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On the phase shift in the Kuzmak–Whitham ansatz for nonlinear waves

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Abstract. We consider one-phase (formal) asymptotic solutions in the Kuzmak–Whitham form for evolutionary nonlinear equations. In this case, the leading asymptotic expansion term has the form $X(S(x, t)/\varepsilon + \Phi(x, t), A(x, t), x, t) + O(\varepsilon)$, where ε is a small parameter and the phase $S(x, t)$ and slowly changing parameters $A(x, t)$ are to be found from the system of averaged Whitham equations. The equation for the phase shift $\Phi(x, t)$ is appearing by studying the *second-order* correction to the leading term. The corresponding procedure for finding the phase shift is then nonuniform with respect to the transition to a linear (and weakly nonlinear) case. We formulate the general conjecture (checked for some examples), which essentially follows from papers by R.Haberman and collaborators, is that if one incorporates the phase shift $\Phi(x, t)$ into the phase and adjust A by setting $S \rightarrow \tilde{S}(x, t, \varepsilon) = S + \varepsilon\Phi + O(h^2)$, $A \rightarrow \tilde{A}(x, t, \varepsilon) = A + \varepsilon a + O(\varepsilon^2)$, then the new functions $\tilde{S}(x, t, \varepsilon)$ and $\tilde{A}(x, t, \varepsilon)$ become solutions of the Cauchy problem for the *same* Whitham system but with modified initial conditions. These functions completely determine the leading asymptotic term in the Whitham method.

1. The Kuzmak–Whitham ansatz

The Kuzmak–Whitham ansatz describes a class of special (formal) asymptotic solutions for a broad class of nonlinear evolution equations, including equations with variable coefficients. Recall some ideas leading to these asymptotics. The plane waves $u = A \cos(\omega t + k \cdot x + \Phi)$ are important solutions of linear evolution equations (or systems of equations). Here ω is the frequency, $k = (k_1, \dots, k_n)$ is the wave vector, A is the amplitude (which is a vector in the vector case A), and the number Φ is a phase correction. The counterpart of this solution in the nonlinear case has the form

$$u = X(\omega t + k \cdot x + \Phi, A), \quad (1)$$

where $X(\theta, A)$ is a smooth 2π -periodic function (or vector function) of the scalar variable θ depending on the parameter(s) A . Usually the existence of a solution of this type implies that ω , k , and the parameters A (which are a nonlinear analog of the amplitude) satisfy some relation of the form $\omega = \Omega(k, A)$, which is known as the *dispersion relation*. In the linear case, Ω is independent of A . In many important physical examples, the functions $X(\theta, A)$ can be expressed via elliptic functions, e.g., the function cn , and the functions (1) are known as *cnoidal waves* in this case.

The exact solution (1) can be generalized in the following way. One can assume that the parameters ω, k, A are smooth functions slowly depending on t and $x = (x_1, \dots, x_n)$. We characterize this slow dependence by a small parameter ε and set $\omega = \omega(\varepsilon t, \varepsilon x)$, $k = k(\varepsilon t, \varepsilon x)$,



$A = A(\varepsilon t, \varepsilon x)$, and $\Phi = \Phi(\varepsilon t, \varepsilon x)$. It is convenient to use a general phase function $S(\varepsilon t, \varepsilon x)$ instead of $\omega(\varepsilon t, \varepsilon x)t + k(\varepsilon t, \varepsilon x) \cdot x$, introduce the variables $t' = \varepsilon t$ and $x' = \varepsilon x$, and set $u = Y(\frac{S(t', x')}{\varepsilon} + \Phi(t', x'), A(t', x'))$. It is also convenient in what follows to omit the prime on the new variables, include small corrections related to the small parameter ε in the solution, and finally write

$$u = Y\left(\frac{S(t, x)}{\varepsilon} + \Phi(t, x), t, x, \varepsilon\right), \quad (2)$$

$$Y = X(\theta, A(t, x), t, x) + \varepsilon Y^1(\theta, t, x) + \varepsilon^2 Y^2(\theta, t, x) + \dots, \quad \theta = \frac{S(t, x)}{\varepsilon} + \Phi(t, x). \quad (3)$$

Here the functions X and Y^j are smooth in all arguments and 2π -periodic in θ . The right-hand side of (2) is called the *Kuzmak–Whitham ansatz*. It describes asymptotic solutions of many important nonlinear problems in mathematical and theoretical physics and appears first in Kuzmak's paper [1] for ordinary differential equation (in this case, the variables x disappear) and in Whitham's paper [2] in the case of partial differential equations. Let us briefly recall the meaning of this ansatz.

2. The Whitham equations

Formally, one can represent the partial differential equation under study in the form

$$\mathcal{L}(t, x, Q(\varepsilon \frac{\partial}{\partial t})u, R(\varepsilon \nabla)u, \varepsilon) = 0, \quad (4)$$

where L is a vector nonlinear (pseudo)differential operator acting on the vector function u , and $Q(\varepsilon \frac{\partial}{\partial t})$ and $R(\varepsilon \nabla)$ are some (pseudo)differential operators acting on the vector function u and giving some new vector function. The main assumption about the last equation is that the ordinary (pseudo)differential equation with fixed (frozen) variables (t, x) ,

$$\mathcal{L}(t, x, Q(\omega(t, x))\frac{\partial}{\partial \theta}X, R(k(t, x))\frac{\partial}{\partial \theta}X, 0) = 0, \quad \omega(t, x) = \frac{\partial S}{\partial t}, \quad k(t, x) = \nabla S, \quad (5)$$

possesses a family of smooth solutions $X(\theta, A, t, x)$ 2π -periodic in θ and also depending on a scalar or vector parameter A . Just as in the case of constant coefficients, the existence of 2π -periodic solutions implies a dispersion relation, which may include dependence on the slow variables (t, x) ,

$$\omega = \Omega(k, A, t, x). \quad (6)$$

Actually, formula (2) gives an asymptotic series, and the function

$$u_0 = X\left(\frac{S(t, x)}{\varepsilon} + \Phi(t, x), A(t, x), t, x\right)$$

is its leading term. As a rule, one can efficiently describe only the leading term X and probably the first correction. *To construct the leading term, one should find the function X , the equations for the phase $S(x, t)$, the parameter(s) A , and the phase shift Φ .*

A scheme for constructing these objects was suggested in [3]. As we mentioned above, to find the function X and the dispersion equation (6), one has to solve equation (5). To derive the equation(s) for the function(s) $A(x)$, one has to consider the equation for the first correction Y^1 , which is a linear inhomogeneous ordinary (pseudo)differential equation for Y^1 with respect to the variable θ varying on the circle $\mathbf{R} \bmod 2\pi$,

$$L(\omega, k, t, x, X, \frac{\partial}{\partial \theta})Y^1 = F_1(X, \omega, k, A, \Phi, t, x), \quad (7)$$

where F is a smooth function depending on X , its θ -derivatives, ω , k , A , Φ , and their derivatives with respect to t and x . Note that determination of F_1 involves two terms of the expansion. The effective variable in this equation is θ , and the variables t, x serve as parameters. The requirement of the existence of 2π -periodic solutions Y^1 of this equation is equivalent to the orthogonality conditions

$$\int_0^{2\pi} F_1(X, \omega, k, A, \Phi, t, x) \bar{Z}(\theta, t, x) d\theta, \quad (8)$$

where Z runs over the solutions of the equation $L^*Z = 0$ with the operator $L^*(\omega, k, t, x, X, \frac{\partial}{\partial\theta})$ adjoint to the operator L . Usually the dimension of the kernel of the operator L^* (as well as of the operator L) is not less than the number m of parameters $A = (A_1, \dots, A_m)$, and the orthogonality conditions, together with the dispersion relation, give a system of equations determining $A_0(x, t) = S(t, x)$, $A_1(t, x), \dots, A_m(t, x)$,

$$A_t = G(\nabla A_0, \dots, \nabla A_m, t, x), \quad (9)$$

where G is a smooth vector function with $m+1$ components, $G = (\Omega(k(t, x), A, t, x), G_1, \dots, G_m)$. Note that the choice of the functions (parameters) A (including A_0) is not unique and is very important: a good choice of the parametrization of the function X may strongly simplify the form of (9). We refer to (9) as the *Whitham equations*. This type of systems originally appeared in the paper [2] by Whitham, who used averaging of conservation laws and variational principle for their derivation. The equivalence of this approach to the orthogonality conditions was shown in [4]. Now it becomes clear that the Whitham are very important objects in mathematical physics, they appear in many important problems and have very interesting properties. For instance they inherit the integrability properties of original system including the case of multi-phase asymptotics (see [5, 6, 7]).

3. Example: The nonlinear Klein-Gordon equation

In this case,

$$\varepsilon^2(u_{tt} - c^2(t, x)\Delta u) + V_u(u, t, x) = 0, \quad V = \frac{1}{2}q^2(t, x)u^2 + \lambda r(t, x)u^4 \quad (10)$$

Here $c(t, x)$, $q(t, x)$, $r(t, x)$ are smooth positive functions, and λ is a parameter. The Whitham equations for the phase S and the scalar function A are (see the specific forms of $X(\theta, A, t, x)$ and $\Omega(A, t, x)$ in [10])

$$S_t^2 - c^2(x, t)(\nabla S)^2 = \Omega(A, t, x), \quad \frac{\partial}{\partial t} \left[\frac{A}{\Omega(A, t, x)} S_t \right] - c^2(x, t) \left\langle \nabla, \frac{A}{\Omega(A, t, x)} \nabla S \right\rangle = 0,$$

4. The phase shift

The Whitham equations do not completely determine the leading term u_0 of the asymptotic solution (2); to this end, one needs to construct an equation for the phase shift $\Phi(t, x)$. But the derivation of the equation for Φ is not the same for the weakly and strongly nonlinear cases; moreover, even the order of this equation with respect to time t is not the same in these two cases. In the weakly nonlinear case (and also in the linear case, parameter $\lambda = \varepsilon$ in example (10)), Eq. (5) is linear, the frequency Ω in the dispersion relation (6) is independent of A , the operators L and L^* in (7) are independent of X , and the dimension of the kernels of the operators L and L^* is equal to $m+1$. Thus, in this case the orthogonality conditions (8) give an equation of the first order in t for the phase shift $\Phi(t, x)$,

$$\Phi_t = \tilde{\Omega}(\nabla\theta, \nabla A_0, \dots, \nabla A_m, t, x), \quad (11)$$

where $\tilde{\Omega}$ is a smooth function of all of its arguments and is linear in θ . In the strongly nonlinear case ($\lambda = 1$ in example (10)), the situation is dramatically different in that the dimension of kernels of L and L^* is equal to m (the number of components of A), and the equation for the phase shift Φ follows from the analysis of the equation for *the second correction* Y^2 . Thus, it includes some characteristics of the first correction Y^1 , and its derivation is much more complicated than that of the Whitham equations. A method for this derivation was presented in [4]. Unfortunately, the final equation for the phase shift Φ in [4] includes an arithmetical mistake, which has been corrected in [8]. In contrast to the weakly nonlinear case, the phase shift now satisfies an equation of the *second order in time* t , which has a form

$$\Phi_{tt} = \mathcal{K}(\Phi_t, \nabla\Phi, A, t, x), \quad (12)$$

where $\mathcal{K}$ is a smooth function of all of its arguments. Thus, we see that the Kuzmak–Whitham type asymptotic solution (2) is not uniform with respect to passage from the weakly nonlinear case to the strongly nonlinear case. Moreover, there exists a “topological” obstruction to this passage: the dimension of the kernel of the adjoint variational operator L^* experiences a jump. Note that the concept of “weak nonlinearity” and “strong nonlinearity” is more or less artificial; the original equation becomes weakly nonlinear for a sufficiently small solution. Thus, the obstruction does not permit us to construct an asymptotic solution decaying outside some compact set describing the wave train propagation. Indeed, the head and tail of this solution are in the framework of the weakly nonlinear situation, and the “main part” is in the framework of the strongly nonlinear situation. Also, there exists an intermediate case, where one should match these two cases which also gives a lot of troubles.

5. “Phase Shift Principle” for the Kuzmak–Whitham ansatz

The beautiful observation made in the paper [8] is that the equation for the phase shift for the nonlinear Klein–Gordon equation can be represented in the form of a system of equations for the vector $(\Phi, \delta A_1, \dots, \delta A_n)$, and this system is none other than the linearization of the Whitham equations (9). The same fact is true for the Korteweg–de Vries equation. Using this fact, we can include the phase shift $\Phi(t, x)$ in the phase S and the corrections $\varepsilon a_j(t, x)$ in the functions A_j , consider these functions $\tilde{S}(t, x, \varepsilon)$ and $\tilde{A}(t, x, \varepsilon)$ as solutions of *the same Whitham equation* (9), and substitute these functions into the Kuzmak–Whitham ansatz

$$\tilde{S}(t, x, \varepsilon) = S(t, x) + \varepsilon\Phi(t, x), \quad \tilde{A}(t, x, \varepsilon) = A(t, x) + \varepsilon a(t, x), \quad (13)$$

$$u = X\left(\frac{\tilde{S}(t, x)}{\varepsilon}, \tilde{A}(t, x, \varepsilon), t, x\right) + \varepsilon\tilde{Y}^1\left(\frac{\tilde{S}(t, x)}{\varepsilon}, t, x\right) + \varepsilon^2\tilde{Y}^2\left(\frac{\tilde{S}(t, x)}{\varepsilon}, t, x\right) + \dots \quad (14)$$

The reconstruction (3), (14) of the Kuzmak–Whitham ansatz is now uniform with respect to the passage from weak nonlinearity to strong nonlinearity and does not need any new objects to construct the leading term of the rapidly oscillating solution.

Now we can state the main result (conjecture) of this paper. We only have a proof of this conjecture for the nonlinear Klein–Gordon equation and the Korteweg–de Vries equation (see [9, 10] and also [11]); thus, we state it in the form of

Phase Shift Principle for the Kuzmak–Whitham Ansatz:

Assume that one constructs a (formal) asymptotic Kuzmak–Whitham solution (2), (3) of some evolution equation (4) with functions $S(t, x)$, $\Phi(t, x)$, and $A(t, x)$ satisfying the Whitham equations (9) and the corresponding equation for the phase shift. Then there exist functions $\tilde{S}(t, x, \varepsilon) = S(t, x) + \varepsilon\Phi(t, x) + O(\varepsilon^2)$ and $\tilde{A}(t, x, \varepsilon) = A(t, x) + O(\varepsilon)$ satisfying the same Whitham equations (9) such that the asymptotic expansions (2)–(3) and (13)–(14) define the same function u .

Recall that Eqs. (13)–(14) describe some special class of asymptotic solutions. To define them, one should add certain additional conditions, for example, add the initial data $A|_{t=0} = A^0(x)$ for the Whitham equations (9), which fix the leading term X of the asymptotics for $t = 0$. From the viewpoint of the original equations (4), one should represent the function $X|_{t=0}$ in the prescribed form, but one should also choose the first correction $Y^1|_{t=0}$ in the prescribed form. (The corresponding formulas for the nonlinear Klein–Gordon and Korteweg–de Vries equations can be found in [9, 10].) The reconstruction (13)–(14) of the Kuzmak–Whitham ansatz is similar to Poincaré–Bogolyubov ideas in the averaging method concerning the construction of asymptotic solutions in the new dependent variables, which are changed at each step of the asymptotic procedure (see [12],[13]). Finally, note that the representation (13)–(14) shows that, owing to the presence of the phase shift $\Phi(t, x)$, the corrections $O(\varepsilon)$ to the initial data in the form of the Kuzmak–Whitham ansatz can strongly change the asymptotic solution in bounded time $t = O(1)$ (see Fig.1). This does not mean that the Kuzmak–Whitham-type solution is not stable, but this only means that, to achieve stability, the change in the initial data should be of the order of $O(h^{1+\kappa})$, $\kappa > 0$. This effect is well known as “orbital stability” in celestial mechanics [14]. One can find a more comprehensive discussion in [9, 10].

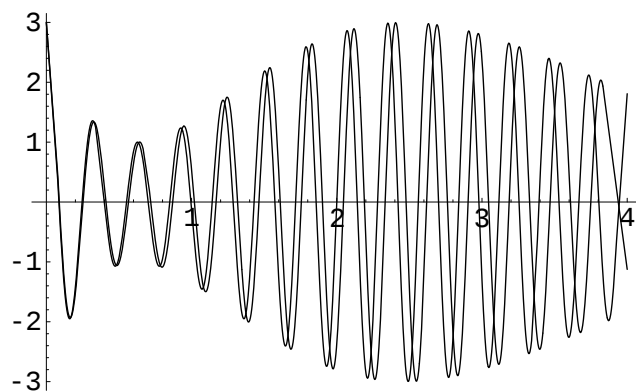


Figure 1. The influence of the phase shift on the asymptotics describing the wave train propagation

6. Some open problems

I think that it is of interest to check (or prove) the *Phase Shift Principle* for other equations where the Kuzmak–Whitham ansatz applies. These include

1. Rapidly oscillating asymptotic solutions of the Navier–Stokes equations constructed in [15].
2. The integrable partial differential equations considered in [16].
3. The Benjamin–Ono equation, which is actually a pseudodifferential equation [17].

At last, it seems that the *Phase Shift Principle* should also work for the asymptotics of soliton-like solutions, and it is of interest to check it for the solutions of the equations considered in [18, 19].

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