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On discrete and continuous nonlinear Fourier transforms

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Abstract. We study a discretization \(DF\) of the nonlinear Fourier transform \(F\) associated with the integrable partial differential equations of AKNS-ZS type. We construct an iterative procedure by means of which we can calculate the inverse transform \(DF(h)\) for an arbitrary small enough argument \(h\) to any desired accuracy. The construction of this procedure is based on the analogous construction for the continuous transform \(F\).

1. Introduction

Integrable nonlinear partial differential equations can be analyzed by means of the inverse scattering method. From its beginnings, this method has been known to be a nonlinear analogue of the well-known procedure for solving the linear partial differential equations by means of the Fourier transform. The first similarity between the two methods that springs to mind lies in the fact that in both methods the time evolution of the transformed data is easier that the evolution of the original (physical) data. More concretely, suppose the time evolution of a linear system is given by a curve \(t \mapsto q(t)\) in some phase space. The problem is to find this curve if the initial condition is given by \(q(0)\). When using the Fourier analysis, we find the solution \(q(t)\) in the following way. We first perform the Fourier transform \(F(q(0))\) of the initial state \(q(0)\), then we find the time evolution \(E_t[F(q(0))]\) of the transformed initial state. Finally, we compute the inverse Fourier transform \(F^{-1}(E_t[F(q(0)])\) of the time-evolved transformed data. This is precisely the solution of our problem,

\[q(t) = F^{-1}(E_t[F(q(0)])].\]

When using the inverse scattering method to search for the solutions of nonlinear integrable systems, the strategy is the same and can be summarized by essentially the same formula

\[q(t) = F^{-1}(E_t[F(q(0)])]\]

in which the linear Fourier transform \(F\) is replaced by the nonlinear operator \(F\) called the inverse scattering transformation. Finding the inverse scattering transform \(F\) is a difficult inverse problem. In this paper we address this issue.

The analogy between the two methods goes beyond this diagramatic description, and the inverse scattering transform is often called the nonlinear Fourier transform. Deep relations...
between the two theories were already studied in the papers [1], [2]. Later important developments were presented in, e.g. [6], [7], [8]. Deep analytical problems of the nonlinear Fourier transform were studied in [3], [4], [5], [9], [10], [11], and in many other sources. In some important cases, the derivative of \( F \) at the origin is the linear Fourier transform. This fact can provide an insight into the Poisson geometric aspect of the problems (see, e.g. [12] and [13]).

Apart from the theory of integrable systems, the nonlinear Fourier transform has interesting applications in many other areas, ranging from the theory of orthogonal polynomials to random matrices. For an interesting introduction, see [15].

In the present paper we shall study the nonlinear Fourier transform associated to the AKNS-ZS systems. Let \( L^2[0,1] \) be the Hilbert space of square-integrable complex functions on the interval \([0,1]\) and let \( l^2_Z \) be the Hilbert space of the bi-infinite square-integrable complex sequences. We define the nonlinear Fourier transform to be the nonlinear operator

\[
F : L^2[0,1] \rightarrow l^2_Z
\]

given as follows. Let \( f(x) \in L^2[0,1] \) and let

\[
L_f(x, z) = \begin{pmatrix}
0 & f(x) e^{-2\pi i x z} \\
-f(x) e^{2\pi i x z} & 0
\end{pmatrix}, \quad x \in [0,1], \ z \in \mathbb{Z}.
\]

The holonomy of \( L_f \) is given by

\[
\text{Hol}_f(z) = \begin{pmatrix}
h^d(z) & h(z) \\
h(z) & h^d(z)
\end{pmatrix} = \Phi(1,z),
\]

where \( \Phi(x,z) \) is the solution of the initial problem

\[
\Phi_x(x,z) = L_f(x,z) \cdot \Phi(x,z), \quad \Phi(0,z) = I.
\]

Let \( P \) denote the projection which assigns the upper anti-diagonal term to a \( 2 \times 2 \) matrix.

**Definition 1** The nonlinear Fourier transform of the a function \( f \in L^2[0,1] \) is given by

\[
F(f)(z) = P(\text{Hol}_f(z)) = h(z).
\]

The holonomy can be given by Dyson’s series, thus we can define \( F \) by

\[
F(f)(z) = P \left[ I + \sum_{k=1}^{\infty} \prod_{i=1}^{k} L_f(x_1, z)L_f(x_2, z) \cdots L_f(x_k, z) \, d\vec{x} \right], \quad (1)
\]

where

\[
\Delta_k = \{ (x_1, x_2, \ldots, x_k); 1 \geq x_1 \geq x_2 \geq \ldots \geq x_k \geq 0 \}
\]

is the so-called ordered \( k \)-dimensional simplex.

In [14], the fundamental analytic properties of \( F \) are studied. In particular, it is proved that the target space of \( F \) is indeed \( l^2_Z \), and that \( F \) is a real analytic operator between the \( L^2 \)-spaces \( L^2[0,1] \) and \( l^2_Z \). Analyticity turns out to be important in calculating the inverse transform \( F^{-1} \). Finding the inverse \( F^{-1} \) is a difficult inverse problem which is equivalent to a suitable Riemann-Hilbert problem. Recasting the inverse problem in the Riemann-Hilbert form is conceptually very important, but it only yields explicit solutions for some very special elements in the target space of \( F \). In [14], we describe an iterative scheme which yields approximations for \( F^{-1}(h) \) to any desired degree of accuracy and for any (small enough) element \( h \) in the target space \( l^2_Z \). To prove the convergence of this scheme, one needs analyticity of \( F \).
The present paper is primarily devoted to the study of a discretization of \( \mathcal{F} \). In section 2, we define the discrete nonlinear transformation \( \mathcal{D}\mathcal{F} \). Our definition of the discretization is perhaps not a very obvious one, therefore one has to show that it makes sense. Let \( f_N \) denote the usual step function approximation of an element \( f \in L^2[0, 1] \) such that \( \lim_{N \to \infty} f_N = f \) in \( L^2[0, 1] \). We show that \( \lim_{N \to \infty} \mathcal{D}\mathcal{F}(f_N) = \mathcal{F}(f) \). We exhibit the relation between \( \mathcal{D}\mathcal{F} \) in two ways. In section 3, we study the inverse nonlinear Fourier transform. First, we consider the continuous case because the idea of the perturbational iterative calculation of \( \mathcal{F}^{-1} \) can be more clearly explained in this case. Then we adapt this procedure to obtain an iterative calculation of the inverse \( \mathcal{D}\mathcal{F}^{-1} \). The formula turns out to be rather complicated, but its calculation demands only calculations of finite sums and of linear inverse discrete Fourier transforms.

2. Discrete nonlinear Fourier transform

The most common way of defining the continuous nonlinear Fourier transform is via the linear system of ordinary differential equations, as we have done above. Clearly, one could define the discrete nonlinear Fourier transform by means of a difference equation. Here we shall propose another definition which is perhaps less obvious, but is in a sense closer to the continuous original, as we shall see later in remark 1.

Let \( a \in \mathbb{C}, b \in \mathbb{R} \) and let us denote

\[
R(a) = \exp\left(\begin{pmatrix} 0 & a \\ -\frac{a}{|a|} \sin |a| & \frac{a}{|a|} \cos |a| \end{pmatrix}\right)
\]

and

\[
E(b) = \exp\left(\begin{pmatrix} -ib & 0 \\ 0 & ib \end{pmatrix}\right) = \begin{pmatrix} e^{-\pi ib} & 0 \\ 0 & e^{\pi ib} \end{pmatrix}.
\]

Let now

\[ f : \mathbb{Z}_N \to \mathbb{C} \]

be a function of discrete variable \( k \in \mathbb{Z} = \mathbb{Z}/N\mathbb{Z} \). Denote by \( C(\mathbb{Z}_N) \) the vector space of all such functions. Clearly \( C(\mathbb{Z}) = \mathbb{C}^N \).

**Definition 2** The discrete nonlinear Fourier transform

\[ \mathcal{D}\mathcal{F} : C(\mathbb{Z}_N) \to C(\mathbb{Z}_N) \]

is given by the formula

\[ \mathcal{D}\mathcal{F}(f)(z) = P(\text{Hol}_{k=0}^{N-1} f(k)) = P \left( \prod_{k=0}^{N-1} \text{Ad}_{E(\frac{k}{N})} R\left(\frac{1}{N}f(k)\right) \right), \quad z = 0, 1, \ldots, N - 1. \]

As before, \( P \) denotes the operation of taking the upper right element of the \( 2 \times 2 \) matrix in the bracket.

The matrix in the above definition can also be written in the form

\[ \text{Hol}_{k=0}^{N-1} f(k) = R\left(\frac{f(N-1)}{N}\right)E(\frac{z}{N}) \cdots R\left(\frac{f(k)}{N}\right)E(\frac{z}{N}) \cdots E(\frac{z}{N})R\left(\frac{f(0)}{N}\right). \]

Despite the unusual appearance of (2), the transformation \( \mathcal{D}\mathcal{F} \) is related to the linear discrete Fourier transform

\[ \mathcal{D}F(f)(z) = \sum_{k=0}^{N-1} f(k) e^{-2\pi i \frac{zk}{N}}. \]
For the derivative of $\mathcal{D}F$ at the origin we have
\[ D_0 \mathcal{D}F(t)(z) = \lim_{s \to 0} \mathcal{D}F(st)(z) = DF(t)(z), \quad \text{for every } t(k). \]

Taking into account the relation
\[ Ad_{E(\frac{zk}{N})} R(\frac{1}{N} f(k)) = R(\frac{1}{N} f(k) e^{-2\pi i \frac{zk}{N}}), \]
we get the following interesting rewriting of $\mathcal{D}F$:
\[ \mathcal{D}F(f)(z) = P \left( \prod_{k=N-1}^{0} R(\frac{1}{N} f(k) \cos (\frac{zk}{N})) \right). \tag{3} \]

This formula suggests the definitions of the discrete nonlinear cosine and sine transforms given by
\[ \mathcal{D}S(f)(z) = P \left( \prod_{k=N-1}^{0} R(\frac{1}{N} f(k) \cos (\frac{zk}{N})) \right) \quad \text{and} \quad \mathcal{D}C(f)(z) = P \left( \prod_{k=N-1}^{0} R(\frac{1}{N} f(k) \sin (\frac{zk}{N})) \right), \]
respectively.

**Proposition 1** Discrete nonlinear sine and cosine transforms of a real valued function give essentially the same information as the discrete linear sine and cosine transforms, provided that the norm of the function is small enough.

**Proof:** The arguments of $R$ in the definitions of $\mathcal{D}C$ and $\mathcal{D}S$ are real. Therefore, for different values of $k$, matrices of the form
\[ R(\frac{1}{N} f(k) \cos (\frac{zk}{N})) = \exp \left( \begin{pmatrix} 0 & 0 \\ -\frac{1}{N} f(k) \cos (\frac{zk}{N}) & \frac{1}{N} f(k) \cos (\frac{zk}{N}) \end{pmatrix} \right) \]
commute. Thus we have
\[ \mathcal{D}C(f) = P \left( \prod_{k=N-1}^{N-1} R(\frac{1}{N} f(k) \cos (\frac{zk}{N})) \right), \]
or more explicitly
\[ \mathcal{D}C(f)(z) = \sin \left( \frac{1}{N} \sum_{k=0}^{N-1} f(k) \cos (\frac{zk}{N}) \right). \]

In other words, we have
\[ \mathcal{D}C(f)(z) = \sin [\mathcal{D}C(f)(z)] \quad \text{and} \quad \mathcal{D}S(f)(z) = \sin [\mathcal{D}S(f)(z)]. \]

From the above we see that the essential difference between the linear and the nonlinear Fourier transforms has its source in the fact that the matrices $R(\frac{1}{N} f(k) \exp (-2\pi i \frac{zk}{N}))$ do not commute for different values of $k$. It is also clear that the above proposition is not true for complex valued functions $f$, since the matrices $R(\frac{1}{N} f(k) \cos (-2\pi i \frac{zk}{N}))$ for different values of $k$ do not commute in this case.
We shall now show that the transformation $\mathcal{DF}$ is indeed a sensible discretization of the continuous nonlinear Fourier transformation $\mathcal{F}$ defined in the introduction. Let $f : [0, 1] \rightarrow \mathbb{C}$ be a continuous function, defined on the interval. For every integer $N$, we shall denote by $f_N$ the discretization $f_N : \mathbb{Z}_N \rightarrow \mathbb{C}$ of $f$, given by $f_N(k) = f\left(\frac{k}{N}\right)$.

We shall prove the following proposition.

**Proposition 2** Let $f$ be a continuous function defined on the interval $[0, 1]$. Then we have

$$\lim_{N \to \infty} \mathcal{DF}(f_N)(z) = \mathcal{F}(f)(z),$$

for every $z \in \mathbb{N}$.

**Proof:** Consider formula (3) for $\mathcal{DF}$. Expanding

$$R\left(\frac{1}{N} f_N(k) e^{-2\pi i \frac{k}{N}}\right) = \text{Exp}\left[\begin{pmatrix} 0 & \frac{1}{N} f_N(k) e^{-2\pi i \frac{k}{N}} \\ \frac{1}{N} f_N(k) e^{2\pi i \frac{k}{N}} & 0 \end{pmatrix}\right] = \text{Exp}\left(\frac{1}{N} \hat{L}(k, z)\right)$$

into power series and putting the expansions into (3) yields

$$\mathcal{DF}(f_N)(z) = P\left[ \prod_{k=N-1}^{0} \left( I + \frac{1}{N} \hat{L}(k, z) + \left(\frac{1}{N}\right)^2 \frac{1}{2!} \hat{L}(k, z)^2 + \ldots \left(\frac{1}{N}\right)^n \frac{1}{n!} \hat{L}(k, z)^n + \ldots \right) \right].$$

Multiplying and arranging with respect to the powers of $\left(\frac{1}{N}\right)$ yields

$$\mathcal{DF}(f_N)(z) = P\left[ I + \frac{1}{N} \sum_{k=0}^{N-1} \hat{L}(k, z) + \left(\frac{1}{N}\right)^2 \left( \sum_{k_1 > k_2} \hat{L}(k_1, z) \hat{L}(k_2, z) + \frac{1}{2!} \sum_{k=0}^{N-1} \hat{L}(k, z)^2 \right) + \ldots \right]. \quad (4)$$

The above sum can be rewritten as follows. Let $\Omega_N$ be the set of all functions $\omega : \{0, 1, \ldots, N-1\} \rightarrow \mathbb{N} \cup \{0\}$ and let

$$|\omega| = \sum_{k=0}^{N-1} \omega(k).$$

Then we have

$$\mathcal{DF}(f_N)(z) = P\left[ \sum_{\omega \in \Omega_N} \left(\frac{1}{N}\right)^{|\omega|} \frac{1}{\omega(0)! \cdots \omega(N-1)!} \hat{L}(N-1, z)^{\omega(N-1)} \cdots \hat{L}(0, z)^{\omega(0)} \right].$$

The above sum can be refined into the following expression

$$\mathcal{DF}(f_N)(z) = P\left[ \sum_{n=0}^{\infty} \left(\frac{1}{N}\right)^{|\omega|} \frac{1}{\omega(0)! \cdots \omega(N-1)!} \hat{L}(N-1, z)^{\omega(N-1)} \cdots \hat{L}(0, z)^{\omega(0)} \right].$$
Let \( \tilde{F} \) be a better approximation of the continuous \( F \) volumes in the approximation grid at the boundary of the simplex \( \Delta \).

The expression (5) is complicated because it takes into account the reduction of volumes in the approximation grid at the boundary of the simplex \( \Delta_n \). For this reason, \( \mathcal{D}F \) is a better approximation of the continuous \( F \) than a discretization, obtained by replacing the differential equation for \( \Phi \) by a suitable difference equation. Such discretization does not take into account the geometry of the boundary \( \partial \Delta_n \).

In order to further exhibit the relation between the discrete transform \( \mathcal{D}F \) and the continuous transform \( F \), one can also derive formula (2) from the definition of \( F \) given in the introduction. Let \( \chi_{[x_0,a]} \) denote the characteristic function of the interval \( [x_0,x_0+a] \). If we apply the Dyson series expression for the nonlinear Fourier transform and the mean value theorem, we get for any complex number \( g \)

\[
\mathcal{F}(g \chi_{[x_0,\frac{1}{N}]})(z) = \frac{g}{|g|} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \left( \frac{|g|}{N} \right)^{2n+1} e^{-2\pi i z (x_1^n - x_2^n + \ldots + x_{2n+1}^n)}.
\]

Here \( (x_1^n, x_2^n, \ldots, x_{2n+1}^n) \) is some point in the ordered standard simplex

\[
\tilde{\Delta}_{2n+1} = \{(x_1, \ldots, x_{2n+1}); x_0 + \frac{1}{N} \geq x_1 \geq \ldots \geq x_{2n+1} \geq x_0 \}.
\]

If we replace the points \( (x_1^n, x_2^n, \ldots, x_{2n+1}^n) \) by \( (x_0, x_0, \ldots, x_0) \) in the above series, we commit only a small mistake, and we get a perturbation \( \tilde{\mathcal{F}}(g \chi_{[x_0,\frac{1}{N}]})(z) \) given by

\[
\tilde{\mathcal{F}}(g \chi_{[x_0,\frac{1}{N}]}) = \frac{g}{|g|} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)!} \left( \frac{|g|}{N} \right)^{2n+1} e^{-2\pi i z x_0} = \frac{g}{|g|} \sin \left( \frac{|g|}{N} \right) e^{-2\pi i z x_0}.
\]

Clearly, we have

\[
\lim_{N \to \infty} \left( \mathcal{F}(g \chi_{[x_0,\frac{1}{N}]})(z) - \tilde{\mathcal{F}}(g \chi_{[x_0,\frac{1}{N}]}) (z) \right) = 0, \quad \text{for every } z \in \mathbb{Z}.
\]
Similar reasoning tells us that the upper diagonal terms \( \mathcal{E}(g \chi_{[x_0, \frac{1}{N}]})(z) \) of the holonomies \( \text{Hol}(g \chi_{[x_0, \frac{1}{N}]})(z) \) are given by
\[
\mathcal{E}(g \chi_{[x_0, \frac{1}{N}]})(z) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n!} \left| \frac{g}{N} \right|^{2n} e^{-2\pi i(x_1^n - x_2^n + \ldots - x_{2n}^n)},
\]
and are well approximated by the values
\[
\tilde{\mathcal{E}}(g \chi_{[x_0, \frac{1}{N}]})(z) = \cos \left( \frac{|g|}{N} \right).
\]
If we replace the points \((x_1^n, x_2^n, \ldots, x_{2n}^n)\) by \((x_0, x_0, \ldots, x_0)\), the sum in the exponent vanishes. So, the holonomies \( \text{Hol}(g \chi_{[x_0, \frac{1}{N}]})(z) \) are well approximated by the matrices
\[
\tilde{H}_{g \chi_{[x_0, \frac{1}{N}]} z}(z) = \left( \begin{array}{cc}
\cos \left( \frac{|g|}{N} \right) & \frac{g}{|N|} \sin \left( \frac{|g|}{N} \right) e^{-2\pi i z x_0} \\
-\frac{g}{|N|} \sin \left( \frac{|g|}{N} \right) e^{2\pi i z x_0} & \cos \left( \frac{|g|}{N} \right)
\end{array} \right) = \text{Ad}_{E(z x_0)} R \left( \frac{g}{N} \right).
\]
Let \( f_1 \) and \( f_2 \) be two functions with disjoint supports, and suppose that the support of \( f_1 \) lies on the left of the support of \( f_2 \). Then we have
\[
\text{Hol}_{f_1+f_2}(z) = \text{Hol}_{f_2}(z) \cdot \text{Hol}_{f_1}(z).
\]
Perhaps the simplest way to see this is to consider the relevant Dyson series. In the context of the scattering transform, this property is often called the superposition law. Let us now approximate a continuous function \( f \) by the step functions
\[
\tilde{f}_N : [0, 1] \rightarrow \mathbb{R},
\]
given by
\[
\tilde{f}_N(x) = \sum_{k=0}^{N-1} f \left( \frac{k}{N} \right) \chi_{[\frac{k}{N}, \frac{k+1}{N}]}.
\]
The nonlinear Fourier transform of \( \tilde{f}_N(x) \) is then well approximated by the expression
\[
\mathcal{F}(\tilde{f}_N)(z) \cong P \left[ \prod_{k=N-1}^{0} \tilde{H} f \left( \frac{k}{N} \chi_{[\frac{k}{N}, \frac{k+1}{N}]} \right)(z) \right] = P \left[ \prod_{k=N-1}^{0} \text{Ad}_{E(\frac{k}{N})} R \left( \frac{1}{N} f \left( \frac{k}{N} \right) \right) \right] .
\]
The expression on the right is precisely the discrete nonlinear Fourier transform of the function \( f_N \),
\[
\mathcal{D}\mathcal{F}(f_N)(z) = P \left[ \prod_{k=N-1}^{0} \text{Ad}_{E(\frac{k}{N})} R \left( \frac{1}{N} f \left( \frac{k}{N} \right) \right) \right] .
\]
From the above it is not difficult to show that we have
\[
\lim_{N \rightarrow \infty} \mathcal{D}\mathcal{F}(f_N)(z) = \mathcal{F}(f)(z), \quad \text{for every } z \in \mathbb{N},
\]
but this was already proved more rigorously and in more detail above.
3. The inverse nonlinear Fourier transform

In this section, we shall describe an algorithm for computing arbitrarily good approximations for the inverse nonlinear Fourier transform of arbitrary function \( h(z) \), provided that it belongs to a suitable space and lies close enough to the origin. We shall first give the construction for the continuous case because it illustrates the problem better. The adaptation of the construction to the case of the discrete nonlinear Fourier transform is rather straightforward.

In order to obtain a perturbational formula, we consider the action of \( \mathcal{F} \) on real analytic curves in the Hilbert space \( L^2[0,1] \). We shall consider curves \( f(x,s) : (-\epsilon, \epsilon) \rightarrow L^2[0,1] \) which start at the origin, i.e., \( f(x,0) = 0 \in L^2[0,1] \) and are real analytic. We can expand \( f(x,s) \) around zero with respect to \( s \). We get the power series

\[
f(x,s) = \sum_{n=1}^{\infty} \frac{s^n}{n!} f_n(x)
\]

and we assume that the convergence radius is \( \alpha > 0 \). The following theorem, proved in \[14\], establishes an essential fact about the nonlinear Fourier transform.

**Theorem 1** Nonlinear Fourier transform

\[
\mathcal{F} : L^2[0,1] \rightarrow l^2_Z
\]

is a real analytic map between the two Hilbert spaces equipped with the appropriate \( L^2 \)-norms.

The real analyticity ensures that the image \( \mathcal{F}(f(x,s)) \) is a real analytic curve in \( l^2_Z \). Since \( \mathcal{F}(0) = 0 \), this curve also starts at the origin. If we expand it in \( l^2_Z \) around \( s = 0 \), we obtain the power series

\[
\mathcal{F}(f(x,s)) = \mathcal{F}\left(\sum_{n=1}^{\infty} \frac{s^n}{n!} f_n(x)\right) = \sum_{n=1}^{\infty} \frac{s^n}{n!} h_n(z).
\]

(6)

It is not difficult to see that the convergence radius of the above series is also equal to \( \alpha \).

The second essential fact about the nonlinear Fourier transform is given by the following proposition.

**Proposition 3** The derivative of \( \mathcal{F} \) at the origin of \( L^2[0,1] \) is the linear Fourier transform

\[
F : L^2[0,1] \rightarrow l^2_Z.
\]

**Proof:** If we evaluate the Dyson series for \( \text{Hol}_{f+st}(z) \), where \( t \in L^2[0,1] \) and \( s \) is real number, we get

\[
\text{Hol}_{f+st}(z) = I + \int_{\Delta_1} L_{f+st}(x) \, dx + \sum_{k=2}^{\infty} \int_{\Delta_k} L_{f+st}(x_1) \cdots L_{f+st}(x_k) \, d\vec{x}
\]

Deriving with respect to \( s \) and evaluating at \( s = 0 \) and at \( f = 0 \), we get

\[
D_0 F(t) = P[\int_{\Delta_1} L_t(\xi, z) \, d\xi] = \int_0^1 t(\xi)e^{-2\pi i z \xi} \, d\xi.
\]

Above we have proved that \( F \) is the weak (directional) derivative of \( \mathcal{F} \) at the origin. It is actually also the strong derivative (differential). This fact is also proved in \[14\]. Now, the linear Fourier transform is an isometry of the \( L^2 \)-spaces \( L^2[0,1] \) and \( l^2_Z \). Therefore, by the inverse function theorem for Hilbert spaces, the map \( \mathcal{F} \) has a local inverse in a neighbourhood of the origin. Moreover, the analytical version of this theorem tells us that this inverse is also a real analytic map. We can summarize this in the following theorem:
Theorem 2 There exist neighbourhoods \( U \subset L^2[0, 1] \) and \( V \subset L^2_{\mathbb{Z}} \) of the respective origins and a real analytic map 
\[ G : V \rightarrow U \]
which is the local inverse of the nonlinear Fourier transform \( F \).

Our next goal is to find an effective way of computing \( G(h) \) of any \( h \) close enough to the origin. First, we shall give explicit formulae for the terms \( h_n \) of series (6). To this end, let us consider the upper-triangular block-Toeplitz matrix
\[
L_f^n(x, z) = \begin{pmatrix}
0 & L_1 & L_2 & \cdots & \frac{1}{m!} L_n \\
0 & 0 & L_1 & \cdots & \frac{1}{(n-1)!} L_{n-1} \\
0 & 0 & 0 & \cdots & \frac{1}{(n-2)!} L_{n-2} \\
& & & \ddots & \\
0 & 0 & 0 & \cdots & L_1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
with blocks given by
\[
L_k = \begin{pmatrix}
0 & e^{-2\pi i z f_k(x)} \\
-2\pi i z f_k(x) & 0
\end{pmatrix}.
\]
Let \( \text{Hol}_f^n(z) = \Phi^n(1, z) \), where \( \Phi^n(x, z) \) is the solution of the initial problem
\[
(\Phi^n)_x = L_f^n \cdot \Phi^n, \quad \Phi^n(0; z) = I.
\]
The coefficient matrix \( L_f^n \) is strictly upper-triangular, therefore for every \( m > n \) and for any choice of \( (x_1, x_2, \ldots, x_m) \) we have
\[
L_f^n(x_1, z) \cdot L_f^n(x_2, z) \cdots L_f^n(x_m, z) = 0.
\]
For this reason the Dyson series for \( \text{Hol}_f^n(z) \) is finite;
\[
\text{Hol}_f^n(z) = I + \sum_{k=1}^{n} \int_{\Delta_k} L_f^n(x_1, z) \cdot L_f^n(x_2, z) \cdots L_f^n(x_k, z) \, d\vec{x}. \tag{7}
\]
Finiteness of the above sum is essential for the derivation of the closed expressions of the terms \( h_n(z) \). The holonomy matrix is again an upper triangular block-Toeplitz matrix of the form
\[
\text{Hol}_f^n(z) = \begin{pmatrix}
I & H_1(z) & \cdots & \frac{1}{m!} H_n(z) \\
0 & I & \cdots & \frac{1}{(n-1)!} H_{n-1}(z) \\
0 & 0 & \cdots & \frac{1}{(n-2)!} H_{n-2}(z) \\
& & & \ddots \\
0 & 0 & \cdots & I
\end{pmatrix}.
\]
On the other hand, the analyticity of \( F \) and of the diagonal term of the holonomy \( \text{Hol}_f \) gives
\[
\text{Hol}_f(z) = I + \sum_{k=1}^{\infty} \frac{s^k}{k!} \hat{H}_k(z).
\]
It is not difficult to see that the terms of this expansion are precisely the blocks of \( \text{Hol}_f^n(z) \),
\[
H_n(z) \equiv \hat{H}_n(z), \quad \text{for every } z \in \mathbb{Z} \text{ and } n \in \mathbb{N}. \tag{8}
\]
Let us denote
\[ \text{Hol}_f(s) = I + \sum_{n=1}^{\infty} \frac{s^n}{n!} H_n(z) = I + \sum_{n=0}^{\infty} \frac{s^n}{n!} \left( \frac{h_n(z)}{h_n(z)} \cdot \frac{h_n(z)}{h_n(z)} \right). \]

A lengthy but straightforward calculation in which we use (7) and (8) gives the following formula

\[ h_n(z) = \sum_{k=1}^{\tilde{n}} (-1)^{k-1} \left( \sum_{|J|=n} \alpha^J \right) \int_{\Delta^{2k-1}} e^{-2\pi i(x_1-x_2+\ldots+x_{2k-1})} f_{j_1}(x_1)f_{j_2}(x_2) \ldots f_{j_{2k-1}}(x_{2k-1}), \]

where \( J = (j_1, j_2, \ldots, j_{2k-1}) \) are multi-indices with \( j_r \geq 1 \)

\[ \alpha^J = \left( \begin{array}{c} 2n+1 \\ j_1, \ldots, j_{2k-1} \end{array} \right) \]

are the multinomial coefficients. The integer \( \tilde{n} \) is equal to \( n/2 \) when \( n \) is even and to \( (n+1)/2 \) when \( n \) is odd. If we separate the term at \( k = 1 \) and the rest in formula (9), we get

\[ h_1(z) = \int_{\Delta_1} \phi_1(\xi) \, d\xi = \int_0^1 f_1(\xi) e^{-2\pi i \xi z} \, d\xi = F(1)(z) \]

for \( n = 1 \)

\[ h_n(z) = F(f_n)(z) + \mathcal{H}_n(f_1, f_2, \ldots, f_{n-2})(z) \]

for \( n > 1 \). The term \( \mathcal{H}_n \) is given by

\[ \mathcal{H}_n(f_1, f_2, \ldots, f_{n-2})(z) = \sum_{k=2}^{\tilde{n}} (-1)^{k-1} \left( \sum_{|J|=n} \alpha^J \right) \int_{\Delta^{2k-1}} \phi_{j_1}(x_1) \phi_{j_2}(x_2) \ldots \phi_{j_{2k-1}}(x_{2k-1}) \, d\vec{x}, \]

where

\[ \phi_{j_r}(x_r) = f_{j_r} e^{-2\pi i x_r z}. \]

The term \( \mathcal{H}_n \) depends only on \( f_1, \ldots, f_{n-2} \). This follows from the fact that we have \( j_r \geq 1 \) for all indices \( j_r \). The products of terms \( \phi_j \) which contain the highest index are of the form \( \phi_1(x_{\sigma(1)}) \phi_2(x_{\sigma(2)}) \phi_{n-2}(x_{\sigma(3)}) \) for any permutation \( \sigma \in S_3 \).

Let \( G \) denote the inverse linear Fourier transform

\[ G(c_n) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z}. \]

Inversion of formulae (10) and (11) yields the following theorem

**Theorem 3** Let

\[ h(z, s) = \sum_{n=1}^{\infty} \frac{s^n}{n!} h_n(z) \]

be a real analytic curve lying in a small enough neighbourhood of the origin of the space \( l^2_z \). Then the inverse nonlinear Fourier transform \( G \) of this curve is the real analytic curve, given by the convergent series

\[ G[h(z, s)](x) = \sum_{n=1}^{\infty} \frac{s^n}{n!} f_n(x) \]
in $L^2[0,1]$ whose terms are given by formulae

\begin{align}
    f_1(x) &= G[h_1(z)](x) \\
    f_n(x) &= G[h_n(z)] - G[\mathcal{H}_n(f_1, \ldots, f_{n-2})(z)](x).
\end{align}

(13)  \hspace{1cm} (14)

Let now $h(z)$ be an arbitrary element in $l^2_Z$ close enough to the origin.

**Theorem 4** The inverse nonlinear Fourier transform $G(h)$ is given by the convergent series

$$G[h(z)](x) = \sum_{n=1}^{\infty} \frac{1}{n!} f_n(x)$$

in $L^2[0,1]$, whose terms can be computed successively by formulae

\begin{align}
    f_1(x) &= G[h(z)](x) \\
    f_{2n}(x) &= 0 \\
    f_{2n+1}(x) &= -G[\mathcal{H}_{2n+1}(f_1, \ldots, f_{2n-1})(z)](x).
\end{align}

**Proof:** Consider the simplest analytic curve $h(z,s)$ in $l^2_{Z}$, given by

$$h(z,s) = s \cdot h(z).$$

In terms of the notation in the previous theorem, we have

$$h_1 = h, \quad h_n = 0, \quad \text{for } n \geq 2.$$  

Provided the norm of $h(z)$ is small enough, theorem 3 ensures the existence of the convergent series

$$G(h)(x) = \sum_{n=1}^{\infty} \frac{s^n}{n!} f_n(x)$$

whose terms are given by equations (13) and (14). If we set $s = 1$, we obtain the expressions for the terms $f_n$ of our series. It only remains to be shown that the even terms $f_{2n}$ are equal to zero. We can prove this by induction. Fix an $n_0 \in \mathbb{N}$. Suppose all $f_{2m} = 0$ for $m < n_0$. We claim that $f_{2n_0} = 0$. If this were not so, then by formula (12) we should have

$$j_1 + j_2 + \ldots + j_{2k-1} = 2n_0.$$  

But by the induction hypothesis, all the indices $j_r$ in the above sum are odd, therefore this sum cannot be even. This shows that $f_{2n_0} = 0$ for all $n$.

It is not difficult to adapt the result of theorems 3 and 4 to the discrete nonlinear Fourier transform $DF$. First, we recall that $D_0DF = DF$. By the inverse function theorem, $DF$ is locally invertible in the vicinity of the origin. Recall also the the continuous nonlinear Fourier transform of a real analytic curve $f(x,s) = \sum_{n=1}^{\infty}(s^n/n!) f_n(x)$ can be written in the form

$$\mathcal{F}\left(\sum_{n=1}^{\infty} \frac{s^n}{n!} f_n(x)\right)(z) = \sum_{n=1}^{\infty} \frac{s^n}{n!} h_n(z),$$

in $L^2[0,1]$.
where $h_n$ are given by (9), or equivalently, by (10) and (11). Formulae (10) and (11) can be adapted to the discrete case. Let $f$ be again a function of the discrete variable

$$f : \mathbb{Z}_N \rightarrow \mathbb{C}.$$  

After some inspection and bookkeeping we find that the discrete versions of (10) and (11) are given by

$$Dh_1(z) = \frac{1}{N} \sum_{k=N-1}^{0} f_1(k) e^{-2\pi i \frac{zk}{N}} = DF(f_1)(z),$$

$$Dh_n(z) = DF(f_n)(z) + D\mathcal{H}_n(f_1, f_2, \ldots, f_{n-2})(z), \text{ for } n > 1.$$  

Here the terms $D\mathcal{H}_n$ are given as follows. Denote

$$L_m(k) = \left( \begin{array}{cc} 0 & \frac{1}{N} f_m(k) e^{-2\pi i \frac{zk}{N}} \\ -\frac{1}{N} f_m(k) e^{2\pi i \frac{zk}{N}} & 0 \end{array} \right).$$  

Then

$$D\mathcal{H}_n(f_1, f_2, \ldots, f_{n-2})(z) = P \left[ \sum_{0}^{0} \prod_{k=N-1}^{1} \frac{1}{n(k)!} P_{n(k)}(f_1, \ldots, f_{n-2}) \right] \quad (15)$$  

and

$$P_{n(k)}(f_1, \ldots, f_{n-2}) = \sum_{r=1}^{n(k)} \frac{1}{r!} \sum_{\sum_{i=1}^{r} m_i = n(k)} \frac{1}{m_1! m_2! \cdots m_r!} L_{m_1}(k) L_{m_2}(k) \cdots L_{m_r}(k). \quad (16)$$  

We note that all the above expressions are finite sums.

Let now

$$DG(h(z))(k) = \sum_{z=0}^{N-1} h(z) e^{2\pi i \frac{zk}{N}}$$  

be the linear discrete inverse Fourier transform. We have the following result.

**Theorem 5** Let

$$h(z, s) = \sum_{n=1}^{\infty} \frac{s^n}{n!} h_n(z) : (-\epsilon, \epsilon) \rightarrow C(\mathbb{Z}_N) = \mathbb{C}^N$$  

be a real analytic curve in $C(\mathbb{Z}_N)$ starting at the origin. Then its inverse image $DG(h(z, s))(k)$ with respect to $DF$ is given by the convergent power series

$$DG(h(z, s))(x) = \sum_{n=1}^{\infty} \frac{s^n}{n!} f_n(k)$$  

whose terms are given by

$$f_1(k) = DG[h_1(z)](k) \quad (17)$$

$$f_n(k) = DG[h_n(z)](k) - DG[D\mathcal{H}_n(z)](k), \quad (18)$$

and $D\mathcal{H}_n$ are given by (15) and (16).
From this we get:

**Theorem 6** Let \( h(z) \in C(\mathbb{Z}) \) be an arbitrary element, close enough to the origin. Then the inverse discrete nonlinear transform \( \mathcal{D}G(h) \) is given by the convergent series

\[
\mathcal{D}G(h)(k) = \sum_{n=1}^{\infty} \frac{s^n}{n!} f_n(k),
\]

where the terms \( f_n \) are given by

\[
f_1(k) = \mathcal{D}G[h(z)](k) \]
\[
f_n(k) = -\mathcal{D}G(\mathcal{D}H_n).
\]

The above theorems are proved in the same way as in the continuous case. The problem of convergence does not really arise here. Transformation \( \mathcal{D}F \) is a map between two finite dimensional spaces and its analyticity is seen rather easily.

References