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## Roll waves in two-layer Hele-Shaw flows

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Abstract. In this paper we study the emergence and development of roll waves in two-layer fluid flow in a Hele-Shaw cell. We propose the mathematical model of such flow and define the conditions of transition from stable state to instability in the form of the roll waves. We find out the physical parameters of flows at which the roll waves exist. A linear stability analysis and the Whitham criterion of roll waves existence are used for solving the problem and arrive to identical conclusions on depths of upper and lower layers at which violation of flow stability occurs. The numerical calculations for the obtained mathematical model at found ratios of densities, viscosities and depths of layers are performed. They confirm development of the roll waves of finite amplitude from small oscillations of the interface.

#### 1. Introduction

A wide class of inhomogeneous fluid flows in long channels allow arising nonlinear regime leading to roll waves. This regime is a quasi periodic channel flow, in which smooth regions of flow are separated by hydraulic jumps. A feature of these flows is a transition from sub-critical flow to supercritical one in a coordinate system moving with the wave. For the first time roll waves were described by R. Dressler [1] in 1949. He found out that a class of travelling waves for shallow water equations consisted of periodic discontinuous solutions. This flow is easy to reproduce experimentally in natural or laboratory conditions. Despite of the long history of roll waves study, there are many open questions. For example it is interested to know how the roll waves arise and develop from small perturbations of interface of two-layer flow under lid. The roll waves change stable stationary flow and can lead to slug regime. That is why their study has a practical value. Different theoretical and experimental aspects of study of roll waves in liquids are considered in monograph [2] and papers [3, 4, 5]. A description and nonlinear analysis of roll waves in the framework of Burgers, Sen-Venane and Kuramoto–Sivashinsky equations can be found in [6, 7].

The mathematical model describing the motion of inhomogeneous immiscible fluid in a Hele-Shaw cell is proposed in recent paper [8]. Moreover, different simple modifications of layered flows are considered in that paper. Now we pay our attention to the study of two-layer flows of fluid and modelling of roll waves. A linear stability analysis is provided for two-layer Hele-Shaw flows. Values of physical parameters are found out for the development of flow instabilities. Velocities of propagation of nonlinear perturbations are defined and hyperbolicity conditions are formulated. Critical parameters and depths, at which the transition to roll wave begins, are found using the Whitham criterion [9]. A numerical modelling of perturbation development

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on the interface is provided. A comparison of conditions of stability violation with numerical results is carried out.

#### 2. Mathematical model of inhomogeneous Hele-Shaw flow

A Hele-Shaw cell is the area between two parallel plates with a small gap between them. Equations of motion of viscous incompressible liquid in the Hele-Shaw cell have the form

$$\rho(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)v) + \nabla p = \mu \mathbf{v}_{zz} + \rho \mathbf{g}, \quad \nabla \cdot \mathbf{v} = 0.$$
<sup>(1)</sup>

Here  $\mathbf{x} = (x, y, z)$  is the coordinate vector, t is the time,  $\mathbf{v} = (u, v, w)$  is the velocity vector, p is the pressure,  $\rho$  is the density,  $\mu$  is the viscosity and  $\mathbf{g} = (0, -g, 0)$  is the acceleration of gravity vector. The operator  $\nabla$  is calculated with respect to the coordinate vector. The characteristic sizes of cell (L, H) in the x and y direction, respectively, are significantly higher than the cell gap a. That is why the summands  $\mathbf{v}_{xx}$  and  $\mathbf{v}_{yy}$  vanish in the momentum equations. They are negligible compared to  $\mathbf{v}_{zz}$ .

We consider the velocity field in the form

$$u = \frac{3}{2} \left( 1 - \left(\frac{2z}{a}\right)^2 \right) u'(t, x, y), \quad v = \frac{3}{2} \left( 1 - \left(\frac{2z}{a}\right)^2 \right) v'(t, x, y), \quad w = 0.$$

It provides the fulfilment of the no-slip conditions on the cell walls  $z = \pm a/2$ . We also assume that the function p does not depend on z. Integrating equations (1) from -a/2 to a/2 leads to the system

$$\rho(u_t + \beta(uu_x + vu_y)) + p_x = -\mu u,$$
  

$$\rho(v_t + \beta(uv_x + vv_y)) + p_y = -\mu v - \rho g,$$
  

$$u_x + v_y = 0.$$
(2)

The primes are omitted. Here and below  $\mu$  denotes the modified fluid viscosity  $12\mu/a^2$ . The coefficient  $\beta$  is equal to 6/5. Listed above simplifications are connected with the inequalities  $a \ll L$ ,  $a \ll H$  and often used for modelling of Hele-Shaw flows [10, 11]. We assume that the horizontal size L of the cell is much more than its depth H, i.e.  $H/L = \varepsilon \ll 1$ . Supposing that

$$t \to \varepsilon^{-1} t, \quad x \to \varepsilon^{-1} x, \quad v \to \varepsilon v, \quad \mu \to \varepsilon \mu$$

and omitting all terms of order  $\varepsilon^2$  we have the long-wave approximation model

$$\rho(u_t + \beta(uu_x + vu_y)) + p_x = -\mu u, \quad p_y = -g\rho,$$
  

$$u_x + v_y = 0; \quad v\big|_{y=0} = v\big|_{y=H} = 0.$$
(3)

Here y = 0 and y = H are the lower and upper boundaries in the y-direction.

#### 2.1. Equations of two-layer flow

Let us consider the class of layered flows with the depths  $h_i(t, x)$ , the velocities  $u_i(t, x)$ , the constant densities  $\rho_i$  and the viscosities  $\mu_i$  in the layers i = 1, 2 (index "1" is for the lower layer and "2" is for the upper one). We assume that the cell height is constant, i.e.  $h_1 + h_2 = H = \text{const.}$  We find the pressure from the second equation of (3) and take into account the kinematic condition on the interface. After some amount of calculations we arrive at the system describing the two-layer flow of viscous liquid in a Hele-Shaw cell [8]

$$\rho_1 (u_{1t} + \beta u_1 u_{1x} + g h_{1x}) + g \rho_2 h_{2x} + p_{0x} = -\mu_1 u_1,$$
  

$$\rho_2 (u_{2t} + \beta u_2 u_{2x}) + p_{0x} = -\mu_2 u_2,$$
  

$$h_{1t} + (u_1 h_1)_x = 0, \quad h_{2t} + (u_2 h_2)_x = 0,$$
(4)

where  $p_0(t, x)$  is the pressure on an upper lid.

Further we use the following notations for the unknown functions, the density ratio and the viscosity ratio in the layers

$$u = u_1, \quad w = u_2, \quad h = h_1, \quad \rho = \rho_2/\rho_1, \quad \mu = \mu_1/\mu_2.$$

Taking into consideration that  $h_2 = H - h$  we multiply the second equation of (4) by  $\rho$  and subtract from the first one. As a result we obtain the system

$$(u - \rho w)_t + \beta (u u_x - \rho w w_x) + b h_x = \frac{\mu_2}{\rho_1} \Big( w - \mu u \Big), \quad h_t + (u h)_x = 0,$$

where  $b = g(1 - \rho)$ . According to the last equations of (4) the flow rate in the channel does not depend on x. We consider flows with known constant flow rate

$$hu + (H - h)w = Hu_m = \text{const},$$

where  $u_m$  is the average flow velocity. We can take H = 1, b = 1 after using corresponding dimensionless variables. The additional scaling of independent variables helps to exclude the parameter  $\mu_2/\rho_1$ . Summarizing given assumptions we take the equations for specification of the depths and velocities in the form

$$(u - \rho w)_t + \beta (u u_x - \rho w w_x) + h_x = w - \mu u,$$
  

$$h_t + (u h)_x = 0, \quad hu + (1 - h)w = u_m.$$
(5)

Further theoretical analysis and modelling of the roll waves are provided in the framework of equations (5).

#### 3. Linear stability analysis

Let us consider the constant solution of system (5)  $u = u_0$ ,  $w = w_0$ ,  $h = h_0$ . We should emphasize that  $u_0$ ,  $w_0$  and  $h_0$  have to satisfy the equations

$$w_0 = \mu u_0, \quad h_0 u_0 + (1 - h_0) w_0 = u_m.$$
 (6)

The velocity components and depth of perturbed flow deviate from the basis flow weakly. That is why these values take the form

$$h = h_0 + \tilde{h}, \quad u = u_0 + \tilde{u}, \quad w = w_0 + \tilde{w}.$$
 (7)

Here  $\tilde{h}$ ,  $\tilde{u}$ ,  $\tilde{w}$  are the functions of all independent variables.

Substitution of solution (7) into equations (5) and linearization of the system obtained give the equations for small perturbations

$$(u - \rho w)_t + \beta (u_0 u_x - \rho w_0 w_x) + h_x = w - \mu u,$$
  

$$h_t + u_0 h_x + h_0 u_x = 0, \quad h_0 u + (1 - h_0) w + (u_0 - w_0) h = 0.$$
(8)

The tildes are omitted. A solution of these equations is sought in the form

$$(h, u, w) = (\hat{h}, \hat{u}, \hat{w}) \exp(ik(x - ct)),$$
(9)

where c is the phase velocity, k > 0 is the wave number, i is the imaginary unit.



Figure 1. Imaginary parts of solutions of equation (10) at  $\beta = 1.2$ ,  $u_m = 0.3$ ,  $\mu = 0.2$  and  $\rho = 0.8$ . They correspond to stable flow at  $h_0 = 0.5$  (a) and unstable flow at  $h_0 = 0.05$  (b).

Substituting anzats (9) into equations (8) and eliminating  $\hat{u}$  and  $\hat{w}$  with the help of  $\hat{h}$  by the formulas

$$\hat{u} = -\frac{u_0 - c}{h_0}\hat{h}, \quad \hat{w} = \frac{w_0 - c}{1 - h_0}\hat{h}$$

we obtain the dispersion relation

$$c^{2} - \frac{(\beta+1)(1-(1-\rho\mu)h_{0})u_{0} - iu_{m}/(ku_{0})}{1-(1-\rho)h_{0}}c + \frac{(1-(1-\rho\mu^{2})h_{0})u_{0}^{2}\beta - (1-h_{0})h_{0} - iw_{0}/k}{1-(1-\rho)h_{0}} = 0.$$
(10)

This relation should be fulfilled for constructing a non-trivial solution of equations (8) in form (9). Here the velocities of the basic stream  $u_0$  and  $w_0$  depend on the depth  $h_0$  only with respect to conditions (6) by the formulas

$$w_0 = \mu u_0, \quad u_0 = \frac{u_m}{\mu + (1 - \mu)h_0}$$

According to classical linear stability theory [12] the flow under study is stable if Im c < 0(c = c(k) is the root of the dispersion relation). The curves Im c(k) are shown in Fig. 1. Here c = c(k) are the branches of solution of quadratic equation (10) at  $\beta = 1.2$ ,  $u_m = 0.3$ ,  $\mu = 0.2$ ,  $\rho = 0.8$  and different values of depth  $h_0$  of the lower layer in the basic flow ( $h_0 = 0.5$  and  $h_0 = 0.05$ ). It can be seen from Fig. 1 (a) that the basic flow is stable when the values of  $h_0$ are close to half of channel width. A development of instability is possible when one of the layers is thin enough at a fixed average velocity, density ratio and viscosity ratio. It is shown in Fig. 1 (b).

In the following sections we demonstrate that proposed linear analysis is in a good agreement with the Whitham conditions of roll waves existence and with numerical experiments.

## 4. Characteristics of equations (5). The Whitham conditions

Using the last equation of (5) we eliminate the velocity in the lower layer

$$w = \frac{u_m - uh}{1 - h}$$

and introduce the equations of motion in the form

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{F},$$

where  $\mathbf{U} = (u, h)^{\mathrm{T}}$  is the vector of unknowns,  $\mathbf{F} = (F, 0)^{\mathrm{T}}$  is the right part. The matrix of coefficients takes the form

$$\mathbf{A} = \left(\begin{array}{cc} M_1 & M_2 \\ h & u \end{array}\right).$$

Values  $M_1$ ,  $M_2$  and F are the following

$$M_{1} = \frac{((1-h)\beta - \rho h)u + (\beta + 1)\rho hw}{1 - (1-\rho)h}, \quad M_{2} = \frac{1 - h - (u - w)(u - \beta w)\rho}{1 - (1-\rho)h},$$
$$F = \frac{(w - \mu u)(1 - h)}{1 - (1-\rho)h}.$$

Eigenvalues  $\lambda^{\pm}$  of matrix **A** have the form

$$\lambda^{\pm} = \frac{1}{2} \Big( M_1 + u \pm \sqrt{(M_1 + u)^2 - 4(uM_1 - hM_2)} \Big)$$

and define the velocities of characteristics propagation  $(dx/dt = \lambda^{\pm})$ . It is necessary to note that equations (5) generate a mixed type system. When the liquid physical parameters are arbitrary, the characteristics  $\lambda^{\pm}$  can be complex. A development of long-wave instability is possible beyond of the interval of system hyperbolicity. The statement of Cauchy problem requires an additional analysis and justification [2].

Together with equations (5) we consider more simple kinematic model, in which the inertial effects are not taken into account. Then the first equation in (5) is replaced by the condition  $w = \mu u$ . So we suppose that the flow rate is constant and the velocity u can be performed with the help of the lower layer of depth h. Make allowance for all aforesaid, we write the kinematic model as one equation

$$\frac{\partial h}{\partial t} + \frac{\partial f(h)}{\partial x} = 0, \quad f(h) = \frac{u_m h}{\mu + (1 - \mu)h}.$$
(11)

The velocity of characteristic of model (11) is

$$\hat{\lambda} = f'(h) = \frac{\mu u_m}{(\mu + (1 - \mu)h)^2}.$$

With respect to the Whitham criterion [9], the roll waves exist in the intervals, in which one of inequalities is valid

$$\hat{\lambda} \ge \lambda^+ \quad \text{or} \quad \hat{\lambda} \le \lambda^-.$$
 (12)

In addition, the solution of equations (5) should not cross over the domain of hyperbolicity. Therefore, a correct statement of Cauchy problem requires not to pass beyond the intersection point of the characteristics  $\lambda^+$  and  $\lambda^-$ .

Characteristics  $\lambda^{\pm}$  and  $\hat{\lambda}$  for different average velocity  $u_m$ , density ratio  $\rho$  and viscosity ratio  $\mu$  are given in Fig. 2. For all cases  $\beta = 1.2$ . For constructing the dependencies  $\lambda = \lambda^{\pm}(h)$  we use the relation u = f(h)/h. Fig. 2 (a) shows that the Whitham criterion (12) is fulfilled for small values of h. Furthermore, the hyperbolicity domain covers the whole interval of depths  $h \in (0, 1)$ . It is necessary to note that the obtained interval for roll waves existence is in a good agreement with the linear stability analysis discussed in the previous section. We can see from both Fig. 1 (b) and Fig. 2 (a) that the instability arises at small depth of the lower layer.



Figure 2. Velocities of characteristics  $\lambda^{\pm}$  and  $\hat{\lambda}$  (curves 1, 2 and 3) depending on h at  $u_m = 0.3$ ,  $\rho = 0.8$ ,  $\mu = 0.2$  (a) and at  $u_m = 2$ ,  $\rho = 0.02$ ,  $\mu = 2$  (b).

The values for constructing velocities of characteristics in Fig. 2 (b) correspond to gas-liquid media. In this case the Whitham criterion  $\hat{\lambda} \geq \lambda^+$  is valid for big enough h. Moreover, the hyperbolicity conditions are kept in not the whole flow region. It requires a more detailed analysis of problem correctness. The study of physical parameters allows to conclude that density ratio affects the critical layer depth, while the viscosity ratio and given flow rate bring to a bare influence on the hyperbolicity conditions of the governing equations.

#### 5. Numerical modelling of the roll waves

Further we carry out a numerical analysis of small non-stationary perturbation development using nonlinear equations of two-layer flow (5). We rewrite equations (5) in conservative form for the calculations. We denote  $s = u - \rho w$  and express the velocities in the layers with the help of s and h:

$$u = \frac{1-h}{1-(1-\rho)h} \left(s + \frac{\rho u_m}{1-h}\right), \quad w = \frac{u_m - uh}{1-h}$$

Then, for the functions s and h we have the conservative form of equations

$$s_t + \left( (u^2 - \rho w^2)\beta/2 + h \right)_x = w - \mu u, \quad h_t + (uh)_x = 0.$$
(13)

Calculations corresponding to model (13) are carried out using MATLAB and TVD Nessyahy– Tadmor scheme of the second-order approximation.

We suppose that the channel length L = 200 and height H = 1. At the initial moment the channel is occupied by the two-layer liquid moving with the average velocity  $u_m = 0.3$ . We assume also that  $\rho = 0.8$ ,  $\mu = 0.2$  and  $\beta = 1.2$ . The depth of the lower layer is  $h_0 = 0.05$ . The initial velocities in the layers are found from (6) with the help of given  $h_0$ . They equal  $u_0 = 1.25$  and  $w_0 = 0.25$ . As mentioned above the emergence of the roll waves is possible in this interval of physical parameters with respect to the Whitham criterion. The soft boundary conditions  $\mathbf{u}_N = \mathbf{u}_{N-1}$  are to be valid at the right boundary of the computational domain. Here  $\mathbf{u}_j$  is the value of function in the nodal point  $x_j$ . Small perturbations  $h(t, 0) = h_0(1 + 0.05 \sin(\omega t))$  and  $s(t, 0) = u_0 - \rho w_0 = 1.05$  are given at the left boundary. A uniform grid with respect to the spatial variable x is used for the calculations, the number of nodes N = 4000. The time step is determined by the Courant condition. The results of calculations at t = 250 are shown in Fig. 3. It can be observed that the roll waves are generated in the considered intervals of parameters. The lengths of the waves depend on frequency of oscillations at the left boundary.



Figure 3. Interface of two-layer flow y = h at t = 250 for  $u_m = 0.3$ ,  $\rho = 0.8$ ,  $\mu = 0.2$  and  $h_0 = 0.05$ . There is small periodic interface perturbation with oscillation frequency  $\omega = 0.7$  (a) and  $\omega = 1$  (b) in the left boundary.

Note that the roll waves move against the flow in the regime under consideration. Thus the calculation using equations (13) demonstrates development of the interface perturbations and the roll waves formation if the Whitham criterion (12) is satisfied.

#### 6. Conclusion

The study is focused on the analysis of two-layer flow equations (5). The solution in the form of normal modes is constructed and dispersion relation (10) is treated for linearized system (8). Stability domains of the flow are identified depending on the depth of the lower layer at a given flow rate and known density ratio and viscosity ratio (Fig. 1). Characteristics for nonlinear system (5) are found. The Whitham conditions for roll waves existence are formulated (Fig. 2). It is shown that the results on linear stability analysis and the Whitham criterion are in a good agreement. A numerical modelling of the roll waves is carried out using TVD scheme of highorder accuracy (Fig. 3). The calculation of interface position demonstrates the emergence of the roll waves for physical parameters found by means of the theoretical analysis.

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