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## Model isothermal internal erosion of soil

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Abstract. The process of internal erosion in a three-phase saturated soil is studied. The problem is described by the equations of mass conservation, Darcy's law and the equation of capillary pressure. The original system of equations is reduced to a system of two equations for porosity and water saturation. In general, the equation of water saturation is degenerate. The degenerate problem in a one-dimensional domain and one special case of the problem in a two-dimensional domain are solved numerically using a finite-difference method. Existence and uniqueness of a classical solution of a nondegenerate problem is proved.

#### 1. Introduction

Various models of internal erosion are used for describing applied problems of the formation of cavities under dam reservoir, the destruction of the wellbore walls as a result of soil erosion, the suffusion craters formation on surface topography. Evaluation of suffusion removal is also a relevant problem in many environmental works. Advanced models of internal erosion are based on approaches of mechanics of multiphase media [1, 2, 3, 4]. These models are intended to describe more detailed picture of the movement of water and fluidized solid particles mixture in soil. Precisely, the determination of a velocity field, porosity, water saturation and pressure of each phase.

These models include phase transition and use filtration approximation. The basic equations are mass conservation law for each phase and Darcy's law for moving phases. This system of equations is similar in structure to the system of Masket-Leverett equations of two-phase filtration for immiscible fluids; for detailed mathematical theory description see [5, 6]. We did not find any study related to the justification problem of the system of equations with porosity to be determined, except a few special cases [7, 8].

The key points of the problem are the degeneracy of the equations of the obtained solution (relative phase permeability coefficients  $\overline{k}_{0i}$  can be equal to 0 if saturation of the *i*th phase  $s_i \leq 0$  in the equation (2)) and the unknown porosity of soil. The problem in this formulation is very difficult to study analytically, especially it is necessary to justify the physical principles of maximum for the porosity and the water saturation  $(0 \le \phi \le 1, 0 \le s_i \le 1)$ .

This approach is also used in the study of two-phase filtration, dissociation of hydrates in natural layers with iced parts, heat and mass transfer in freezing or melting soil.

#### 2. Formulation of the problem

We consider a mathematical model of internal erosion of soil in a finite domain  $Q \subset \mathbb{R}^n$  (where Q and n will be specified further). Saturated soil is a three-phase porous medium [1] consisting of water (i = 1, the first phase), fluidized solid particles (i = 2, the second phase) and solid

skeleton of soil (i = 3, the third phase). The process is described by the equations of mass conservation for each phase with phase transitions, generalized Darcy's law and the equation of capillary pressure for water and fluidized solid particles [2, 3, 4]:

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \vec{u}_i) = \sum_{j=1}^3 I_{ij}, \quad i = 1, 2, 3, \quad I_{ji} = -I_{ij}, \quad \sum_{i,j=1}^3 I_{ij} = 0;$$
(1)

$$\vec{v}_i = s_i \phi \vec{u}_i = -K_0(\phi) \frac{k_{0i}}{\mu_i} (\nabla p_i + \rho_i^0 \vec{g}), \ i = 1, 2; \quad p_2 - p_1 = p_c(s_1) \ge 0, \quad s_1 + s_2 = 1.$$
(2)

Here  $\vec{u}_i$  is a velocity of the *i*th phase ( $\vec{u}_3 = 0$ , the third phase is considered to be stationary),  $\rho_i$  is a reduced density related to a true density  $\rho_i^0$  and a volumetric concentration  $\alpha_i$  by the formula  $\rho_i = \alpha_i \rho_i^0$  (the condition  $\sum_{i=1}^3 \alpha_i = 1$  is a consequence of the definition of  $\rho_i$ ),  $I_{ji}$  is a rate of mass transfer from the *j*th to the *i*th phase per unit volume in unit of time,  $\vec{v}_i = \phi_{s_i} \vec{u}_i$ is a filtration velocity of the *i*th phase (i=1,2),  $\phi$  is porosity,  $s_1$  and  $s_2$  are water and fluidized solid particles saturations ( $\alpha_1 = \phi s_1, \alpha_2 = \phi s_2, \alpha_3 = 1 - \phi$ ),  $K_0$  is the filtration tensor;  $k_{0i}$  is a relative phase permeability ( $\bar{k}_{0i} = \bar{k}_{0i}(s_i) \ge 0, \bar{k}_{0i}|_{s_i=0} = 0$ ),  $\mu_i$  is a dynamic viscosity,  $p_i$  is a phase pressure (*i*=1,2);  $p_c$  is the capillary pressure,  $\vec{g}$  is the acceleration vector due to gravity.

The functions of the system (1), (2) satisfy the assumptions:  $\rho_i^0 = const$ ,  $\rho_2^0 = \rho_3^0$ ,  $I_{ij} = -I_{ji}$ . Moreover,  $I_{12} = I_{13} = 0$ ,  $I_{23} = \rho_3^0 I$ , I is a prescribed function of  $s_1, \phi, v_1, v_2$ .

If porosity is a known function, then system (1), (2) will read as the Masket — Leverett equations of two-phase filtration for immiscible fluids. We shall investigate the problem with the porosity as a function to be determined.

We shall prove the unique solvability of the nondegenerate problem (it means a(s) > 0 for any s) in a one-dimensional domain, check validity of physical maximum principles for the porosity and the water saturation in the same problem and solve numerically the one-dimensional degenerate problem and one special case of the problem in a two-dimensional domain.

In the one-dimensional domain the system (1), (2) reads [4, 9]

$$\frac{\partial s\phi}{\partial t} = \frac{\partial}{\partial x} (K_0(\phi)a(s)\frac{\partial s}{\partial x} - b(s)v(t) + F(s,\phi)), \tag{3}$$

$$\frac{\partial \phi}{\partial t} = I(s,\phi). \tag{4}$$

Here  $s \equiv s_1$ ,  $v(t) = v_1 + v_2$  is a mixture filtration velocity (prescribed function of time) [5]. The rate of mass transfer is given as  $I = \lambda \delta(s) R(\phi) \max\{|v(t)| - v_k, 0\}$  [2, 10], where  $\delta(s) = 0$  if  $s \ge 1$ ,  $\delta(s) = 1 - s$  if 0 < s < 1,  $\delta(s) = 1$  if  $s \le 0$ ;  $R(\phi) = 1$  if  $\phi \ge 1$ ,  $R(\phi) = \phi(1 - \phi)$  if  $0 < \phi < 1$ ,  $R(\phi) = 0$  if  $\phi \le 0$ ,  $\lambda > 0$  is a dimension constant [1/m],  $v_k$  is the critical velocity. The critical velocity indicates whether the erosion process has started or has not. If the velocity v reaches this critical speed, the theory suggests the erosion process will begin (the critical speed can be evaluated using experimental methods, see, e.g. [11]).

The coefficients of the system (3), (4) are given in following form

$$a(s) = -\frac{k_{01}k_{02}}{k}\frac{\partial p_c}{\partial s} > 0, \quad k_{0i}(s) = \frac{\overline{k}_{0i}}{\mu_i} \ge 0, \quad k(s) = k_{01} + k_{02} > 0,$$
$$F(s,\phi) = K_0(\phi)\frac{k_{01}k_{02}}{\mu_i}(\rho_1^0 - \rho_2^0)g, \quad b(s) = \frac{k_{01}}{\mu_i} \ge 0.$$

The functions s(x,t) and  $\phi(x,t)$  satisfy the initial and boundary conditions

$$s(0,t) = s_0(t), \quad s(1,t) = s_1(t), \quad s(x,0) = s^0(x), \quad \phi(x,0) = \phi^0(x).$$
 (5)

In further analysis we use notation of functional domains described in [12].

**Definition.** We call a pair of functions s(x,t) and  $\phi(x,t)$  as a classical solution of the problem (3)–(5), where  $s(x,t), \phi(x,t) \in C^{2+\alpha,(2+\alpha)/2}(Q_T), Q_T = Q \times (0,T), Q = (0,1)$  and satisfy the equations (3), (4) and the initial and boundary conditions (5) as continuous functions in  $Q_T$ .

**Theorem** We suppose that functions, coefficients and parameters of the problem (3)–(5) satisfy the following conditions:

1) the functions  $K_0(\phi), a(s), b(s), F(s, \phi), p_c(s)$  and their derivatives up to the second order are continuous in  $s \in [0, 1], \phi \in [0, 1]$  and satisfy the conditions

$$0 < m \le K_0(\phi), a(s) \le M < \infty, \quad |F(s,\phi)|, b(s) \le M, \quad p_c \ge 0, \quad \frac{\partial p_c}{\partial s} < 0,$$

where  $F(s, \phi) = 0$  if s < 0, s > 1;

2) the functions v(t),  $s_0(t)$ ,  $s_1(t)$ ,  $s^0(x)$ ,  $\phi^0(x)$  satisfy following smoothness conditions

 $v(t), s_0(t), s_1(t) \in C^{2+\alpha}[0, T]; \quad s^0(x), \phi^0(x) \in C^{2+\alpha}(\overline{Q})$ 

and the matching conditions

$$s_0(0) = s^0(0), \quad s_1(0) = s^0(1),$$

as well as the inequalities

 $|v(t)| \ge v_k, \quad 0 \le s^0(x) \le 1, \quad 0 < m_0 \le \phi^0(x) \le M_0 < 1,$  $0 \le s_0(t) \le 1, \quad 0 \le s_1(t) \le 1,$ 

where  $m_0$ ,  $M_0$ ,  $m_1$ ,  $M_1$  are given positive constants.

Then the problem (3)–(5) has only one classical solution for any finite interval (0,T]. Moreover,

$$0 \le s(x,t) \le 1, 0 < \phi(x,t) < 1, (x,t) \in Q_T.$$

We shall establish the proof of the theorem with the help of the Schauder fixed-point theorem [13] and standard auxiliary constructions [14].

**Lemma 1.** Let the assumptions of the theorem be satisfied and a pair of function  $(s, \phi)$  be the classical solution of the problem (3)–(5). Then  $\phi$  and s satisfy inequalities  $\phi_0 \leq \phi \leq 1$  and  $0 \leq s \leq 1$ .

Proof. The inequality for  $\phi$  is easily obtained from the fact that  $I(s, \phi)$  is nonnegative. In the purpose to obtain the inequality for s we introduce a cut-off function  $\overline{s} = max\{s - 1, 0\}$ . Multiplying the equation (3) by  $\overline{s}$  and integrating the result in  $Q_t$ , we derive the equality

$$\int_{0}^{t} \int_{0}^{1} s\phi_{\tau}\overline{s}dxd\tau + \int_{0}^{t} \int_{0}^{1} (\phi\overline{s})_{\tau}\overline{s}dxd\tau = \int_{0}^{t} \int_{0}^{1} (K_{0}a\overline{s}_{x} + F)\overline{s}_{x}dxd\tau - \int_{0}^{t} \int_{0}^{1} v(\tau)b_{s}\overline{s}_{x}\overline{s}dxd\tau.$$
(6)

Due to the definition of  $\overline{s}$ , all integrals in (6) should be taken in the domain  $Q_t^* = \{(x,t) \in Q_t, s > 1\}$ , where  $\overline{s} = s - 1$ ,  $s_t = \overline{s}_t, s_x = \overline{s}_x, \phi_t > 0$  and F = 0. Taking this into account in (6), we arrive at the following estimate

$$||\sqrt{\phi}\overline{s}||_{2,Q}^2 \le C \int_0^t ||\sqrt{\phi}\overline{s}||_{2,Q}^2 d\tau,$$

where C is a positive constant. Hence, it follows that  $\overline{s} \equiv 0$ . The inequality  $s \geq 0$  is proved in complete analogy using the same method with  $\overline{s} = max\{-s, 0\}$ . Lemma 1 is proved.

Following [4], we come to the Lemma 2.

**Lemma 2.** Let two pairs  $(s_i, \phi_i)$ , i = 1, 2, be two different classical solutions of the system (3)–(4) with initial and boundary conditions equal to  $s_i^0, \phi_i^0$  and  $s_0^{(i)}, s_1^{(i)}$  respectively. Let

$$||s_0^1 - s_0^2|| + ||s_1^1 - s_1^2|| + ||s_1^0 - s_2^0|| + ||\phi_1^0 - \phi_2^0|| = \delta.$$

Then we have the estimate

$$\int_{0}^{1} (s^{2} + \phi_{t}^{2} + \phi^{2}) dx \le C\delta, \quad \int_{0}^{t} \int_{0}^{1} (\phi_{x}^{2} + \phi_{xt}^{2} + s_{x}^{2}) dx d\tau \le C\delta,$$

where C is a constant depends only on  $m_0, m, M, v_k$ .

Using the substitution  $V = \int_{0}^{s} a(\tau) d\tau$  in (3) we arrive at the equation

$$V_t - \frac{K_0(\phi)}{\phi}a(s(V))V_{xx} = f,$$

$$f = -a(s(V))s(V)\frac{\phi_t}{\phi} + a\frac{1}{\phi}\frac{\partial K_0}{\partial \phi}\phi_x V_x + \frac{1}{\phi}\frac{\partial F}{\partial V}V_x + a\frac{1}{\phi}\frac{\partial F}{\partial \phi}\phi_x - a\frac{v(t)}{\phi}\frac{\partial b}{\partial V}V_x.$$

Following [5], we come to the Lemma 3.

**Lemma 3.** Let  $\phi$  and s be the classical solution of the problem (3)–(4), then we have the inequality

$$\int_{0}^{1} (V_x^2 + V_t^2) dx + \int_{0}^{t} \int_{0}^{1} (V_{xx}^2 + V_{xt}^2) dx dt \le C_1(m_0, m, M, v_k).$$

Proving of the Lemmas 1 and 2 is discussed in [4].

As a consequence of lemma 3 we have the estimate

$$(||s||_{Q_T}^{(\alpha)} + ||\phi||_{Q_T}^{(\alpha)}) \le C_2(m_0, m, M, v_k).$$

**Proof of the theorem.** Substituting any continuous function  $\tilde{s}$ , which satisfies the inequality  $|\tilde{s}| \leq M_1$  (where the constant  $M_1 > 0$  is any nonnegative constant and will be chosen further), into coefficients of the equation (4), we arrive at the problem for  $\tilde{\phi}$ 

$$\frac{\partial \widetilde{\phi}}{\partial t} = I(\widetilde{\phi}, \widetilde{s}), \quad \widetilde{\phi}|_{t=0} = \phi^0(x).$$
(7)

Considering that the problem (7) has the unique classical solution and following the Lemma 1 we come to the estimate  $\phi_0 \leq \tilde{\phi} \leq 1$ . Substituting  $\tilde{s}(x,t)$  and  $\tilde{\phi}(x,t)$  into coefficients of the equation (3), we arrive at the linear equation for s(x,t)

$$\frac{\partial s\widetilde{\phi}}{\partial t} = \frac{\partial}{\partial x} (K_0(\widetilde{\phi})a(\widetilde{s})\frac{\partial s}{\partial x} - b(\widetilde{s})v(t) + F(\widetilde{s},\widetilde{\phi})), \tag{8}$$

with the boundary and initial conditions described by the equation (5). The solvability of this problem is evidently established following the well-known results about the solvability of the

linear parabolic equations [12]. Moreover, following the proof of the Lemmas 1–3, we obtain, in particular, the estimate of the Holder constant

$$||s||_{Q_T}^{(\alpha)} \le C_3(m_0, m, M, v_k).$$
(9)

Taking  $M_1 = C_3$  due to the fact that  $M_1$  can be chosen freely, we shall consider compact mapping  $U: \tilde{s} \to s$ . From the inequality (9) it follows that this mapping in the domain  $C_{0,0}(Q)$ transforms the solid sphere  $\{|\tilde{s}| \leq M_1\}$  into itself. Besides, the mapping is continuous in the domain  $C_{0,0}(Q)$ . Moreover, the mapping U is a completely continuous mapping: any continuous function is transformed to a function from the class  $C^{\alpha,\alpha/2}(Q)$  satisfying the inequality (9). Consequently, all conditions of the Schauder fixed-point theorem are satisfied for our mapping U. Hence, there exist at least one fixed-point  $s(x,t) \equiv \tilde{s}(x,t)$  of the mapping U and the corresponding classical solution  $\phi(x,t) \equiv \tilde{\phi}(x,t)$  of the problem (7)–(8). It is evident that the pair  $(\phi(x,t)s(x,t))$  is the classical solution of the problem (3)–(5). The uniqueness is achieved with the help of the Lemma 2.

#### 3. Method of the numerical solution

The system of the equations (1)-(4) is solved numerically with the given initial and boundary conditions in the form (5) in a finite one-dimensional domain. Numerical study aims to investigate the behavior of the porosity and water saturation as functions of space variables and time depending on the given parameters of the problem.

The numerical solution of the problem (3)–(5) is obtained with the help of finite-difference method. Taking this into account, it is convenient to introduce a finite-difference grid with constant step  $x_i = ih$ , i = 0...N and  $t_n = n\tau$ , n = 0...T, h is a mesh size, N is the number of spatial steps,  $\tau$  is a time step, T is the number of time steps. Hence, the system (3), (4) reduced to the system of finite-difference equations

$$\begin{split} \phi_{i}^{n} \frac{s_{i}^{n+1} - s_{i}^{n}}{\tau} &= \frac{1}{h^{2}} (K_{0i+\frac{1}{2}}^{n} a_{i+\frac{1}{2}}^{n} (s_{i+1}^{n+1} - s_{i}^{n+1}) - K_{0i-\frac{1}{2}}^{n} a_{i-\frac{1}{2}}^{n} (s_{i}^{n+1} - s_{i-1}^{n+1})) + \\ &+ \left( \frac{\partial F}{\partial s} (s_{i}^{n}, \phi_{i}^{n}) - v(t_{n}) \frac{\partial b}{\partial s} (s_{i}^{n}) \right) \frac{s_{i+1}^{n+1} - s_{i-1}^{n+1}}{2h} - s_{i}^{n+1} I_{i}^{n}, \\ &a_{i+\frac{1}{2}}^{n} &= \frac{a(s_{i}^{n} + s_{i+1}^{n})}{2}, \quad a_{i-\frac{1}{2}}^{n} &= \frac{a(s_{i}^{n} + s_{i-1}^{n})}{2}, \\ &K_{0i+\frac{1}{2}}^{n} &= \frac{K_{0}(\phi_{i}^{n} + \phi_{i+1}^{n})}{2}, \quad K_{0i-\frac{1}{2}}^{n} &= \frac{K_{0}(\phi_{i}^{n} + \phi_{i-1}^{n})}{2}, \quad I_{i}^{n} = I(s_{i}^{n}, \phi_{i}^{n}). \end{split}$$

Here the index i refers to space and index n refers to time.

We shall find  $\phi$  in the form [15]

$$\phi = \frac{\phi^0}{\int_{0}^{t} \delta(s) \max\{|\vec{v}(\tau)| - v_k, 0\} d\tau},$$
(11)  

$$\phi^0 + (1 - \phi^0) e^{-\int_{0}^{t} \delta(s) \max\{|\vec{v}(\tau)| - v_k, 0\} d\tau},$$

which can be obtained from (4) with  $R(\phi) = (1 - \phi)\phi$ .

The numerical algorithm consists of several steps. First, we substitute the initial condition for the porosity  $\phi^0$  into the coefficients of the finite-difference scheme (10). Using sweep method in the scheme (10) we find the values of the saturation s. Then substituting found the values of the saturation into the given equation (11) for  $\phi$ , we find the values of the porosity  $\phi_i^1$  in each grid point. Finally, repeating the algorithm, we find the values of porosity and concentration at next time step and so on.

Numerical results

The porosity and the water saturation are calculated with  $(g = 0, v_k = 0, v(t) = 1, p_c(s) = (1/s^2 - 1), K_0(\phi) = \phi^3, k_{0i} = s_i^2$  if  $0 \le s \le 1, k_{0i} = 0$  if  $s_i \le 0, k_{0i} = 1$  if  $s_i \ge 1$ ). The initial and boundary conditions are  $s(x, 0) = (x0.5)/l + 0.5, s(0, t) = 0.5, s(l, t) = 1, \phi^0(x) = 0.3$ . At the points x = l there is no fluidized solid particles. The initial condition for s assumed to be a linear function. In our calculations we keep  $h = 0.01, \tau = 0.01, N = 100, T = 100, l = 1$ . The numerical results are shown in figure 1–2. It is shown that if saturation s reaches 1 the erosion process will stop and the porosity will be unchanged.



**Figure 1.** Porosity as a function of time and x.



Figure 2. Water saturation as a function of time and x.

#### 4. Two dimensional case of the problem

We shall describe one special case of the profile problem and solve it numerically in the purpose of finding 2D area affected by the erosion process.

We consider a two-dimensional movement of the water and fluidized solid particles with given concentration of the fluidized solid particles  $s_2 = c = const \in [0, 1]$  and with the equality

of both pressures  $p_1 = p_1 = p$ . The intensity of the phase transition is taken in the form  $I_{23} = \rho_3^0 \lambda \phi c (1 - \phi) max(|v_1| - v_k, 0).$ 

With such assumptions in the system (1)-(2) we arrive at the following system

$$div(a(c,\phi)(\nabla p + \rho_1^0 \vec{g})) = 0, \quad \frac{\partial \phi}{\partial t} = I,$$
(12)

where  $a(c, \phi) = (c\lambda + 1 - c)\phi k_f$  and all derivatives are taken with respect to two space variables. The flow region Q is a rectangular cross section with sides  $0 \le x \le 25, 0 \le y \le 1$ . The boundary of Q consists of 5 parts:  $\Gamma_1 = \{0 \le x \le 25, y = 0\}, \Gamma_2 = \{0 \le y \le 0.25, x = 0\} \cup \{0.29 \le y \le 1, x = 0\}, \Gamma_3 = \{0 \le x \le 25, y = 1\}, \Gamma_4 = \{0 \le y \le 1, x = 25\}, \Gamma_5 = \{0.25 \le y \le 0.29, x = 0\}.$ The velocity is satisfied to the conditions of impermeability on  $\Gamma_1, \Gamma_2, \Gamma_3$ . Deriving equations for p from these conditions with the help of the Darcy's law and adding conditions for p on  $\Gamma_4, \Gamma_5$ , we come to the boundary conditions for pressure p

$$\frac{\partial p}{\partial y}\Big|_{\Gamma_1} = -g\rho_1^0; \quad \frac{\partial p}{\partial x}\Big|_{\Gamma_2} = 0; \quad \frac{\partial p}{\partial y}\Big|_{\Gamma_3} = -g\rho_1^0; \quad p\Big|_{\Gamma_4} = -g\rho_1^0y + p_0; \quad p\Big|_{\Gamma_5} = -g\rho_1^0y.$$

Here  $p_0$  is a given pressure of the saturated soil in the point of intersection of  $\Gamma_1$  and  $\Gamma_4$ . The equation (12) is solved numerically using the similar method described in the section 1, but with respect to two space variables, i.e. at first, we solve the finite-difference equation for pressure p in x direction at a half time step n + 1/2, then we solve the equation for p in y direction at next time step n + 1. The equation (12) is investigated numerically for  $\phi^0 = 0, 25, g = 9.81 \text{ m/}c^2$ ,  $k_f = 10^{-2} \text{ m/s}$ . The critical speed  $v_k$  is 0.11 m/s. Numerical results are shown in figure 3. The area affected by the process of erosion is shown by the orange color.



Figure 3. The area affected by the process of erosion (orange).

## 5. Conclusion

The global solvability of the nondegenerate one dimensional problem is proved. The degenerate one dimensional problem and one special case of the problem in a two dimensional domain with given concentration of the fluidized solid particles are solved numerically by the finite-difference method. The area affected by the process of erosion is founded.

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