contradiction via

$$\int_{\partial D} \lambda \, ds = \int_{\partial D} \{\nu \cdot d + \lambda\} ds = 0.$$

Hence, non-uniqueness can occur only when $\overline{D_1} \cap \overline{D_2} \neq \emptyset$, and, presently, this case cannot be excluded on the knowledge of the far field pattern for scattering of one incident plane wave only. However, when we have overdetermined data in the sense that the far field pattern is known for all incident directions we have the following classical uniqueness result for sound-soft scatterers due to Schiffer.

Theorem 2.1 *Assume that D_1 and D_2 are two sound-soft scatterers such that their far field patterns coincide for an infinite number of incident plane waves with distinct directions and one fixed wave number. Then $D_1 = D_2$.*

Proof. Assume that $D_1 \neq D_2$. By Rellich's lemma for each incident plane wave u^i the scattered waves u_1^s and u_2^s for the obstacles D_1 and D_2 coincide in the unbounded component G of the complement of $D_1 \cup D_2$. Without loss of generality, we can assume that $D^* := (\mathbb{R}^3 \setminus G) \setminus \bar{D}_1$ is nonempty. Then u_1^s is defined in D^*, and the total field $u = u^i + u_1^s$ satisfies the Helmholtz equation in D^* and the homogeneous boundary condition $u = 0$ on ∂D^*. Hence, u is a Dirichlet eigenfunction of $-\Delta$ in the domain D^* with eigenvalue k^2. The proof can now be completed by showing that the total fields for distinct incoming plane waves are linearly independent, since this contradicts the fact that for a fixed eigenvalue there exist only finitely many linearly independent Dirichlet eigenfunctions of $-\Delta$ in $H_0^1(D^*)$. □

Schiffer's uniqueness result was obtained around 1960 and appeared as a private communication in the monograph by Lax and Philipps [31]. We note that the proof presented in [31] contains a slight technical fault since the above argument does not work if D^* is replaced by $D_2 \setminus (D_1 \cap D_2)$ for the case where the complement of $D_1 \cup D_2$ is not connected.

By analyticity the far field pattern is completely determined on the whole unit sphere by only knowing it on some surface patch. Therefore, Schiffer's result and, simultaneously, all other results of this section carry over to the case of limited aperture problems where the far field is only known on some open subset of Ω.

Using the strong monotonicity property of the Dirichlet eigenvalues of $-\Delta$, extending Schiffer's ideas, Colton and Sleeman [10] showed that a sound-soft scatterer is uniquely determined by the far field pattern for one incident plane wave under the a priori assumption that it is contained in a ball of radius R such that $kR < \pi$. More recently, exploiting the fact that the wave functions are complex-valued, this bound was improved to $kR < 4.49$ by Gintides [13].

Schiffer's proof cannot be generalized to other boundary conditions. This is due to the fact that the finiteness of the dimension of the eigenspaces for eigenvalues of $-\Delta$ for the Neumann or impedance boundary condition requires the boundary of the intersection D^* from the proof of the Theorem 2.1 to be sufficiently smooth. Therefore, a different approach is required for establishing uniqueness for the inverse scattering problem for other boundary conditions. Assuming two different scatterers that produce the same far field patterns for all incident directions, Isakov [19] obtained a contradiction by considering a sequence of solutions with a singularity moving towards a boundary point of one scatterer that is not contained in the other scatterer. He used weak solutions and the analysis is technically involved. Later on, Kirsch and Kress [24] realized that the proof can be simplified by using classical solutions rather than weak solutions and by obtaining the contradiction by considering pointwise limits of the singular solutions rather than limits of L^2 integrals. Only after this new uniqueness proof was published, it was also observed by the authors that for scattering from impenetrable objects it is not required to know the boundary condition for the scattered wave as stated in the following theorem.

In the proof of that theorem, in addition to scattering of plane waves, we also need to consider scattering of point sources $\Phi(\cdot\,, z)$ with source location $z \in \mathbb{R}^3 \setminus \bar{D}$ given through the fundamental solution

$$\Phi(x, z) := \frac{e^{ik|x-z|}}{4\pi|x-z|}, \quad x \neq z,$$

to the Helmholtz equation in $\mathbb{R}^3$. We denote the corresponding scattered wave by $w^s(\cdot\,, z)$ and its far field pattern by $w_\infty(\cdot\,, z)$. Scattering by plane waves and by point sources is related through the *mixed reciprocity relation* (see [26, 37])

$$u^s(z, d) = 4\pi w_\infty(-d, z), \quad z \in \mathbb{R}^3 \setminus \bar{D},\, d \in \Omega, \tag{2.1}$$

which is valid both for the sound-soft and impedance boundary condition.

Theorem 2.2 *Assume that D_1 and D_2 are two scatterers with boundary conditions B_1 and B_2 such that the far field patterns coincide for all incident directions and one fixed wave number. Then $D_1 = D_2$ and $B_1 = B_2$*

Proof. Following Potthast [37] we simplify the approach of Kirsch and Kress through the use of the mixed reciprocity relation (2.1). Let $u_{\infty,1}$ and $u_{\infty,2}$ be the far field patterns for plane wave incidence and let w_1^s and w_2^s be the scattered waves for point source incidence corresponding to D_1 and D_2, respectively. With (2.1) and two applications of Rellich's lemma, first for scattering of plane waves and then for scattering of point sources, from the assumption $u_{\infty,1}(\hat{x}, d) = u_{\infty,2}(\hat{x}, d)$ for all $\hat{x}, d \in \Omega$ it can be concluded that $w_1^s(x, z) = w_2^s(x, z)$ for all $x, z \in G$. Here, as in the previous proof, G denotes the unbounded component of the complement of $D_1 \cup D_2$.

Now assume that $D_1 \neq D_2$. Then, without loss of generality, there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \bar{D}_2$. In particular, denoting by ν the outward unit normal to ∂D_1, we have

$$z_n := x^* + \frac{1}{n}\,\nu(x^*) \in G, \quad n = 1, 2, \dots,$$

for sufficiently large n. Then, on one hand we obtain that

$$\lim_{n\to\infty} B_1 w_2^s(x^*, z_n) = B_1 w_2^s(x^*, x^*),$$

since $w_2^s(x^*, \cdot)$ is continuously differentiable in a neighborhood of $x^* \notin \bar{D}_2$ due to reciprocity and the well-posedness of the direct scattering problem with boundary condition B_2 on ∂D_2. On the other hand we find that

$$\lim_{n\to\infty} B_1 w_1^s(x^*, z_n) = \infty,$$

because of the boundary condition $B_1 w_1^s(x^*, z_n) = -B_1\Phi(x^*, z_n)$ on ∂D_1. This contradicts $w_1^s(x^*, z_n) = w_2^s(x^*, z_n)$ for all sufficiently large n, and therefore $D_1 = D_2$.

Finally, to establish that $\lambda_1 = \lambda_2$ for the case of two impedance boundary conditions B_1 and B_2 we set $D = D_1 = D_2$ and assume that $\lambda_1 \neq \lambda_2$. Then from Rellich's lemma and the boundary conditions, considering one incident field, we have that

$$\frac{\partial u}{\partial \nu} + ik\lambda_1 u = \frac{\partial u}{\partial \nu} + ik\lambda_2 u = 0 \quad \text{on } \partial D$$

for the total wave $u = u_1 = u_2$. Hence, $(\lambda_1 - \lambda_2)u = 0$ on ∂D. From this, in view of the fact that $\lambda_1 \neq \lambda_2$, by Holmgren's theorem (see [26]) and the boundary condition we obtain that $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. This leads to the contradiction that the incident field must satisfy the radiation condition. Hence, $\lambda_1 = \lambda_2$. The case when one of the boundary conditions is the sound-soft boundary condition is dealt with analogously. □

Although there is widespread belief that the far field pattern for one single direction and one single wave number determines the scatterer without any additional a priori information, establishing this result still remains a challenging open problem. To illustrate the difficulty of a proof, we consider scattering of the entire solution v^i given by (1.5) from a sound-soft ball D of radius R centered at the origin. Then from (1.7) we observe that the total field v vanishes on the spheres with radius $R + m\pi/k$ centered at the origin for all integers m. This indicates that proving uniqueness of the inverse obstacle scattering problem with one single incident plane wave needs to incorporate special features of the incident field.

Some progress has recently be obtained by Cheng and Yamamoto [4], Alessandrini and Rondi [1], and Liu and Zou [32] who established uniqueness with one incident plane wave for polyhedral scatterers. Assuming that there exist two polyhedral scatterers producing the same far field pattern for one incident plane wave, the main idea of their proofs is to use the reflexion principle to construct a zero field line extending to infinity. However, in view of the fact that the scattered wave tends to zero uniformly at infinity, this contradicts the property that the incident plane wave has modulus one everywhere.

3. Iterative methods

We now turn to reconstruction methods and as a first group we describe iterative methods. Here the inverse problem is interpreted as a nonlinear ill-posed operator equation which is solved by iteration methods such as regularized Newton methods, Landweber iterations or conjugate gradient methods. The solution to the direct scattering problem with a fixed incident plane wave u^i defines an operator

$$A : \partial D \mapsto u_\infty$$

that maps the boundary ∂D of the scatterer D onto the far field pattern u_∞ of the scattered wave. In terms of this operator, given a far field pattern u_∞, the inverse problem just consists in solving the nonlinear and ill-posed operator equation

$$A(\partial D) = u_\infty \tag{3.1}$$

for the unknown surface ∂D.

In order to define the operator A rigorously, the most appropriate approach is to choose a fixed reference domain D of class C^2 and consider a family of scatterers D_h with boundaries represented in the form

$$\partial D_h = \{x + h(x) : x \in \partial D\},$$

where $h : \partial D \to \mathbb{R}^3$ is of class C^2 and sufficiently small in the C^2 norm on ∂D. Then we may consider the operator A as a mapping from a ball $V := \{h \in C^2(\partial D) : \|h\|_{C^2} < a\} \subset C^2(\partial D)\}$ with sufficiently small radius $a > 0$ into $L^2(\Omega)$. However, for ease of presentation, we proceed differently and consider only starlike domains, i.e., domains D_r that allow a parameterization of the form

$$\partial D_r = \{r(\hat{x})\, \hat{x} : \hat{x} \in \Omega\} \tag{3.2}$$

where $r : \Omega \to \mathbb{R}$ is a positive function representing the radial distance from the origin. Then, we may interpret the operator A as a mapping

$$A : \{r \in C^2(\Omega) : r > 0\} \to L^2(\Omega), \quad A : r \mapsto u_\infty,$$

and, consequently, the inverse obstacle scattering problem consists in solving

$$A(r) = u_\infty \tag{3.3}$$

for the unknown radial function r.

Since A is nonlinear, we may linearize

$$A(r + q) = A(r) + A'(r)q + O(q^2)$$

in terms of a Fréchet derivative $A'(r)$. Then given a current approximation r for the solution of (3.3) in order to obtain an update $r + q$ instead of solving the full equation $A(r + q) = u_\infty$ we solve the approximate linear equation

$$A(r) + A'(r)q = u_\infty \tag{3.4}$$

for q. We note that the linearized equation inherits the ill-posedness of the nonlinear equation and thererfore regularization is required. As in the classical Newton iterations this linearization procedure is iterated until some stopping criteria is satisfied.

The Fréchet differentiabitlity of the operator A is addressed in the following theorem.

Theorem 3.1 *The boundary to far field mapping $A : r \mapsto u_\infty$ is Fréchet differentiable and the derivative is given by*

$$A'(r) : q \mapsto v_{q,\infty}$$

where $v_{q,\infty}$ is the far field pattern of the solution v_q to the Dirichlet problem for the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{D}_r$ satisfying the Sommerfeld radiation condition and the boundary condition

$$v_q = -\nu \cdot \hat{x}\, \frac{\partial u}{\partial \nu}\, q \quad \text{on } \partial D_r \tag{3.5}$$

with $u = u^i + u^s$ the total wave for scattering from the domain D_r.

The boundary condition (3.5) for the derivative can be obtained formally by differentiating the boundary condition $u = 0$ on ∂D_r with respect to ∂D_r by the chain rule. It was obtained by Roger [41] who first employed Newton type iterations for the approximate solution of inverse obstacle scattering problems. Rigorous foundations for the Fréchet differentiability were given by Kirsch [20] in the sense of a domain derivative via variational methods and by Potthast [33] via boundary integral equation techniques. Alternative proofs were contributed by Kress and Päivärinta [28] based on Green's theorems and a factorization of the difference of the far field for the domains D_r and D_{r+q} and by Hohage [16] and Schormann [42] via the implicit function theorem.

To justify the application of regularization methods for stabilizing (3.4) one has to establish injectivity and dense range of the operator $A'(r) : L^2(\Omega) \to L^2(\Omega)$. This is settled for the Dirichlet and impedance boundary condition for large λ and remains an open problem for the Neumann boundary condition [29]. In the classical Tikhonov regularization, (3.4) is replaced by solving

$$\alpha q + \left[A'(r)\right]^* A'(r) q = \left[A'(r)\right]^* \{u_\infty - A(r)\} \tag{3.6}$$

with some positive regularization parameter α and the L^2 adjoint $[A'(r)]^*$ of $A'(r)$. For details on the numerical implementation, in particular on the choice of the regularization parameter, and numerical examples in two dimensions we refer to [7, 15, 20, 25, 27] and the references therein. The numerical examples strongly indicate that it is advantageous to use some Sobolev norm instead of the L^2 norm as penalty term in the Tikhonov regularization. Numerical examples in three dimensions have been more recently reported by Farhat et al [12] and by Harbrecht and Hohage [14].

In closing the section on Newton iterations we note as their main advantages that this approach is conceptually simple and, as the numerical examples indicate, leads to highly accurate reconstructions with reasonable stability against errors in the far field pattern. On the other hand, it should be noted that for the numerical implementation an efficient forward solver is needed and good a priori information is required in order to ensure convergence. In addition, on the theoretical side, although some progress has been made through the work of Hohage [16] and Potthast [38] the convergence of regularized Newton iterations for inverse obstacle scattering problems has not been completely settled.

4. Decomposition methods

The main idea of so-called decomposition methods is to break up the inverse obstacle scattering problem into two parts: the first part deals with the ill-posedness by constructing the scattered wave u^s from its far field pattern u_∞ and the second part deals with the nonlinearity by determining the unknown boundary ∂D of the scatterer as the location where the boundary condition for the total field $u^i + u^s$ is satisfied in a least-squares sense. In the potential method due to Kirsch and Kress [23], for the first part, enough a priori information on the unknown scatterer D is assumed so one can place a closed surface Γ inside D. Then the scattered field u^s is sought as a single-layer potential

$$u^s(x) = \int_\Gamma \varphi(y)\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{4.1}$$

with an unknown density $\varphi \in L^2(\Gamma)$. In this case the far field pattern u_∞ has the representation

$$u_\infty(\hat{x}) = \frac{1}{4\pi} \int_\Gamma e^{-ik\,\hat{x}\cdot y}\varphi(y)\,ds(y), \quad \hat{x} \in \Omega.$$

Given the far field pattern u_∞, the density φ is now found by solving the integral equation of the first kind

$$S_\infty \varphi = u_\infty \tag{4.2}$$

with the compact integral operator

$$(S_\infty \varphi)(\hat{x}) := \frac{1}{4\pi} \int_\Gamma e^{-ik\,\hat{x}\cdot y}\varphi(y)\,ds(y), \quad \hat{x} \in \Omega.$$

Due to the analytic kernel of S_∞, the integral equation (4.2) is severely ill-posed. For a stable numerical solution of (4.2) Tikhonov regularization can be applied, that is, the ill-posed equation (4.2) is replaced by

$$\alpha\varphi_\alpha + S_\infty^* S_\infty \varphi_\alpha = S_\infty^* u_\infty \tag{4.3}$$

with some positive regularization parameter α and the adjoint S_∞^* of $S_\infty : L^2(\Gamma) \to L^2(\Omega)$.

Given an approximation of the scattered wave u_α^s by inserting a solution φ_α of (4.3) into the potential (4.1), the unknown boundary ∂D is then determined by requiring the sound-soft boundary condition

$$u^i + u^s = 0 \quad \text{on } \partial D \tag{4.4}$$

to be satisfied in a least-squares sense, i.e., by minimizing the L^2 norm of the defect

$$\left\| u^i + u_\alpha^s \right\|_{L^2(\Lambda)}$$

over a suitable set of admissible surfaces Λ. Of course, instead of solving this minimization problem we also can confine ourselves to visualizing ∂D by color coding the values the modulus $|u|$ of the total field $u \approx u^i + u_\alpha^s$ on a sufficiently fine grid over $\mathbb{R}^3$.

Clearly, we can expect (4.2) to have a solution $\varphi \in L^2(\Omega)$ if and only if u_∞ is the far field of a radiating solution to the Helmholtz equation in the exterior of Γ with sufficiently smooth boundary values on Γ. Hence, the solvability of (4.2) is related to regularity properties of the scattered wave which, in general, cannot be known in advance for the unknown scatterer D. Nevertheless, it is possible to provide a solid theoretical foundation to the above procedure (see [7, 23]).

The point source method of Potthast [34, 35, 37] can also be interpreted as a decomposition method. Its motivation is based on Green's representation for the scattered wave for a sound-soft obstacle

$$u^s(x) = -\int_{\partial D} \frac{\partial u}{\partial \nu}(y)\,\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{4.5}$$

and its far field pattern

$$u_\infty(\hat{x}) = -\frac{1}{4\pi} \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\,e^{-ik\,\hat{x}\cdot y}\,ds(y), \quad \hat{x} \in \Omega, \tag{4.6}$$

that is, Huygen's principle (see Theorem 3.12 in [7]). For $z \in \mathbb{R}^3 \setminus \bar{D}$ we choose a domain B_z such that $z \notin B_z$ and $\bar{D} \subset B_z$ and approximate the point source $\Phi(\cdot, z)$ by a Herglotz wave function

$$\Phi(y,z) \approx \int_\Omega e^{ik\, y\cdot d} g_z(d)\, ds(d), \quad y \in B_z \tag{4.7}$$

with kernel g_z. Under the assumption that there does not exist a nontrivial solution to the Helmholtz equation in B_z with homogeneous Dirichlet boundary condition on ∂B_z, the Herglotz wave functions are dense in $H^{1/2}(\partial B_z)$ [8, 11] and consequently the approximation (4.7) can be achieved uniformly with respect to y on compact subsets of B_z. Then we can insert (4.7) into (4.5) and use (4.6) to obtain

$$u^s(z) \approx 4\pi \int_\Omega g_z(\hat{x}) u_\infty(-\hat{x})\, ds(\hat{x}) \tag{4.8}$$

as an approximation for the scattered wave u^s. Knowing an approximation for the scattered wave, the boundary ∂D can be found as above from the boundary condition (4.4).

The approximation (4.7), for example, can be obtained by solving the ill-posed linear integral equation

$$\int_\Omega e^{ik\, y\cdot d} g_z(d)\, ds(d) = \Phi(y,z), \quad y \in \partial B_z, \tag{4.9}$$

via Tikhonov regularization and the Morozov discrepancy principle. Note that although the integral equation (4.9), in general, is not solvable the approximation property (4.8) is ensured through the above denseness result on Herglotz wave functions.

As a first advantage of the decomposition methods we note that with the idea of separating the ill-posedness and the nonlinearity again they are conceptually straightforward. The second and main advantage consists of the fact that their numerical implementation does not require a forward solver. As a disadvantage, as in the Newton method of the previous section, if we go beyond visualization of the level surfaces of $|u|$ and proceed with the minimization, a good a priori information on the unknown scatterer is needed. Furthermore, the accuracy of the reconstructions is slightly inferior to that of the Newton iterations.

More recently a hybrid method combining ideas of decomposition methods and Newton iterations of the previous section have been suggested [27, 30, 43]. In principle, this approach may be considered as a modification of the potential method due to Kirsch and Kress in the sense that the auxiliary surface Γ is viewed as an approximation for the unknown boundary and, keeping φ_α fixed as a regularized solution of (4.2), update Γ via linearizing the boundary condition (4.4) around Γ. For its brief description we assume the scatterer to be starlike and recall the representation (3.2). Given a far field u_∞ and a current approximation ∂D_r with radial function r for the boundary surface, we solve ill-posed integral equation

$$\frac{1}{4\pi} \int_{\partial D_r} e^{-ik\, \hat{x}\cdot y} \varphi(y)\, ds(y) = u_\infty(\hat{x}), \quad \hat{x} \in \Omega,$$

by Tikhonov regularization and set

$$u^s(x) = \int_{\partial D_r} \frac{e^{ik|x-y|}}{4\pi\, |x-y|}\, \varphi(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D_r.$$

Then we evaluate the boundary values of $u = u^i + u^s$ and its derivatives on ∂D_r via the jump relations and find an update $r + q$ by linearizing the boundary condition $u|_{\partial D_{r+q}} = 0$, that is, by solving

$$u|_{\partial D_r} + \hat{x} \cdot \operatorname{grad} u|_{\partial D_r}\, q = 0$$

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for q. In an obvious way, these two steps are iterated. Clearly, this approach does not require a forward solver and connects ideas of Newton iterations and decomposition methods. From numerical examples (see [27, 30, 43]) it can be concluded that the quality of the reconstructions is similar to that of Newton iterations in the spirit of the previous section.

Without giving any details on the computations, in Fig. 1–4 we present some examples for reconstructions by the above hybrid method obtained by Pedro Serranho. The numerical quadratures were based on Wienert's method [44] as described in section 3.6 of [7] and the radial distance functions were approximated by linear combinations of spherical harmonics up to order eight. In each example the figure on the left hand side gives the exact boundary shape, the figure in the middle the reconstruction with one incident wave in direction of the arrow and the figure on the right hand side gives the difference between the exact and the approximate radial function. The reconstructions are obtained with 2% random noise added to the synthetic far field pattern. The wave number is $k = 1$.

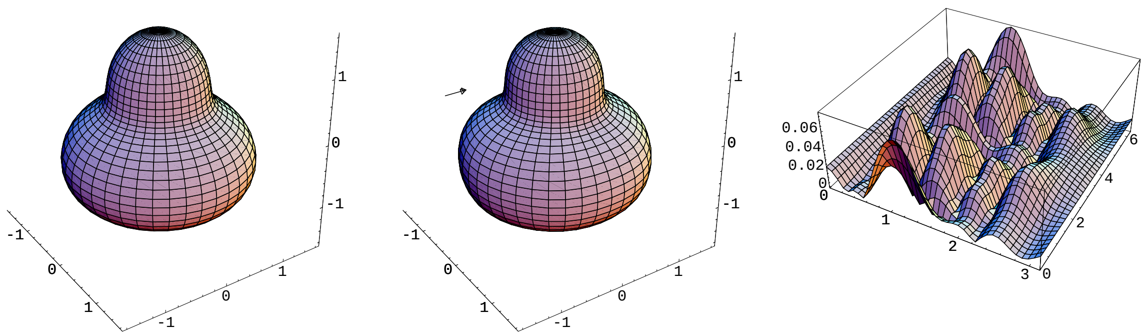

Figure 1. Reconstruction of an acorn shaped domain.

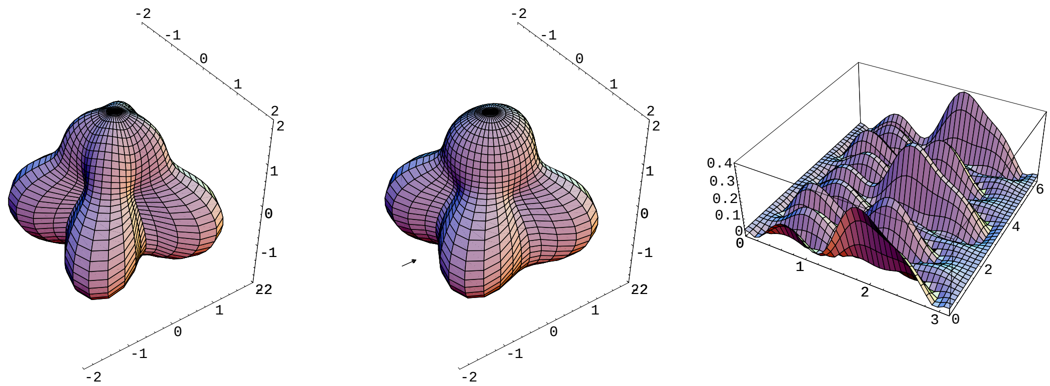

Figure 2. Reconstruction of a pinched acorn shaped domain.

5. Sampling methods

The main idea of sampling methods is to choose an indicator function f on $\mathbb{R}^3$ such that its value $f(z)$ decides whether z lies inside or outside the scatterer D. In most cases, the indicator function is designed in terms of the behavior of an ill-posed linear integral equation. To obtain

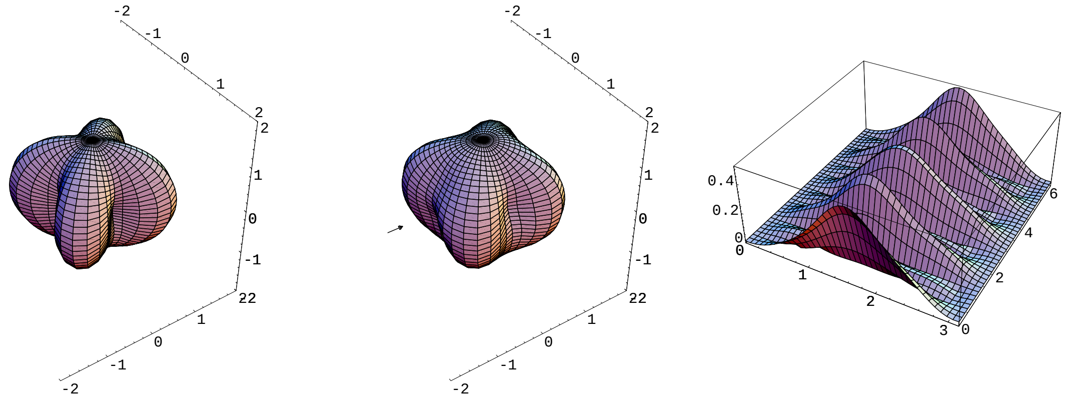

Figure 3. Reconstruction of a star shaped domain.

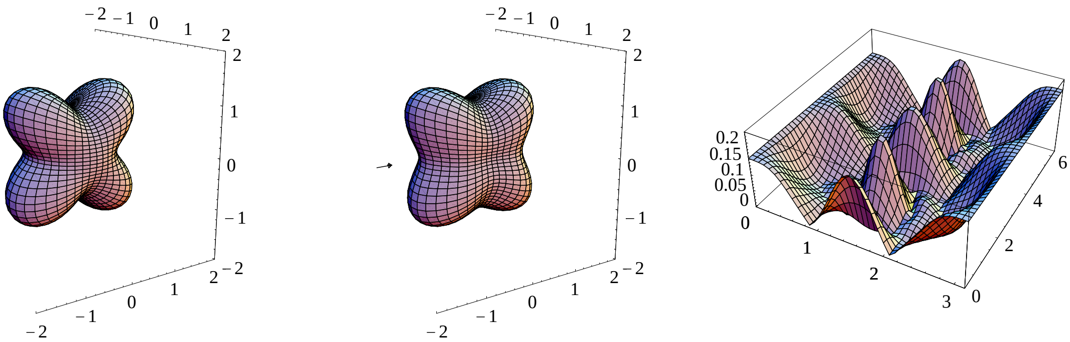

Figure 4. Reconstruction of a cushion shaped domain.

reconstructions the criterion is evaluated numerically for a grid of points. As opposed to the two previous groups of methods, in principle, the sampling methods need full data $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \Omega$.

We begin by describing the *linear sampling method* as suggested by Colton and Kirsch [6]. Its basic idea is to find a Herglotz wave function

$$v^i(x) = \int_\Omega e^{ik\,x\cdot d} g(d,z)\, ds(d), \quad x \in \mathbb{R}^3,$$

with kernel g, i.e., a superposition of plane waves, such that the corresponding scattered wave v^s coincides with a point source $\Phi(\cdot\,, z)$ located at a point $z \in D$. To this aim we define the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ as integral operator with kernel given through the far field pattern by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \Omega. \tag{5.1}$$

Obviously, by superposition Fg is the far field pattern corresponding to scattering of the Herglotz wave function with kernel g. Then, to achieve the above goal, we have to find the kernel $g(\cdot\,, z)$ of the Herglotz wave function as a solution to the integral equation of the first kind

$$Fg(\cdot\,, z) = \Phi_\infty(\cdot\,, z) \tag{5.2}$$

with the far field of the fundamental solution given by

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi}\, e^{-ik\, z\cdot\hat{x}}.$$

Assume that g solves equation (5.2). Then, by Rellich's lemma, we have that

$$\int_\Omega u^s(x; d) g(d, z)\, ds(d) = \Phi(x, z), \quad x \in \mathbb{R}^3 \setminus \bar{D}. \tag{5.3}$$

Letting x tend to the boundary and using the boundary condition $u^i + u^s = 0$ on ∂D we conclude that the Herglotz wave function v^i with kernel g is a solution to the interior Dirichlet problem

$$\Delta v^i + k^2 v^i = 0 \quad \text{in } D \tag{5.4}$$

with boundary condition

$$v^i + \Phi(\cdot\,, z) = 0 \quad \text{on } \partial D. \tag{5.5}$$

Conversely, if the Herglotz wave function v^i with kernel g solves (5.4)–(5.5) then its kernel g is a solution of (5.2). Hence, if a solution $g(\cdot\,, z)$ to the integral equation (5.2) of the first kind exists for all $z \in D$, then from the boundary condition (5.5) for the Herglotz wave function we conclude that

$$\|g(\cdot\,, z)\|_{L^2(\Omega)} \to \infty$$

as the source point z approaches the boundary ∂D. Therefore, in principle, the boundary ∂D may be found by solving the integral equation (5.2) for z taken from a sufficiently fine grid in $\mathbb{R}^3$ and determining ∂D as the location of those points z where $\|g(\cdot\, z)\|_{L^2(\Omega)}$ becomes large.

However, in general, the solution to the interior Dirichlet problem (5.4)–(5.5) will have an extension as a Herglotz wave function across the boundary ∂D only in very special cases (for example if D is a ball with center at z). Hence, the integral equation of the first kind (5.2) will have a solution only in special cases. Nevertheless, by making use of the denseness properties of the Herglotz wave functions as mentioned above, the following result can be established (see [6]).

Theorem 5.1 *Under the assumption that there does not exist a nontrivial solution to the Helmholtz equation in D with homogeneous Dirichlet boundary condition on ∂D, for every $\varepsilon > 0$ and $z \in D$ there exists a function $g(\cdot\,, z) \in L^2(\Omega)$ such that*

$$\|Fg(\cdot\,, z) - \Phi_\infty(\cdot\,, z)\|_{L^2(\Omega)} \leq \varepsilon$$

and

$$\|g(\cdot\,, z)\|_{L^2(\Omega)} \to \infty, \quad z \to \partial D,$$

and the Herglotz wave function v^i with kernel $g(\cdot\,, z)$ becomes unbounded

$$\|v^i\|_{L^2(D)} \to \infty,\ z \to \partial D.$$

From this it can be expected that solving the integral equation (5.2) and scanning the values for $\|g(\cdot\,, z)\|_{L^2(\Omega)}$ will yield an approximation for ∂D through those points where the norm of g is large. A possible procedure with noisy data $\|u_{\infty,\delta} - u_\infty\|_{L^2(\Omega\times\Omega)} \leq \delta$ with error level δ is as follows. Denote by F_δ the far field operator F with the kernel u_∞ replaced by the data $u_{\infty,\delta}$. Then for each z from a grid in $\mathbb{R}^3$ determine $g^\delta = g^\delta(\cdot\,, z)$ by minimizing the Tikhonov functional

$$\|F_\delta g^\delta(\cdot\,, z) - \Phi_\infty(\cdot\,, z)\|^2_{L^2(\Omega)} + \alpha\|g^\delta(\cdot\,, z)\|^2_{L^2(\Omega)},$$

where the regularization parameter α is chosen according to Morozov's generalized discrepancy principle, i.e., $\alpha = \alpha(z)$ is chosen such that

$$\|F_\delta g^\delta(\cdot\,, z) - \Phi_\infty(\cdot\,, z)\|_{L^2(\Omega)} \approx \delta\|g^\delta(\cdot\,, z)\|_{L^2(\Omega)}.$$

Then the unknown boundary is determined by those points where $\|g^\delta(\cdot\,, z)\|_{L^2(\Omega)}$ sharply increases.

However there is a problem with the linear sampling method since is not clear whether the regularized solution obtained for (5.2) by Tikhonov regularization via the discrepancy principle actually provides an approximation in the sense of Theorem 5.1. A remedy, at least for the case of a sound-soft obstacle, has been provided by Arens [2] through connecting the linear sampling method to the *factorization method* that we are now going to describe as a second example of a sampling method.

The drawback that for $z \in D$ the integral equation (5.2), in general, is not solvable is remedied through the factorization method due Kirsch [21]. In this method, (5.2) is replaced by

$$(F^*F)^{1/4}g(\cdot\,, z) = \Phi_\infty(\cdot\,, z) \tag{5.6}$$

leading to the following characterization of the scatterer D.

Theorem 5.2 *Assume that there does not exist a nontrivial solution to the Helmholtz equation in D with homogeneous Dirichlet boundary condition on ∂D. Then $z \in D$ if and only if* (5.6) *is solvable in $L^2(\Omega)$.*

For a proof we refer to [21, 22]. Comparing equations (5.2) and (5.6), the above results can be interpreted in the sense that as compared with $(F^*F)^{1/4}$ the operator F itself is too much smoothing since $\Phi_\infty(\cdot\,, z)$ does not belong to its range $F(L^2(\Omega))$ if $z \in D$. The results also imply that, in contrast to the linear sampling method, if Tikhonov regularization with the regularization parameter chosen by the Morozov discrepancy principle is used to solve equation (5.6) with noisy data u_∞, then $\|g(\cdot, z)\|$ converges as the noise level tends to zero if and only if $z \in D$. The most convenient approach to a numerical implementation of Theorem 5.2 is via Picard's criterion for the solvability of ill-posed linear operator equations in terms of a singular system of F.

Both for the linear sampling method and the factorization method the indicator function f is given through the norm $f(z) := \|g(\cdot\,, z)\|_{L^2(\Omega)}$ of the solutions to (5.2) and (5.6), respectively. For Potthast's [36, 37, 40] *singular source method*, that we now will consider as a third and final example for sample methods, the indicator function is given by $f(z) := w^s(z, z)$ through the value of the scattered wave $w^s(\cdot\,, z)$ for the singular source $\Phi(\cdot\,, z)$ as incident field evaluated at the source point z. The values $w^s(z, z)$ will be small for points $z \in \mathbb{R}^3 \setminus \bar{D}$ that are away from the boundary and will blow up when z approaches the boundary due to the singularity of the incident field. Clearly, the singular source method can be viewed as a straightforward numerical implementation of the uniqueness proof for Theorem 2.2.

Assuming the far field pattern for plane wave incidence to be known for all incident and observation directions, the indicator function $w^s(z, z)$ can be obtained by two applications of (4.8) and the mixed reciprocity principle (2.1). Combining (2.1) and (4.8) we obtain the approximation

$$w_\infty(-d, z) = \frac{1}{4\pi}\, u^s(z, d) \approx \int_\Omega g_z(\hat{x}) u_\infty(-\hat{x}, d)\, ds(\hat{x}).$$

Inserting this into (4.8) as applied to w^s yields the approximation

$$w^s(z, z) \approx 4\pi \int_\Omega \int_\Omega g_z(d) g_z(\hat{x}) u_\infty(-\hat{x}, d)\, ds(\hat{x})\, ds(d). \tag{5.7}$$

The probe method as suggested by Ikehata [17, 18] uses as indicator function an energy integral for $w^s(\cdot, z)$ instead of the point evaluation $w^s(z, z)$. In this sense, it follows the uniqueness proof of Isakov whereas the singular source method mimics the uniqueness proof of Kirsch and Kress.

The theoretical foundation of sampling methods provides beautiful and exciting mathematics. Their main advantage consists of their simple implementation and the fact that no a priori information on the shape and location of the obstacle is required. In addition, in general, also the boundary condition need not to be known in advance. On the other hand, as a disadvantage the sampling methods require a lot of data and do not provide very sharp boundaries due to the need to decide numerically the question on how large infinity is.

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