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# Free-surface potential flow of an ideal fluid due to a singular sink 

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#### Abstract

A two-dimensional problem of a potential free-surface flow of an ideal incompressible fluid caused by a singular sink is considered. The sink is placed at the horizontal bottom of the channel. By employing a conformal map, the problem is equivalently rewritten in the unit circle. After that, it is investigated by the Levi - Civita technique with the extraction of the singular part of the flow that corresponds to the sink. We derive a Nekrasov type equation that describes exactly the form of the free boundary. This equation is studied at first numerically and then by an exact mathematical technique. It is shown that for the Froude number greater than some particular value, there exists a unique solution of the problem such that the free surface decreases monotonically when moving from the infinity to the sink. At the point over the sink, the free surface has a cusp.


## 1. Introduction

In this paper, we study a two-dimensional steady problem of a potential free surface flow of an ideal incompressible fluid over a flat horizontal bottom. The flow is caused by a singular sink of the strength $m>0$ that is located at the bottom. We assume that at infinity the flow velocity tends to a constant $V$ that is determined by the strength of the sink and by the depth $h$ of the fluid layer. The problem has only one non-dimensional parameter, the Froude number $\operatorname{Fr}=V / \sqrt{g h}$, where $g$ is the gravity acceleration. Since the statement of the problem does not include the value of $V$, it would be more appropriate to define the Froude number in terms of the sink strength $m$. Due to the incompressibility of the fluid, $m=4 h V$ and we have

$$
\mathrm{Fr}^{2}=\frac{m^{2}}{16 g h^{3}}
$$

In this paper, for brevity, we use the parameter $\alpha=\operatorname{Fr}^{2} / 2$ instead of $\operatorname{Fr}$.
In the non-dimensional statement, the problem reads as follows. Let $(x, y)$ be rectangular Cartesian coordinate system with the origin at the point $O$, where the sink is located. The flow domain $D$ is then between the free boundary $\Gamma=\{(x, y) \mid y=\eta(x)\}$ and the bottom $\Gamma_{0}$ that coincides with the horizontal axis $O x$ (see Fig. 1). The problem is to find the function $\eta$ as well as the velocity vector field $\boldsymbol{v}=\left(v_{x}, v_{y}\right)$ and the pressure $p$ that satisfy in $D$ the stationary Euler equations of an ideal incompressible fluid. These functions satisfy also the following conditions:

$$
\eta(x) \rightarrow 1 \quad \text { and } \quad \boldsymbol{v}(x, y) \rightarrow(\mp 1,0) \quad \text { as } \quad x \rightarrow \pm \infty
$$

$$
\begin{gathered}
\Gamma_{0} \backslash\{O\}: \quad v_{y}=0, \\
\left(\boldsymbol{v}(x, y)+\frac{2}{\pi} \frac{\boldsymbol{r}}{|\boldsymbol{r}|^{2}}\right) \rightarrow 0 \quad \text { as } \quad|\boldsymbol{r}| \rightarrow 0, \quad \boldsymbol{r}=(x, y) \in D, \\
\Gamma: \quad \boldsymbol{v} \cdot \boldsymbol{n}=0, \quad p=0,
\end{gathered}
$$

where $\boldsymbol{n}$ is the normal vector to $\Gamma$. We assumed that the pressure is equal to zero at $\Gamma$ since the pressure is constant on the free boundary and the Euler equations include only the gradient of $p$. Notice that, as a consequence of the Euler equations and the conditions at the boundary $\Gamma$, the functions $\boldsymbol{v}$ and $\eta$ satisfy the Bernoulli law:

$$
\begin{equation*}
\alpha|\boldsymbol{v}(x, \eta(x))|^{2}+\eta(x)=\alpha+1 \quad \text { for } \quad x \in(-\infty, \infty) \tag{1}
\end{equation*}
$$



Figure 1. General flow pattern in the physical plane.

Due to the incompressibility of the fluid and the potentiality of the flow we can exclude the pressure from the statement of the problem. Namely, there exist defined in $D$ functions $\varphi$ and $\psi$ called the potential and the stream function respectively such that

$$
\begin{equation*}
v_{x}=\partial_{x} \varphi=\partial_{y} \psi \quad \text { and } \quad v_{y}=\partial_{y} \varphi=-\partial_{x} \psi \quad \text { for } \quad(x, y) \in D . \tag{2}
\end{equation*}
$$

These equations mean that $\varphi$ and $\psi$ are adjoint harmonic functions in $D$. Notice also that the potential and the stream function are defined up to an additive constant.

There are quite a large number of works in which this and similar problems are studied numerically and from a mechanical point of view. A lot of remarkable results are obtained and we are not able to mention all of them. We restrict ourselves to papers directly related to our study. It seems to us that there are two key properties of the flow under consideration. The first one consists in the fact that, for sufficiently large values of the Froude number, there are no waves going to infinity. The monotonicity of the free surface has already been obtained numerically in the first papers on the subject (see, e.g., [1-4]). It can be explained by the following reason. The wave velocity is equal to $\sqrt{g h}$ and, for large Froude numbers, this quantity is less than the velocity of the flow which is directed towards the sink. Notice that in the case of the source the waves do exist (see [5]). In general, the waves occur in the flows with small Froude numbers, i.e., in the so called subcritical case. This is not the subject of the present paper and we refer to the monograph [6], where an extensive bibliography on the question and the solution of various free surface flow problems can be found. In particular, the existence of waves for the flow around the point vortex was proved. We have another point singularity but the technique suggested there seems to be applicable also in our case.

The second and not so obvious property of the flow is that the free boundary has a cusp over the sink for the sufficiently large Froude number. This fact was discovered numerically in the above-cited works $[1-4]$. Here, we should notice that for small Froude numbers the stagnation
point can occur over the sink. At this point the velocity of the fluid is equal to zero. The presence of the stagnation point is typical for the problem with the sink at the sloping bottom (see $[7]$ and also $[1,2]$ ).

In this paper, we consider the case of the Froude number greater than some particular value. Thus, the free boundary possesses both these properties. The novelty of our work consists in the following. The papers cited above investigate numerically approximate equations describing the free surface. Usually they use a truncated power series. We derive a Nekrasov type equation that is exactly satisfied by the free boundary and then look for its numerical solutions. Moreover, we are able to proof with the mathematical rigor that this equation has a unique solution. This fact enables us to be sure that our numerical solution is close to the exact solution of the problem. The mathematical part of this work will be published later. In the present paper, we formulate the results only.

## 2. The statement of the problem in the unit semicircle

Since we consider a two-dimensional problem, it is convenient to make use of the complex variable approach. Together with the complex variable $z=x+i y$ we introduce the complex velocity $v=v_{x}-i v_{y}$ and the complex potential $w=\varphi+i \psi$. As it follows from (2), $w$ is a holomorphic function in $D$ and $v=d w / d z$. Let us denote by $F$ the conformal mapping of the flow domain $D$ onto the upper unit semicircle $D^{*}$. We denote by $G$ the inverse mapping and by $t$ the complex variable in $D^{*}$. The images of the point of the $\operatorname{sink} z=0$ and the points $A$ and $B$ at infinity are the points $t=0, t=1$ and $t=-1$ respectively (see Fig. 2). Besides that, $F(\Gamma)=\Gamma^{*}=\{t| | t \mid=1, \operatorname{Im} t>0\}$. The mapping $F$ as well as $G$ is unknown and should be found. However, we can easily determine the flow in the semicircle. It is not difficult to see that


Figure 2. The flow in the unit semicircle $D^{*}$.
we have a singular sink at the point $t=0$ and the same singular sources at $t= \pm 1$. Therefore, the complex potential $w^{*}$ in $D^{*}$ is given by the following expression:

$$
w^{*}(t)=\frac{2}{\pi} \log \left(\frac{t^{2}-1}{2 t}\right)-i=\frac{2}{\pi}(\log (t+1)+\log (t-1)-\log 2 t-i \pi / 2) .
$$

Notice, that $w(z)=w^{*}(F(z))$ and

$$
v(G(t))=\frac{d w^{*}}{d t}(t) \frac{d F}{d z}(G(t))=\frac{d w^{*}}{d t}(t)\left(\frac{d G(t)}{d t}\right)^{-1} .
$$

This implies that

$$
\begin{equation*}
\frac{d G(t)}{d t}=\frac{1}{v(G(t))} \frac{d w^{*}}{d t}(t) \tag{3}
\end{equation*}
$$

If we could determine the function $v(G(t))$, then the last equation would make it possible to determine $G$ and, as a consequence, to solve the problem. The function $v(G(t))$ is holomorphic in $D^{*}$ and we denote by $u(t)$ its holomorphic extension to the unit circle, which is possible due to the Schwarz reflection principle. Notice that $u(t)$ has a pole at the point $t=0$. Following the Levi-Civita approach, we introduce a holomorphic in the unit circle $\{t||t|<1\}$ function $\Omega(t)=\hat{\theta}(t)+i \hat{\tau}(t)$ with real $\hat{\theta}$ and $\hat{\tau}$ such that

$$
u(t)=-\frac{2}{\pi} \frac{1}{t} e^{-i \Omega(t)}
$$

The function $\hat{\theta}$ is defined up to an additive constant $2 \pi k$, where $k$ is an integer number. We make this function single-valued by fixing it at the point $t=1: \hat{\theta}(1)=0$. In order to find $\Omega$ in the unit circle, it is sufficient to know its boundary value at $|t|=1$, i.e., for $t=e^{i \sigma}, \sigma \in[0,2 \pi)$. Denote

$$
\tau(\sigma)=\hat{\tau}\left(e^{i \sigma}\right), \quad \theta(\sigma)=\hat{\theta}\left(e^{i \sigma}\right)
$$

These functions satisfy the following boundary conditions:

$$
\begin{equation*}
\theta(0)=\theta(\pi / 2)=\theta(\pi)=\theta(3 \pi / 2)=0 \quad \text { and } \quad \tau(0)=\tau(\pi)=\log \frac{\pi}{2} . \tag{4}
\end{equation*}
$$

Besides that, due to the symmetry of the problem, we have:

$$
\begin{gathered}
\tau(\pi / 2+\sigma)=\tau(\pi / 2-\sigma), \quad \theta(\pi / 2+\sigma)=-\theta(\pi / 2-\sigma), \\
\tau(\sigma)=\tau(-\sigma), \quad \theta(\sigma)=-\theta(-\sigma)
\end{gathered}
$$

for all $\sigma \in[0,2 \pi]$.
In order to obtain an equation for $\tau$ and $\theta$ on $\Gamma^{*}$, we employ the Bernoulli equation (1). The substitution $t=e^{i \sigma}$ and the differentiation with respect to $\sigma$ give:

$$
\alpha \frac{d\left|u\left(e^{i \sigma}\right)\right|^{2}}{d \sigma}+\operatorname{Im} \frac{d G\left(e^{i \sigma}\right)}{d \sigma}=0 \quad \text { for } \quad \sigma \in(0, \pi) .
$$

As it follows from (3),

$$
\begin{equation*}
\frac{d G\left(e^{i \sigma}\right)}{d \sigma}=-e^{-\tau(\sigma)} e^{i(\sigma+\theta(\sigma))} \cot \sigma \tag{5}
\end{equation*}
$$

After some calculation we obtain the Nekrasov type equation:

$$
\begin{equation*}
\tau^{\prime}(\sigma)=\frac{\sin (\sigma+\theta(\sigma)) \cot \sigma}{\alpha \pi+3 \int_{0}^{\sigma} \sin (s+\theta(s)) \cot s d s} \quad \text { for } \quad \sigma \in(0, \pi) \tag{6}
\end{equation*}
$$

This equation contains two unknown functions. To obtain one more equation, we can use the Hilbert transform. Taking into account the symmetry properties of the functions $\tau$ and $\theta$, we derive the following relation:

$$
\begin{equation*}
\theta(\sigma)=\frac{1}{\pi} \int_{0}^{\pi / 2} \tau(s)(\cot (s-\sigma)-\cot (s+\sigma)) d s, \quad \sigma \in[0, \pi / 2] . \tag{7}
\end{equation*}
$$

Equations (6) and (7) together with the boundary conditions (4) enable us to find the functions $\tau$ and $\theta$ and to solve the problem. However, it would be more convenient to rewrite (7) in another form. Since $\cot \sigma=(\log |\sin \sigma|)^{\prime}$, we have

$$
\begin{equation*}
\theta(\sigma)=\frac{1}{\pi} \int_{0}^{\pi / 2} K(s, \sigma) \tau^{\prime}(s) d s, \quad \sigma \in[0, \pi / 2], \tag{8}
\end{equation*}
$$

where

$$
K(s, \sigma)=\log \left|\frac{\sin (s+\sigma)}{\sin (s-\sigma)}\right|=\log \left|\frac{\tan s+\tan \sigma}{\tan s-\tan \sigma}\right|=2 \sum_{k=1}^{\infty} \frac{\sin 2 k s \sin 2 k \sigma}{k} .
$$

Notice that $K(s, \sigma) \geq 0$ for all $s$ and $\sigma$ in $[0, \pi / 2]$.

## 3. Numerical simulations

If we find the functions $\tau$ and $\theta$ from equations (6) and (8), we can determine the velocity vector field in $D$ and the free boundary $\Gamma$. In this section, we present the numerical calculations of $\Gamma$. This boundary can be defined in the parametric form:

$$
\Gamma=\left\{(x, y) \mid x=x_{\Gamma}(\sigma), y=y_{\Gamma}(\sigma), \sigma \in(0, \pi)\right\} .
$$

This definition in the complex variables is as follows: $\Gamma=\left\{z \in \mathbb{C} \mid z=G\left(e^{i \sigma}\right), \sigma \in(0, \pi)\right\}$. Due to (5) we have

$$
\begin{array}{ll}
\frac{d x_{\Gamma}(\sigma)}{d \sigma}=\operatorname{Re} \frac{d G\left(e^{i \sigma}\right)}{d \sigma}=-e^{-\tau(\sigma)} \cos (\sigma+\theta(\sigma)) \cot \sigma, & x_{\Gamma}(\pi / 2)=0 \\
\frac{d y_{\Gamma}(\sigma)}{d \sigma}=\operatorname{Im} \frac{d G\left(e^{i \sigma}\right)}{d \sigma}=-e^{-\tau(\sigma)} \sin (\sigma+\theta(\sigma)) \cot \sigma, & y_{\Gamma}(0)=1
\end{array}
$$

These equations, in particular, yield that

$$
\frac{d y_{\Gamma}(\sigma)}{d \sigma}\left(\frac{d x_{\Gamma}(\sigma)}{d \sigma}\right)^{-1}=\tan (\sigma+\theta(\sigma)) .
$$

It means that $\sigma+\theta(\sigma)$ is the slope angle of the tangent to $\Gamma$ at the point $\left(x_{\Gamma}(\sigma), y_{\Gamma}(\sigma)\right)$.
In order to find the solution of equations (6) and (8), we have used the method of successive approximations. We take $\theta \equiv 0$ as the first approximation and calculate until the difference between two successive approximations is greater than $10^{-6}$. In Fig. 3, the free boundary for


Figure 3. The free boundaries for two values of the Froude number.


Figure 4. The dependence of the distance between the sink and the cusp point at the free boundary on the Froude number.
$\alpha=0.9$ and $\alpha=100$ is shown. It should be noted that the iterative process diverges for sufficiently small $\alpha$, namely, when $\alpha$ is less than $1 / 2$ approximately. By this reason, the graph in Fig. 4 begins at a non-zero Froude number. This graph shows the dependence of the distance between the sink and the cusp point at the free boundary on the Froude number. As seen, this distance decreases very slowly starting from the value 5 of the Froude number.

## 4. Mathematical treatment of the problem

In this section, we formulate the mathematical results related to the solvability of the problem. These results will be proved in a separate publication. Notice that we have essentially used some ideas from [8] and especially from [9].

Let us rewrite equations (6) and (8) as an operator equation for the function $\zeta(\sigma)=3 \tau^{\prime}(\sigma)$ :

$$
\begin{equation*}
\zeta=\Phi(\zeta) \tag{9}
\end{equation*}
$$

where $\Phi$ is a nonlinear operator that acts as follows

$$
\begin{gathered}
\Phi(\zeta)(\sigma)=\frac{3}{\alpha \pi} \frac{\sin (\sigma+\theta(\sigma)) \cot \sigma}{\exp \int_{0}^{\sigma} \zeta d s}, \quad \sigma \in[0, \pi / 2] \\
\theta(\sigma)=\frac{1}{3}(H \zeta)(\sigma), \quad(H \zeta)(\sigma)=\frac{1}{\pi} \int_{0}^{\pi / 2} K(s, \sigma) \zeta(s) d s
\end{gathered}
$$

The kernel $K$ is defined after equation (8). We look for a solution of equation (9) in the Banach space $L^{2}(0, \pi / 2)=\{\zeta(\sigma) \mid\|\zeta\|<\infty\}$,

$$
\|\zeta\|^{2}=\int_{0}^{\pi / 2}|\zeta(\sigma)|^{2} d \sigma
$$

In order to prove the solvability of equation (9), we have used two classical fixed point theorems. The first one is the Schauder theorem. Let us take the following closed convex set in $L^{2}(0, \pi / 2)$ :

$$
B_{R}^{+}=\left\{\zeta \in L^{2}(0, \pi / 2) \mid\|\zeta\| \leqslant R, \zeta \geq 0\right\}
$$

It is not difficult to establish that $\Phi$ is continuous and compact as an operator in $L^{2}(0, \pi / 2)$. It remains only to prove the existence of $R>0$ such that $\Phi\left(B_{R}^{+}\right) \subset B_{R}^{+}$. By employing the Fourier analysis, we have found that this condition is satisfied if

$$
R \geq \frac{3}{2} \sqrt{\frac{\pi}{2}} \frac{1}{\alpha \pi-2}
$$

The positiveness of $\Phi$ follows from the positiveness of the kernel $K$. Thus, equation (9) is solvable if

$$
\begin{equation*}
\alpha>\frac{2}{\pi} \tag{10}
\end{equation*}
$$

This inequality in terms of the Froude number approximately looks as $\mathrm{Fr}>1.128$.
The Schauder fixed point theorem says nothing about the uniqueness of the solution. In order to establish this fact, we have to use other methods, for instance, the Banach contraction mapping principle. This principle gives not only the existence but also the uniqueness of the solution. We have proved that

$$
\left\|\Phi\left(\zeta_{1}\right)-\Phi\left(\zeta_{2}\right)\right\| \leq \frac{8}{\alpha \pi}\left\|\zeta_{1}-\zeta_{2}\right\|, \quad \text { for all } \quad \zeta_{1}, \zeta_{2} \in B_{R}^{+}
$$

where $R$ is the same as above. Thus, $\Phi$ is a contraction and the solution of $(9)$ is unique if

$$
\begin{equation*}
\alpha>\frac{8}{\pi} \tag{11}
\end{equation*}
$$

This inequality in terms of the Froude number approximately looks as $\mathrm{Fr}>2.257$.
We do not claim that (10) and (11) are the best possible conditions for the existence and the uniqueness. Surely, more precise estimates will improve these conditions. Notice that, in the numerical simulations, we were able to calculate the solution for smaller values of $\alpha$. Of course, it can also be explained by a not sufficient accurate approximation of the problem.

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