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Digraph-Based Algorithm for Finding Minimal Positive Realisation of Two-Dimensional Linear System with Delays

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Digraph-Based Algorithm for Finding Minimal Positive Realisation of Two-Dimensional Linear System with Delays

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Abstract. In this paper, the new method of the positive minimal realisation two-dimensional linear system with delays described by general model using multi-dimensional digraphs theory $\mathfrak{D}^{(n)}$ has been presented. For the proposed method, a digraph-based algorithm was constructed. The algorithm is based on a parallel computing method to gain needed speed and computational power for such a solution. The proposed solution allows minimal digraphs construction for any positive two-dimensional linear system with delays. The proposed method was discussed and illustrated with some numerical examples.

1. Introduction

The most popular models of two-dimensional linear system have been introduced by Roesser [1], Fornasini and Marchesini [2], [3] and Kurek [4]. In recent years, many researchers have been interested in positive linear systems. In this type of the system, state variables and outputs take only non-negative values. Analysis of positive one-dimensional (1D) systems is more difficult than of standard systems. Examples of positive systems include industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. An overview of the state of the art in the positive systems theory is given in [5], [6] and [7].

The realisation problem is a very difficult task. In many research studies, we can find the canonical form of the system, i.e. constant matrix form, which satisfies the system described by the transfer function. With the use of this form, we are able to write only one realisation (or some by the transformation matrices) of the system, while there exist many sets of matrices which fit into the system transfer function. The realisation problem for positive discrete-time systems without and with delays was considered in [6], [8], [9], [10], while in [11], [12] a solution for finding a set of possible realisations of the characteristic polynomial was proposed, that allows for finding many sets of matrices. In paper [13], [14], [15], [16], [17] the proposed method for finding a minimal positive realisations is an extension of the method for finding realisation of the characteristic polynomial. The optimisation of the proposed algorithm is presented in the paper [18] and [19].

The digraphs theory was applied to the analysis of dynamical systems. The use of the multi-dimensional theory was proposed for the first time in the paper [20], [21], [22] for analysis of positive two-dimensional systems.

In this paper, a new method of determination of a positive minimal realisation for the two-dimensional linear system with delay will be proposed, and the procedure for computation of the minimal realisation will be given. The proposed method is based on multi-dimensional digraphs theory, and it will be illustrated with a numerical example.

This work has been organised as follows: Section 2 presents some notations, defines a positive two-dimensional linear system with delays as the state-space representation, presents basic definitions of the digraph theory and introduces the concept of a multi-dimensional digraph and formulates the problem statement. In Section 3, we construct an algorithm for the determination of a positive minimal realisation of the linear two-dimensional system with delays described by the general model. Finally, we demonstrate the working of the algorithm on a numerical example in Section 4, and at the end we present some concluding remarks, open problems and bibliography positions.

2. Preliminaries and Problem Formulation

2.1. Notion

In this paper, the following notion will be used. The matrices will be denoted by a bold font (for example $\mathbf{A}$, $\mathbf{B}$, ...), the sets by the double line (for example $\mathbb{A}$, $\mathbb{B}$, ...), lower/upper indices and polynomial coefficients will be written using a small font (for example a, b, ...) and the digraph will be denoted using a mathfrak font $\mathfrak{D}$. The set $n \times m$ real matrices will be denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. If $\mathbf{G} = [g_{ij}]$ is a matrix, we write $\mathbf{G} \gg 0$ (matrix $\mathbf{G}$ is called strictly positive), if $g_{ij} > 0$ for all i, j; $\mathbf{G} > 0$ (matrix $\mathbf{G}$ is called positive), if $g_{ij} > 0$ for all i, j; $\mathbf{G} \geqslant 0$ (matrix $\mathbf{G}$ is called non-negative), if $g_{ij} \geqslant 0$ for all i, j. The set of $n \times m$ real matrices with non-negative entries wll be denoted by $\mathbb{R}_+^{n\times m}$ and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$. The $n \times n$ identity matrix will be denoted by $\mathbf{I}_n$. For more information about matrix theory, an interested reader is referred, for instance, to: [23], [24].

2.2. Positive Two-Dimensional Linear System with Delays

Let be given the model described by the equations:

$$x_{i+1,j+1} = \sum_{k_1=0}^{q_1}\sum_{l_1=0}^{q_2}\Big(\mathbf{A}^0_{k_1l_1}x_{i-k_1,j-l_1} + \mathbf{A}^1_{k_1l_1}x_{i-k_1+1,j-l_1} + \mathbf{A}^2_{k_1l_1}x_{i-k_1,j-l_1+1}\Big) + \\ + \sum_{k_2=0}^{p_1}\sum_{l_2=0}^{p_2}\Big(\mathbf{B}^0_{k_2l_2}u_{i-k_2,j-l_2} + \mathbf{B}^1_{k_2l_2}u_{i-k_2+1,j-l_2} + \mathbf{B}^2_{k_2l_2}u_{i-k_2,j-l_2+1}\Big), \tag{1a}$$

$$y_{ij} = \mathbf{C}x_{ij} + \mathbf{D}u_{ij}, \qquad i, j \in \mathbb{Z}_+ = \{0, 1, \ldots\}. \tag{1b}$$

The model described by the equations (1a)–(1b) is called the two-dimensional general model with delays, where $x_{ij} \in \mathbb{R}^n$, $u_{ij} \in \mathbb{R}^m$, $y_{ij} \in \mathbb{R}^p$ are state, input and output vectors respectively and $\mathbf{A}^t_{k_1l_1} \in \mathbb{R}^{n\times n}$, $\mathbf{B}^t_{k_2l_2} \in \mathbb{R}^{n\times m}$, $t = 0, 1, 2$; $k_1 = 0, 1, \ldots, q_1$, $l_1 = 0, 1, \ldots, q_2$,$k_2 = 0, 1, \ldots, p_1$, $l_2 = 0, 1, \ldots, p_2$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times m}$.

Boundary conditions for model described by the equations (1a)–(1b) have the form

$$x_{i-k_1,-l_1},\;\; x_{-k_1,j-l_1},\;\; k_1 = 0, 1, \ldots, q_1;\;\; l_1 = 0, 1, \ldots, q_2;\;\; i, j \in \mathbb{Z}_+. \tag{2}$$

The model described by the equations (1a)–(1b) is called a two-dimensional linear system with (q_1, q_2)–delays in state vector and (p_1, p_2)–delays in input vector.

Using to (1a) $\mathcal{Z}$–transform with zero boundary conditions (2) and taking into account that $Y(z_1, z_2) = \mathbf{C}X(z_1, z_2) + \mathbf{D}U(z_1, z_2)$ we can determine the transfer matrix in the following form:

$$\begin{aligned}\mathbf{T}(z_1, z_2) = \quad & \mathbf{C}\left[\mathbf{I}z_1z_2 - \sum_{k_1=0}^{q_1}\sum_{l_1=0}^{q_2} z_1^{-k_1}z_2^{-l_1}\left(\mathbf{A}_{k_1l_1}^0 + \mathbf{A}_{k_1l_1}^1 z_1 + \mathbf{A}_{k_1l_1}^2 z_2\right)\right]^{-1} \times \\ & \left[\sum_{k_2=0}^{p_1}\sum_{l_2=0}^{p_2} z_1^{-k_2}z_2^{-l_2}\left(\mathbf{B}_{k_2l_2}^0 + \mathbf{B}_{k_2l_2}^1 z_1 + \mathbf{B}_{k_2l_2}^2 z_2\right)\right] + \mathbf{D} = \frac{\mathbf{N}(z_1, z_2)}{d(z_1, z_2)},\end{aligned} \tag{3}$$

where

$$\mathbf{N}(z_1, z_2) = \begin{bmatrix} n_{11}(z_1, z_2) & \dots & n_{1m}(z_1, z_2) \\ \vdots & \ddots & \vdots \\ n_{p1}(z_1, z_2) & \dots & n_{pm}(z_1, z_2) \end{bmatrix},$$

$$\begin{aligned}n_{ij}(z_1, z_2) = \quad & \sum_{\substack{k=0 \\ k+l \le N_1+N_2}}^{N_1}\sum_{l=0}^{N_2} n_{kl}^{ij} z_1^k z_2^l = n_{N_1,N_2}^{ij} z_1^{N_1} z_2^{N_2} + n_{N_1,N_2-1}^{ij} z_1^{N_1} z_2^{N_2-1} + \\ & + n_{N_1-1,N_2}^{ij} z_1^{N_1-1} z_2^{N_2} + \ldots + n_{10}^{ij} z_1 + n_{01}^{ij} z_2 + n_{00}^{ij},\end{aligned} \tag{4}$$

$$\begin{aligned}d(z_1, z_2) = \quad & z_1^{nq_1} z_2^{nq_2} - \sum_{\substack{k=0 \\ k+l < n(q_1+q_2)}}^{nq_1}\sum_{l=0}^{nq_2} d_{kl} z_1^k z_2^l = z_1^{nq_1} z_2^{nq_2} - d_{nq_1,nq_2-1} z_1^{nq_1} z_2^{nq_2-1} - \\ & - d_{nq_1-1,nq_2} z_1^{nq_1-1} z_2^{nq_2} - \ldots - d_{10} z_1 - d_{01} z_2 - d_{00}.\end{aligned} \tag{5}$$

Definition 1 *The system described by the equations* (1a)–(1b) *is called internally positive if and only if* $x(i, j) \in \mathbb{R}_+^n$ *and* $y(i, j) \in \mathbb{R}_+^p$, $i, j \in \mathbb{Z}_+$ *for any boundary conditions* $x_{i-k_1,-l_1} \in \mathbb{R}_+^n$, $x_{-k_1,j-l_1} \in \mathbb{R}_+^n$, $k_1 = 0, 1, \ldots, q_1$, $l_1 = 0, 1, \ldots, q_2$, $i, j \in \mathbb{Z}_+$ *and all input sequence* $u(i, j) \in \mathbb{R}_+^m$, $i = -p_1, -p_1 + 1, \ldots;\ j = -p_2, -p_2 + 1, \ldots$.

Theorem 1 *The two-dimensional linear system with delay described by the equations* (1a)–(1b) *is positive if and only if*

$$\begin{aligned}&\mathbf{A}_{k_1l_1}^k \in \mathbb{R}_+^{n\times n},\ \ k_1 = 0, 1, \cdots, q_1;\ \ l_1 = 0, 1, \cdots, q_2;\ k = 0, 1, 2 \\ &\mathbf{B}_{k_2l_2}^k \in \mathbb{R}_+^{n\times m},\ \ k_2 = 0, 1, \cdots, p_1;\ \ l_2 = 0, 1, \cdots, p_2;\ k = 0, 1, 2, \\ &\mathbf{C} \in \mathbb{R}_+^{p\times n},\ \ \mathbf{D} \in \mathbb{R}_+^{p\times m}.\end{aligned} \tag{6}$$

The proof of the Theorem 1 is given in [25].

Matrices (6) are called positive realisations of the transfer matrix if they satisfy the equality (3). The realisation is called minimal if the dimension of the state matrix $\mathbf{A}_{k_1l_1}^k$, $k_1 = 0, 1, \ldots, q_1$; $l_1 = 1, 2, \ldots, q_2$; $k = 0, 1, 2$ is minimal among all possible realisations of $\mathbf{T}(z_1, z_2)$. In [26] the basic definitions of the positive realisation problem have been presented in detail.

2.3. Multi-Dimensional Digraphs

A directed graph (or just digraph) $\mathfrak{D}$ consists of a non-empty finite set $\mathbb{V}(\mathfrak{D})$ of elements called vertices and a finite set $\mathbb{A}(\mathfrak{D})$ of ordered pairs of distinct vertices called arcs [27]. We call $\mathbb{V}(\mathfrak{D})$ the vertex set and $\mathbb{A}(\mathfrak{D})$ the arc set of $\mathfrak{D}$. We will often write $\mathfrak{D} = (\mathbb{V}, \mathbb{A})$ which means that $\mathbb{V}$ and $\mathbb{A}$ are the vertex set and arc set of $\mathfrak{D}$, respectively. The order of $\mathfrak{D}$ is the number of vertices in $\mathfrak{D}$. The size of $\mathfrak{D}$ is the number of arc in $\mathfrak{D}$. For an arc (v_1, v_2) the first vertex v_1 is its tail and the second vertex v_2 is its head.

A two-dimensional digraph $\mathfrak{D}^{(2)}$ is a directed graph with two types of arcs and input flows. For the first time, this type of digraph was presented in papers [20], [21] and [22]. When we generalise this approach, we can define multi-dimensional digraphs $\mathfrak{D}^{(n)}$ in the following form.

Definition 2 *A n-dimensional digraph $\mathfrak{D}^{(n)}$ is a directed graph with q types of arcs and input flows. In detail, it is $(\mathbb{S}, \mathbb{V}, \mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_q, \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_q)$, where $\mathbb{S} = \{s_1, s_2, \dots, s_m\}$ is the set of sources, $\mathbb{V} = \{v_1, v_2, \dots, v_n\}$ is the set of vertices, $\mathfrak{A}_1$, $\mathfrak{A}_2$, ..., $\mathfrak{A}_q$ are the subsets of $\mathbb{V} \times \mathbb{V}$ whose elements are called $\mathfrak{A}_1$-arcs, $\mathfrak{A}_2$-arcs, ..., $\mathfrak{A}_q$-arcs respectively, $\mathfrak{B}_1$, $\mathfrak{B}_2$, ..., $\mathfrak{B}_q$ are the subsets of $\mathbb{S} \times \mathbb{V}$ whose elements are called $\mathfrak{B}_1$-arcs, $\mathfrak{B}_2$-arcs, ..., $\mathfrak{B}_q$-arcs respectively.*

When we consider multi-dimensional digraphs $\mathfrak{D}^{(n)}$, we write arc in the following form: $(v_i, v_j)_{\mathfrak{A}_q}$ or $(v_i, v_j)_{\mathfrak{B}_q}$, where the first vertex v_i is its tail, the second vertex v_j is its head and $\mathfrak{A}_q$ or $\mathfrak{B}_q$ corresponding to the matrix $\mathbf{A}_q$ and $\mathbf{B}_q$ respectively.

Remark 1 *$\mathfrak{A}_q$ -arcs and $\mathfrak{B}_q$ -arcs, are drawn by the other colour or line style. In this paper, $\mathfrak{A}_1$ -arc and $\mathfrak{B}_1$-arc is drawn by the solid line, $\mathfrak{A}_2$-arc and $\mathfrak{B}_2$ -arc is drawn by the dashed line and $\mathfrak{A}_3$ -arc and $\mathfrak{B}_3$ -arc is drawn by dotted line.*

Example 1 *For the system described by the matrices*

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{A}_1}, \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}_2}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A}_3}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{B}_1}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{B}_3} \tag{7}$$

we can draw three-dimensional digraphs $\mathfrak{D}^{(3)}$ consisting of vertices v_1, v_2, v_3 and source s_1, s_2. A three-dimensional digraph corresponding to system (7) is presented in Figure 1.

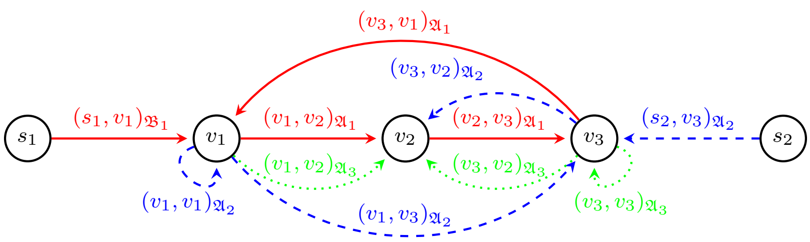

Figure 1. Three-dimensional digraph $\mathfrak{D}^{(3)}$ corresponding to matrices (7)

We present below some basic notions from the graph theory which are used in farther considerations [27], [28], [29], [30]. A walk in a multi-dimensional digraph $\mathfrak{D}^{(n)}$ is a nite sequence of arcs in which every two vertices v_i and v_j are adjacent or identical. A walk in which all of the arcs are distinct is called a path. The path, that goes through all vertices, is called a nite path. If the initial and the terminal vertices of the path are the same, then the path is called a cycle.

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2.4. Problem Formulation

For the given transfer matrix (3), determine a minimal positive realisation of the system (1a)–(1b) using the multi-dimensional $\mathfrak{D}^{(n)}$ digraphs theory. The dimension of the system must be the minimal among possible.

3. Problem Solution

The essence of the proposed method for determining a minimal realisation for a positive linear two-dimensional general model with delays described by the equations (1a)–(1b) will be presented in single-input single-output (SISO) systems. The transfer matrix (3) can be written in the following form:

$$\begin{aligned}\mathbf{T}(z_1,z_2) = & \quad \mathbf{C}\left[\mathbf{I}z_1z_2 - \sum_{\substack{k_1=-1\\k_1+l_1>-2}}^{q_1}\sum_{l_1=-1}^{q_2}\mathbf{A}_{k_1l_1}z_1^{-k_1}z_2^{-l_1}\right]^{-1}\left[\sum_{\substack{k_2=-1\\k_2+l_2>-2}}^{p_1}\sum_{l_2=-1}^{p_2}\mathbf{B}_{k_2l_2}z_1^{-k_2}z_2^{-l_2}\right]+\mathbf{D}= \\ = & \quad \frac{\mathbf{N}(z_1,z_2)}{d(z_1,z_2)}, \end{aligned} \tag{8}$$

where

$$\begin{aligned}&\mathbf{A}_{-1,0}=\mathbf{A}^1_{00},\ \ \mathbf{A}_{0,-1}=\mathbf{A}^2_{00},\ \ \mathbf{A}_{-1,1}=\mathbf{A}^1_{01},\ \ \mathbf{A}_{1,-1}=\mathbf{A}^2_{10},\\ &\mathbf{A}_{k_1l_1}=\mathbf{A}^0_{k_1l_1}+\mathbf{A}^1_{k_1+1,l_1}+\mathbf{A}^2_{k_1,l_1+1},\ \ k_1=0,1,\dots,q_1-1;\ l_1=0,1,\dots,q_2-1,\\ &\mathbf{A}_{q_1-1,q_2}=\mathbf{A}^0_{q_1-1,q_2}+\mathbf{A}^1_{q_1,q_2},\ \ \mathbf{A}_{q_1,q_2-1}=\mathbf{A}^0_{q_1,q_2-1}+\mathbf{A}^2_{q_1,q_2},\ \mathbf{A}_{q_1,q_2}=\mathbf{A}^0_{q_1,q_2},\\ &\mathbf{B}_{-1,0}=\mathbf{B}^1_{00},\ \ \mathbf{B}_{0,-1}=\mathbf{B}^2_{00},\ \ \mathbf{B}_{-1,1}=\mathbf{B}^1_{01},\ \ \mathbf{B}_{1,-1}=\mathbf{B}^2_{10},\\ &\mathbf{B}_{k_2l_2}=\mathbf{B}^0_{k_2l_2}+\mathbf{B}^1_{k_2+1,l_2}+\mathbf{B}^2_{k_2,l_2+1},\ \ k_2=0,1,\dots,q_1-1;\ l_2=0,1,\dots,q_2-1,\\ &\mathbf{B}_{q_1-1,q_2}=\mathbf{B}^0_{q_1-1,q_2}+\mathbf{B}^1_{q_1,q_2},\ \ \mathbf{B}_{q_1,q_2-1}=\mathbf{B}^0_{q_1,q_2-1}+\mathbf{B}^2_{q_1,q_2},\ \mathbf{B}_{q_1,q_2}=\mathbf{B}^0_{q_1,q_2}.\end{aligned} \tag{9}$$

From (8) we have

$$\mathbf{D}=\lim_{z_1\to\infty,z_2\to\infty}\mathbf{T}(z_1,z_2)=\begin{bmatrix} n^{ij}_{N_1,N_2} & \cdots & n^{1m}_{N_1,N_2}\\ \vdots & \ddots & \vdots \\ n^{p1}_{N_1,N_2} & \cdots & n^{pm}_{N_1,N_2}\end{bmatrix}, \tag{10}$$

since $\lim_{z_1\to\infty,\ z_2\to\infty}[\mathbf{I}z_1z_2-\sum_{k_1=-1}^{q_1}\sum_{l_1=-1}^{q_2}z_1^{-k_1}z_2^{-l_1}\mathbf{A}_{k_1l_1}z_1^{-k_1}z_2^{-l_1}]^{-1}=0$. The strictly proper transfer function is given by the equation

$$\mathbf{T}_{sp}(z_1,z_2)=\mathbf{T}(z_1,z_2)-\mathbf{D}=\frac{\widetilde{\mathbf{N}}(z_1,z_2)}{d(z_1,z_2)}=\frac{1}{d(z_1,z_2)}\begin{bmatrix}\widetilde{n}_{11}(z_1,z_2) & \dots & \widetilde{n}_{1m}(z_1,z_2)\\ \vdots & \ddots & \vdots\\ \widetilde{n}_{p1}(z_1,z_2) & \dots & \widetilde{n}_{pm}(z_1,z_2)\end{bmatrix}, \tag{11}$$

where

$$\begin{aligned}\widetilde{n}_{ij}(z_1,z_2) = & \quad \sum_{\substack{k=0\\k+l<N_1+N_2}}^{N_1}\sum_{l=0}^{N_2}\widetilde{n}^{ij}_{kl}z_1^kz_2^l = \\ = & \quad \widetilde{n}^{ij}_{N_1,N_2-1}z_1^{N_1}z_2^{N_2-1}+\widetilde{n}^{ij}_{N_1-1,N_2}z_1^{N_1-1}z_2^{N_2}+\ldots+\widetilde{n}^{ij}_{10}z_1+\widetilde{n}^{ij}_{01}z_2+\widetilde{n}^{ij}_{00},\end{aligned} \tag{12}$$

and a characteristic polynomial is described by the equation (5).

After multiplying the nominator and denominator of the transfer function (3) by $z_1^{-nq_1} z_2^{-nq_2}$, we obtain:

$$\begin{aligned} \widetilde{n}_{ij}(z_1, z_2) = \quad & \widetilde{n}^{ij}_{N_1,N_2-1} z_1^{N_1-nq_1} z_2^{N_2-nq_2-1} + \widetilde{n}^{ij}_{N_1-1,N_2} z_1^{N_1-nq_1-1} z_2^{N_2-nq_2} + \dots \\ & \dots + \widetilde{n}^{ij}_{10} z^{1-nq_1} z_2^{-nq_2} + \widetilde{n}^{ij}_{01} z_1^{-nq_1} z_2^{1-nq_2} + \widetilde{n}^{ij}_{00} z_1^{-nq_1} z_2^{-nq_2}, \end{aligned} \tag{13}$$

$$\begin{aligned} d(z_1^{-1}, z_2^{-1}) = \quad & 1 - d_{nq_1,nq_2-1} z_2^{-1} - d_{nq_1-1,nq_2} z_1^{-1} - \dots \\ & \dots - d_{10} z_1^{1-nq_1} z_2^{-nq_2} - d_{01} z_1^{-nq_1} z_2^{1-nq_2} - d_{00} z_1^{-nq_1} z_2^{-nq_2}. \end{aligned} \tag{14}$$

In the first step, we must find matrices $\mathbf{A}_{k_1 l_1} \in \mathbb{R}_+^{n\times n}$, $k_1 = -1, 0, 1, \cdots, q_1$; $l_1 = -1, 0, 1, \cdots, q_2$; using a decomposition characteristic polynomial (14). We decompose the polynomial into a set of simple monomials

$$d(z_1, z_2) = (1 - d_{M_1}(z_1, z_2)) \cup (1 - d_{M_2}(z_1, z_2)) \cup \cdots \cup \big(1 - d_{M_p}(z_1, z_2)\big) \tag{15}$$

where p is a number of simple monomials in the characteristic polynomial (14). For each simple monomial, we create digraphs representations. Then we can determine all possible characteristic polynomial realisations using all combinations of the digraphs monomial representations. Finally, we combine received digraphs in one digraph which is corresponding to a characteristic polynomial $d(z_1, z_2)$.

Theorem 2 *There exists positive matrices* $\mathbf{A}_{k_1 l_1} \in \mathbb{R}_+^{n\times n}$*,* $k_1 = -1, 0, 1, \cdots, q_1$*;* $l_1 = -1, 0, 1, \cdots, q_2$ *of the linear two-dimensional positive system with delays described by the equations* (1a)–(1b) *corresponding to the characteristic polynomial* $d(z_1, z_2)$ *if*

(C1) *the coefficients of the polynomial* $d(z_1, z_2)$ *are non-negative.*

(C2) *the sets* $\mathbb{D}_{M_1} \cap \mathbb{D}_{M_2} \cap \cdots \cap \mathbb{D}_{M_p}$*, where* p *is a number of simple monomials in polynomial* $d(z_1, z_2)$ *corresponding to multi-dimensional digraphs are not disjoint.*

(C3) *the obtained multi-dimensional digraph does not have additional cycles.*

Proof: *Condition (C1):* This condition came from an internal positivity of the system (Definition 1) and must be satisfied if we consider positive systems. If coefficients of the characteristic polynomial are negative, then in state matrices negative elements appear. The proof is given in paper [25]. *Condition (C2):* The sets $\mathbb{D}_{M_1} \cap \mathbb{D}_{M_2} \cap \cdots \cap \mathbb{D}_{M_p}$, where p is number of simple monomials in polynomial $d(z_1, z_2)$ are disjoint if $\mathbb{D}_{M_1} \cap \mathbb{D}_{M_2} \cap \cdots \cap \mathbb{D}_{M_p} = \emptyset$ then we have a digraph whose vertices can be divided into two disjoint sets. It means that we obtain an additional simple monomial in a characteristic polynomial $d(z_1, z_2)$. In this situation, we obtain a new characteristic polynomial $\overline{d}(z_1, z_2)$, $d(z_1, z_2) \neq \overline{d}(z_1, z_2)$. *Condition (C3):* Each monomial is represented by one cycle. If after combining all digraphs (each corresponding to one monomial) we obtain an additional cycle, this means that in the polynomial an additional simple monomial appears. □

Using the Theorem 2 we can construct the Algorithm 1.

Let us assume that the matrix $\mathbf{B}_{k_2 l_2}$, $k_2 = -1, 0, 1, \cdots, p_1$, $l_2 = -1, 0, 1, \cdots, p_2$ and matrix $\mathbf{C}$ have the following form:

$$\mathbf{B}_{k_2 l_2}|_{k_2=-1,0,1,\cdots p_1\ l_2=-1,0,1,\cdots p_2;} = \begin{bmatrix} b_{11}^{k_2 l_2} & b_{12}^{k_2 l_2} & \dots & b_{1,m}^{k_2 l_2} \\ b_{21}^{k_2 l_2} & b_{22}^{k_2 l_2} & \dots & b_{2,m}^{k_2 l_2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1}^{k_2 l_2} & b_{n,2}^{k_2 l_2} & \dots & b_{n,m}^{k_2 l_2} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1,n} \\ c_{21} & c_{22} & \dots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p,1} & c_{p,2} & \dots & c_{p,n} \end{bmatrix}. \tag{16}$$

After determining the state matrix $\mathbf{A}_{k_1 l_1} \in \mathbb{R}_+^{n\times n}$, $k_1 = -1, 0, 1, \cdots, q_1$; $l_1 = -1, 0, 1, \cdots, q_2$ from the Algorithm 1 and inserting matrices (16) to the equation (8) we obtain the polynomial $\overline{n}_{ij}(z_1, z_2)$. After comparing variables with the same power of z_1 and z_2 polynomials $\widetilde{n}_{ij}(z_1, z_2) = \overline{n}_{ij}(z_1, z_2)$ we receive the set of equations. After solving the equation and inserting to (9), we obtain matrices (6).

Algorithm 1 $DetermineMinimalRealisation()$

Determine matrix $\mathbf{D}$ using (10);
Determine strictly proper transfer function/matrix using (11);
if $\mathbf{D} => 0$ **then**
  $monomial = 1$;
  Determine number of $cycles$ in characteristic polynomial (14);
  for $monomial = 1$ **to** $cycles$ **do**
    Determine multi-dimensional digraph $\mathfrak{D}^{(n)}$ for all monomial;
    $MonomialRealisation(monomial)$;
  end for
  for $monomial = 1$ **to** $cycles$ **do**
    Determine digraph as a combination of the digraph monomial representation;
    $PolynomialRealisation(monomial)$;
    if $PolynomialRealisation\ != \ cycles$ **then**
      Digraph contains additional cycles or digraph contains disjoint union;
      BREAK
    else if $PolynomialRealisation\ == \ cycles$ **then**
      Digraph satisfies characteristic polynomial;
      Determine weights of the arcs in digraph;
      Write state matrix $\mathbf{A}_{k_1 l_1}|_{k_1=-1,0,1,\cdots,q_1;\ l_1=-1,0,1,\cdots,q_2}$;
      return $(PolynomialRealisation,\ \mathbf{A}_{k_1 l_1}|_{k_1=-1,0,1,\cdots,q_1,\ l_1=-1,0,1,\cdots,q_2})$;
    end if
  end for
  for $PolynomialRealisation = 1$ **to** j **do**
    Input - state matrix $\mathbf{A}_{k_1 l_1}|_{k_1=-1,0,1,\cdots,q_1;\ l_1=-1,0,1,\cdots,q_2}$;
    Determine polynomial $\widetilde{n}(z_1, z_2)$;
    Compare variables with the same power of the $n(z_1, z_2)$;
    Solve non-linear set of the equations;
    if Matrix $\mathbf{B}_{k_2 l_2}|_{k_2=-1,0,1,\cdots p_1\ l_2=-1,0,1,\cdots p_2} \geqslant 0$ AND matrix $\mathbf{C} \geqslant 0$ **then**
      return $(MinimalRealisation,\ \mathbf{A}, \mathbf{B}, \mathbf{C})$;
    else if Matrix $\mathbf{B}_{k_2 l_2}|_{k_2=-1,0,1,\cdots p_1\ l_2=-1,0,1,\cdots p_2} < 0$ OR matrix $\mathbf{C} < 0$ **then**
      Realisation is not positive;
      BREAK
    end if
  end for
  BREAK
else if $\mathbf{D} < 0$ **then**
  Realisation is not positive;
  BREAK
end if

4. Numerical Example

For the given transfer function

$$T(z_1, z_2) = \tag{17}$$
$$= \frac{2z_1^3 z_2^6 + 5z_1^3 z_2^5 + z_1^3 z_2^4 + 4z_1^2 z_2^6 + 4z_1^2 z_2^5 + z_1^2 z_2^4 + 3z_1^2 z_2^3 + 3z_1 z_2^6 + z_1 z_2^3 + 2z_1 z_2^2 + 2z_1 + 2}{z_1^3 z_2^6 - 2z_1^2 z_2^5 - z_1^2 z_2^3 - z_1 z_2^2 - z_1 z_2 - 1},$$

find all possible realisations of the two-dimensional positive linear system with $(q_1 = 1, q_2 = 1)$ delays in state vector and $(p_1 = 1, p_2 = 0)$ delays in input vector using multi-dimensional digraph theory.

Solution. In the first step, after using formula (10) and transfer function (17), we compute

$$\mathbf{D} = \lim_{z_1\to\infty,\, z_2\to\infty} T(z_1, z_2) = 2 \tag{18}$$

and a strictly proper transfer function (using (11) and (18)):

$$\begin{aligned} T_{sp}(z_1, z_2) &= T(z_1, z_2) - \mathbf{D} = \frac{n(z_1, z_2) - 2 \cdot d(z_1, z_2)}{d(z_1, z_2)} = \\ &= \frac{5z_1^3 z_2^5 + 4z_1^2 z_2^6 + 3z_1 z_2^6 + z_1^3 z_2^4 + z_1^2 z_2^4 + z_1^2 z_2^3 + z_1 z_2^3}{z_1^3 z_2^6 - 2z_1^2 z_2^5 - z_1^2 z_2^3 - z_1 z_2^2 - z_1 z_2 - 1} = \frac{\widetilde{n}(z_1, z_2)}{d(z_1, z_2)}. \end{aligned} \tag{19}$$

In the next step, after using (8), we write the transfer function in the following form:

$$\begin{aligned} \mathbf{T}_{sp}(z_1, z_2) = \; & \mathbf{C}\left[\mathbf{I} - \mathbf{A}_{-1,0} z_2^{-1} - \mathbf{A}_{-1,1} z_2^{-2} - \mathbf{A}_{0,-1} z_1^{-1} - \mathbf{A}_{0,0} z_1^{-1} z_2^{-1} - \mathbf{A}_{0,1} z_1^{-1} z_2^{-2} - \right. \\ & \left. \mathbf{A}_{1,-1} z_1^{-2} - \mathbf{A}_{1,0} z_1^{-2} z_2^{-1} - \mathbf{A}_{1,1} z_1^{-2} z_2^{-2}\right] \times \\ & \times \left(\mathbf{B}_{-1,0} z_2^{-1} + \mathbf{B}_{0,-1} z_1^{-1} + \mathbf{B}_{0,0} z_1^{-1} z_2^{-1} + \mathbf{B}_{1,-1} z_1^{-2} + \mathbf{B}_{1,0} z_1^{-2} z_2^{-1}\right) + \mathbf{D} \end{aligned} \tag{20}$$

After using (20), we determine possible weights from which we will build digraphs:

$$z_1^{-1}, \quad z_1^{-2}, \quad z_2^{-1}, \quad z_2^{-2}, \quad z_1^{-1} z_2^{-1}, \quad z_1^{-1} z_2^{-2}, \quad z_1^{-2} z_2^{-1}, \quad z_1^{-2} z_2^{-2}. \tag{21}$$

After multiplying the nominator and denominator of the strictly proper transfer function (19) by $z_1^{-3} z_2^{-6}$, we obtain:

$$T_{sp}(z_1, z_2) = \frac{5z_2^{-1} + 4z_1^{-1} + 3z_1^{-2} + z_2^{-2} + z_1^{-1} z_2^{-2} + z_1^{-1} z_2^{-3} + z_1^{-2} z_2^{-3}}{1 - 2z_1^{-1} z_2^{-1} - z_1^{-1} z_2^{-3} - z_1^{-2} z_2^{-4} - z_1^{-2} z_2^{-5} - z_1^{-3} z_2^{-6}} = \frac{\widetilde{n}(z_1, z_2)}{d(z_1, z_2)}, \tag{22}$$

where $d(z_1, z_2)$ is the characteristic polynomial. The proposed method finds state matrices $\mathbf{A}_{k_1 l_1}$, $k_1 = -1, 0, 1$; $l_1 = -1, 0, 1$ using the decomposing characteristic polynomial (15).

Using (21) and characteristic polynomial $d(z_1, z_2)$, we write initial conditions:

- number of vertices in digraph: $vertices = 3$;
- number of colours and line style in digraph: $colour = 4$, $line\_style = 4$;
- possible weights from which we will build digraphs: $(v_i, v_j)_{\mathfrak{A}_{-1,0}} z_2^{-1}$ (blue line), $(v_i, v_j)_{\mathfrak{A}_{-1,1}} z_2^{-2}$ (blue dashed line), $(v_i, v_j)_{\mathfrak{A}_{0,-1}} z_1^{-1}$ (red line), $(v_i, v_j)_{\mathfrak{A}_{0,0}} z_1^{-1} z_2^{-1}$ (black line), $(v_i, v_j)_{\mathfrak{A}_{0,1}} z_1^{-1} z_2^{-2}$ (blue dotted line), $(v_i, v_j)_{\mathfrak{A}_{1,-1}} z_1^{-2}$ (red dashed line), $(v_i, v_j)_{\mathfrak{A}_{1,0}} z_1^{-2} z_2^{-1}$ (red dotted line), $(v_i, v_j)_{\mathfrak{A}_{1,1}} z_1^{-2} z_2^{-2}$ (black dashed line);
- monomials: $M_1 = 1 - 2z_1^{-1} z_2^{-1}$, $M_2 = 1 - z_1^{-1} z_2^{-3}$, $M_3 = 1 - z_1^{-2} z_2^{-4}$, $M_4 = 1 - z_1^{-2} z_2^{-5}$, $M_5 = z_1^{-3} z_2^{-6}$.

Then, we determine all possible realisations of the:

- monomial M_1 presented in: Figure 2(a), for $i = 1, 2, 3$; Figure 2(b), for $i, j = 1, 2, 3$, $i \neq j$;,
- monomial M_2 presented in: Figure 2(c), for $i, j = 1, 2, 3$; $i \neq j$; Figure 2(d), for $i, j = 1, 2, 3$; $i \neq j$; Figure 2(e), for $i, j, k = 1, 2, 3$; $i \neq j$, $j \neq k$, $i \neq k$;
- monomial M_3 presented in Figure 2(f), for $i, j, k = 1, 2, 3$; $i \neq j$, $j \neq k$, $i \neq k$; Figure 2(g), for $i, j = 1, 2, 3$, $i \neq j$;

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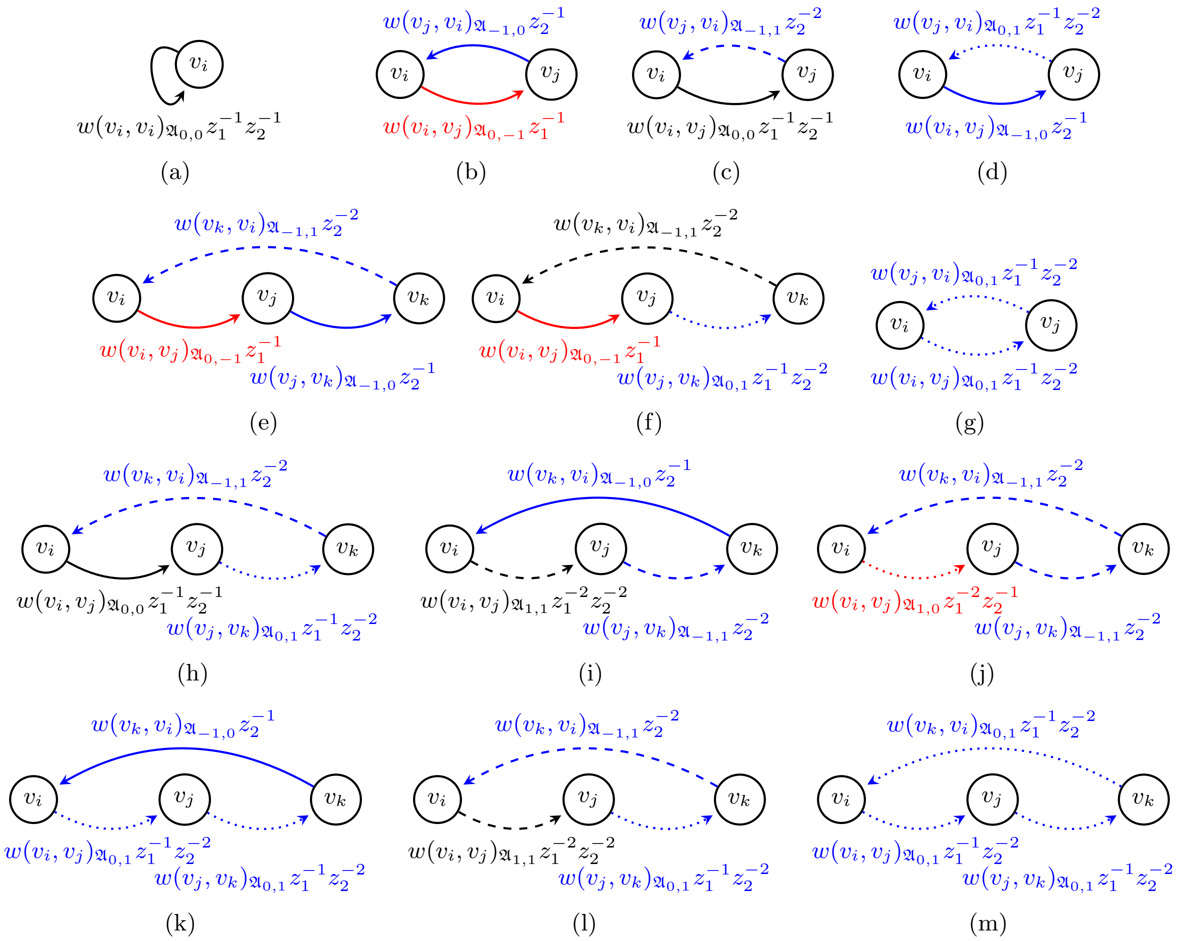

Figure 2. Multi-dimensional digraphs corresponding to the monomials: M_1:2(a), 2(b); M_2: 2(c), 2(d), 2(e); M_3: 2(f), 2(g); M_4: 2(h), 2(i), 2(j), 2(k); M_5: 2(l), 2(m).

- monomial M_4 presented in: Figure 2(h), Figure 2(i), Figure 2(j) and Figure 2(k) for $i, j, k = 1, 2, 3$; $i \neq j$, $j \neq k$, $i \neq k$;
- monomial M_5 presented in: Figure 2(l) and Figure 2(m) for $i, j, k = 1, 2, 3$; $i \neq j$, $j \neq k$, $i \neq k$;

In the defined problem, we assume that dimension of the state matrices must be the minimal among possible. Taking into account this condition, we have 23 328 variants of possible minimal polynomial realisations consisting of three vertices.

Consider the following two possible realisations:

Realisation 1 presented in Figure 3 consists of the digraphs presented in: Figure 2(a) for $i = 3$; Figure 2(d) for $i = 2$, $j = 1$; Figure 2(g) for $i = 1$, $j = 2$; Figure 2(k) for $i = 1$, $j = 2$, $k = 3$; Figure 2(m) for $i = 1$, $j = 2$, $k = 3$. After using Theorem 2, we can check the conditions. The coefficients of the characteristic polynomial satisfy the condition (C1). To verify the second condition, we must compare sets $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$, $\mathbb{D}_4$ and $\mathbb{D}_5$ corresponding to representation of all simple monomial digraphs. The sets are disjoint $\mathbb{D}_1 \cap \mathbb{D}_2 \cap \mathbb{D}_3 \cap \mathbb{D}_4 \cap \mathbb{D}_5 = \emptyset$. Described realisation does not satisfy the condition (C2). The obtained digraphs presented in Figure 4 do not appear in additional cycles. The Condition (C3) is satisfied. The realisation does not satisfy all conditions and is rejected.

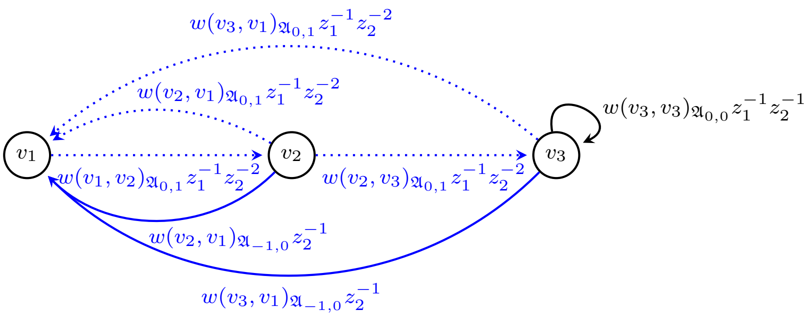

Figure 3. Multi-dimensional digraph corresponding to characteristic polynomial $d(z_1, z_2)$

Realisation 2 presented in Figure 4 consists of the digraphs presented in: Figure 2(a) for $i = 1$; Figure 2(d) for $i = 2$, $j = 1$; Figure 2(g) for $i = 1$, $j = 2$; Figure 2(k) for $i = 1$, $j = 2$, $k = 3$; Figure 2(m) for $i = 1$, $j = 2$, $k = 3$. After using Theorem 2, we can check the conditions. The coefficients of the characteristic polynomial satisfy the condition (C1). To verify the second condition, we must compare sets $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$, $\mathbb{D}_4$ and $\mathbb{D}_5$ corresponding to representation of all simple monomial digraphs. The sets are not disjoint $\mathbb{D}_1 \cap \mathbb{D}_2 \cap \mathbb{D}_3 \cap \mathbb{D}_4 \cap \mathbb{D}_5 = \{v_1\}$ (vertex v_1 in the Figure 3). Described realisation satisfies the condition (C2). The obtained digraphs presented in Figure 3 do not appear in additional cycles. The condition (C3) is satisfied. The realisation does satisfy all conditions and is correct.

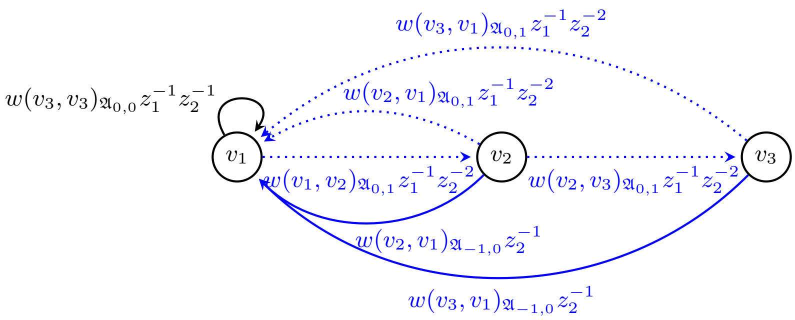

Figure 4. Multi-dimensional digraph corresponding to characteristic polynomial $d(z_1, z_2)$

From the obtained digraphs, we can write state matrices $\mathbf{A}_{q_1,q_2}$, $q_1 = -1, 0, 1$ and $q_2 = -1, 0, 1$ in the form:

$$
\begin{aligned}
\mathbf{A}_{0,0} &= \begin{bmatrix} w(v_1,v_1)_{\mathfrak{A}_{0,0}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{A}_{-1,0} &= \begin{bmatrix} 0 & w(v_2,v_1)_{\mathfrak{A}_{-1,0}} & w(v_3,v_1)_{\mathfrak{A}_{-1,0}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{A}_{0,1} &= \begin{bmatrix} 0 & w(v_2,v_1)_{\mathfrak{A}_{0,1}} & w(v_3,v_1)_{\mathfrak{A}_{0,1}} \\ w(v_1,v_2)_{\mathfrak{A}_{0,1}} & 0 & 0 \\ 0 & w(v_2,v_3)_{\mathfrak{A}_{0,1}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\end{aligned} \tag{23}
$$

After inserting matrices (16) and (23) to the equation (20), we obtain the polynomial $\overline{n}(z_1, z_2)$. After comparison of the coefficients of the same powers of z_1 and z_2 polynomials $\widetilde{n}(z_1, z_2) = \overline{n}(z_{1,2})$, we receive the set of the equations. After solving them, we obtain the following matrices:

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} c_{11} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}; \\ \mathbf{B}_{-1,0} &= \begin{bmatrix} b_{11}^{-1,0} \\ 0 \\ b_{31}^{-1,0} \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}; \quad \mathbf{B}_{0,-1} = \begin{bmatrix} b_{11}^{0,-1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}; \\ \mathbf{B}_{0,0} &= \begin{bmatrix} 0 \\ 0 \\ b_{31}^{0,0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad \mathbf{B}_{1,-1} = \begin{bmatrix} b_{11}^{1,-1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{B}_{1,0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{24}$$

The desired positive realisation of the (17) is given by (18), (23) and (24).

5. Concluding Remarks

The paper presents a method, based on the multi-dimensional digraph theory, for finding the complete set of multi-dimensional characteristic polynomial realisations, that can be used to solve the minimal positive realisation problem of a two-dimensional linear system with delays described by the general model which includes single-input and single-output (SISO). The difference between the proposed algorithm in this paper and currently used methods based on canonical forms of the system (i.e. constant matrix forms) is the creation of not one or few minimal realisations, but a set of possible minimal realisations. A number of colours used in the digraphs, increases with the number of delays (q_1, q_2) in the state vector, is a huge problem. It should be noted, that a large number of colours generates an additional number of potential solutions that must be analysed by the algorithm. This leads to a problem of allocating a cube on the graphics card.

Further work includes extension of the algorithm to find all possible solutions, solving the realisation problem, reachability and controllability of systems using the fast graph-based method. There is also a very difficult open problem of the analysis of systems dynamics for realisations on a different number of nodes in digraphs (not only minimal number of nodes). Currently, the method of determining a positive realisation using a parallel computing method and digraphs methods is optimised, and in the next step it will be implemented in a memory-efficient way.

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