Journal of Physics: Conference Series

Max-out-in pivot rule with Dantzig's safeguarding rule for the simplex method

To cite this article: Monsicha Tipawanna and Krung Sinapiromsaran 2014 *J. Phys.: Conf. Ser.* **490** 012196

Max-out-in pivot rule with Dantzig's safeguarding rule for the simplex method

Monsicha Tipawanna and Krung Sinapiromsaran

Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok, 10330, Thailand

E-mail: ktmonsic@kmitl.ac.th (M. Tipawanna) and Krung.s@chula.ac.th (K. Sinapiromsaran)

Abstract. The simplex method is used to solve linear programming problem by improving the current basic feasible solution. It uses a pivot rule to guide the search in the feasible region. The pivot rule is used to select an entering index in simplex method. Nowadays, many pivot rule have been presented, but no pivot rule shows superior performance than other. Therefore, this is still an active research in linear programming. In this research, we present the max-out-in pivot rule with Dantzig's safeguarding for simplex method. This rule is based on maximum improvement of objective value of the current basic feasible point similar to the Dantzig's rule. We can illustrate by Klee and Minty problems that our rule outperforms that of Dantzig's rule by the number of iterations for solving linear programming problems.
Keywords: Linear programming, Simplex method, Pivot rule, Dantzig's rule

1. Introduction

The simplex method is the solution method for solving a linear programming problem (LP) (Dantzig, 1949). This method starts at a feasible solution of a feasible region and moves to the better solution by increasing value of a nonbasic variable called the entering variable and decreasing value of a basic variable called the leaving variable [2].The criteria to select the entering and leaving variable is called *the pivot rule*. Many researchers have attempted to improve the simplex performance , for examples, see Pan (1990) [9], Junior and Lins (2005) [4], Hu (2007) [5] and Arsham (2007) [3]. The pivot rule affects the simplex performance. So a lot of researches have been proposed new pivot rules such as Devex rule [7], Steepest-edge rule [1, 6], and a largest-distance pivot rule [8].

In this paper, we propose the out-in pivot rule called max-out-in pivot rule safeguarding with Dantzig's rule for the simplex method. This rule is first selecting the leaving variable that has the maximum value from the current basic variable set. Then it chooses the best corresponding entering variable that gives the smallest positive contribution to the binding constraint of the leaving variable. If the selected basic variable and nonbasic variable can not swap or there is no corresponding nonbasic candidate, then we use Dantzig's rule instead. We can show by Klee and Minty problem [10] that our method improves Dantzig's rule.

This paper is organized as follows. In section 2, we explain the main idea of our pivot rule and give the max-out-in pivot rule. We give the example to illustrate the implementation our rule in section 3. The application of our rule to Klee and Minty problem is given in section 4. Finally, the conclusion is given in section 5.

2. Max-Out-In Pivot Rule

2.1. The Main Idea

Consider the linear programming problem in the following form:

$$\begin{aligned} \text{Maximize} \quad & \mathbf{1}^{\mathbf{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{7}$$

where $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{1} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m\times n} (m < n)$ and $\text{rank}(\mathbf{A}) = m$. In this form, the value of each decision variable is equal to its contribution to the objective value. Since the leaving variable decreases to zero when it changes to the nonbasic variable, and the increment in value of the entering variable depends on the decrement in value of the leaving variable. Then we should select the leaving variable that has the maximum value among all basic variables and select the nonbasic variable that can improve the objective value and allows the maximum increase. Rewrite the system (7) into the tableau

	x_B	x_N	**RHS**
	$\mathbf{0}^T$	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N^T$	$\mathbf{c}_B^T\mathbf{B}^{-1}\bar{\mathbf{b}}$
x_B	I_m	$\bar{\mathbf{A}} = \mathbf{B}^{-1}\mathbf{N}$	$\bar{\mathbf{b}}$

where I_m is an identity matrix of size m, $\bar{\mathbf{A}} = (\bar{a}_{ij}) \in \mathbb{R}^{m\times(n\text{-}m)}$, $\mathbf{c}_B = \mathbf{1} \in \mathbb{R}^m$, $\mathbf{c}_N = \mathbf{1} \in \mathbb{R}^{n\text{-}m}$, $\bar{\mathbf{b}} = \mathbf{B}^{-\mathbf{1}}\mathbf{b} \in \mathbb{R}^m$, $\bar{\mathbf{b}} \geq \mathbf{0}$ and $\mathbf{0} \in \mathbb{R}^m$. If we select x_r to be the leaving variable. Consider the binding constraint of x_r:

$$x_r + \sum_{i\in I_N\setminus J} \bar{a}_{ri}x_i + \sum_{j\in J} \bar{a}_{rj}x_j = \bar{b}_r \tag{8}$$

In order to improve objective value, we select the entering variable form the set J that allows the maximum increase. We select the leaving variable $x_{\tilde{j}}$ such that

$$\tilde{j} = \text{Argmin}\left\{\bar{a}_{rj} \mid \bar{a}_{rj} > 0, j \in J\right\}.$$

2.2. Max-Out-In Pivot Rule (with Dantzig's rule safeguarding)

From our main idea, we state our proposed pivot rule called *the max-out-in pivot rule* as following.

Max-out-in pivot rule: If $J \neq \phi$.

Select x_r to leave the basic by $x_r = \mathbf{max}\left\{x_i \mid i \in I_B\right\}$.
Select $x_{\tilde{j}}$ to enter the basic by $\tilde{j} = \mathbf{Argmin}\{\bar{a}_{rj} \mid \bar{a}_{rj} > 0,\ j \in J\}$.

When it performs the max-out-in pivot rule, there are the possible two cases.

Case 1. One basic variable x_r and one nonbasic variable $x_{\tilde{j}}$ can be swapped.

Case 2. The max-out-in pivot rule can not be used.

- If the selected basic variable violates the minimum ratio test.
- If no corresponding nonbasic variable exists.

In case 1, we can use the max-out-in pivot rule to perform the pivot step. However, we cannot perform the pivot step if case 2 occurs. We apply Dantzig's rule as the safeguarding rule instead.

3. Illustrative Examples

In this section, we give the example to illustrate the implementation of the proposed method.

Example 1. Consider the following problem:

$$\begin{aligned} \text{Maximize} \quad & -86x_1 - 6x_2 - 28x_3 + 2x_4 + 5x_5 \\ \text{subject to:} \quad & 54x_1 + 84x_2 + 46x_3 + 16x_4 + 90x_5 \le 14000 \\ & 51x_1 + 59x_2 + 34x_3 + 3x_4 + 94x_5 \le 12000 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 \ge 0. \end{aligned}$$

Replacing x_j by $|c(j)|\,\tilde{x}_j$ for $j = 1, 2, \ldots, 5$, and let $\tilde{x}_6$ and $\tilde{x}_7$ be slack variables associate with the first and the second constraint, respectively. Then the initial tableau for above problem is:

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$	**RHS**
z	1	1	1	-1	-1	0	0	0
x_6	$\frac{27}{43}$	14	$\frac{23}{14}$	8	18	1	0	14000
x_7	$\frac{51}{86}$	$\frac{59}{6}$	$\frac{17}{14}$	$\frac{3}{2}$	$\frac{94}{5}$	0	1	12000

Since the maximum value among all basic variables is at $x_6 = 14000$ corresponding to $r = 1$. Then $\{\bar{a}_{1j} \mid j \in J \text{ and } \bar{a}_{1j} > 0\} = \{\bar{a}_{14},\ \bar{a}_{15}\} = \{8,\ 18\}$. $\tilde{j} = 4 = \text{Argmin}\,\{\bar{a}_{14},\ \bar{a}_{15}\}$. Since $\left\{\frac{b_k}{\bar{a}_{k4}} |\bar{a}_{k4} > 0, \text{for } k = 1, 2\right\} = \left\{\frac{b_1}{\bar{a}_{14}}, \frac{b_2}{\bar{a}_{24}}\right\} = \{1750, 8000\}$. From case 1, we select x_6 to be the leaving variable and select x_4 to be the entering variable. After pivoting, the simplex tableau is

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$	**RHS**
z	$\frac{371}{344}$	$\frac{11}{4}$	$\frac{135}{112}$	0	$\frac{5}{4}$	$\frac{1}{8}$	0	1750
x_4	$\frac{27}{344}$	$\frac{7}{4}$	$\frac{23}{112}$	1	$\frac{9}{4}$	1750	0	1750
x_6	$\frac{327}{688}$	$\frac{173}{24}$	$\frac{29}{32}$	0	$\frac{617}{40}$	$\frac{-3}{16}$	1	9375

This is the optimal tableau. The optimal solution is $\tilde{x}_4 = 1750$, $\tilde{x}_j = 0$ for $j = 1, \ldots, 5$ and $j \neq 4$. Then we get $x_4 = 875$, $x_j = 0$ for $j = 1, \ldots, 5$ and $j \neq 4$ with the optimal value 1750 and the number of iteration is 1. By using the simplex method with Dantzig's pivot rule, the number of iterations is 3.

4. Application of the max-out-in pivot rule to Klee and Minty problem

Klee and Minty proposed some linear programming problems that the simplex method has exponential worst-case running time in 1972[10]. This problem is called the Klee and Minty problem, which is stated as below.

Klee and Minty problem:

$$\begin{aligned} \text{Maximize} \quad & \sum_{j=1}^{n} 10^{n-j} x_j \\ \text{subject to:} \quad & 2\sum_{j=1}^{i-1} 10^{i-j} x_j + x_i \le 100^{i-1},\ i = 1, \ldots, n \qquad (9) \\ & x_i \ge 0,\ i = 1, \ldots, n \end{aligned}$$

The simplex method with Dantzig's pivot rule requires $2^n - 1$ iterations to solve Klee and Minty problem [10]. Next, we show that our rule obtains the optimal solution in 1 iteration.

We first normalize the objective coefficients to be +1 or 0 or −1 and we let $x_{n+1}, \cdots, x_{2n}$ be slack variables associate with the first to the n^{th} constraint, respectively. Rewrite this problem into the tableau

	x_1	$\cdots$	x_n	x_{n+1}	$\cdots$	x_{2n}	**RHS**
z	-1	$\cdots$	-1	0	$\cdots$	0	0
x_{n+1}	$\frac{1}{10^{n-1}}$	$\cdots$	0	1	$\cdots$	0	1
$\vdots$						$\vdots$	$\vdots$
x_{2n}	2	2	1	0	$\cdots$	1	10^{n-1}

By using the max-out-in pivot rule, x_{2n} is selected to be the leaving variable and x_n is selected to be the entering variable.

	x_1	$\cdots$	x_n	x_{n+1}	$\cdots$	x_{2n}	**RHS**
z	1	1	0	0	$\cdots$	0	10^{n-1}
x_{n+1}	$\frac{1}{10^{n-1}}$	$\cdots$	0	1	$\cdots$	0	1
$\vdots$						$\vdots$	$\vdots$
x_{2n}	2	2	1	0	$\cdots$	1	10^{n-1}

This is the optimal tableau. Then the optimal solution is obtained in 1 iteration for Klee and Minty problem of any size n.

5. Conclusions

The objective of this paper is to propose a new pivot rule called the max-out-in pivot rule safeguarding with Dantzig's rule for the simplex method. This rule first normalizes the objective coefficients to be +1 or 0 or −1. The keys of this rule are that the maximum basic variable is selected to leave basis, and the corresponding nonbasic variable which allowed the maximum increase is selected to enter basis. We can illustrate by the examples that the max-out-in pivot rule can achieve good results. In addition, our rule is better than Dantzig's rule over the Klee and Minty problems [10]. Further work will be done for the improvement of the max-out-in pivot rule.

References

[1] D. Goldfarb and J. Ried, A practical steepest-edge simplex algorithm. *Mathematical Programming*, **12** (1977), 361-371.
[2] G.B. Dantzig, *Linear Programming and Extensions*. Princeton University Press, Princeton, NJ, 1963.
[3] H. Arsham, A computationally stable solution algorithm for linear programs. *Applied Mathematics and Computation*, **188** (2007), 1549-1561.
[4] H.V. Junior and M.P.E. Lins, An improved initial basis for simplex algorithm. *Computers and and Operations Research*, **32** (2005), 1983-1993.
[5] J.F. Hu, A note on "an improved initial basis for simplex algorithm". *Computers and Operations Research*, **34** (2007), 3397-3401.
[6] J.J.H. Forrest and D. Goldfarb, Steepest-edge simplex algorithms for linear programming. *Mathematical Programming*, **57** (1992), 341-374.
[7] P.M.J. Harris, Pivot selection methods of the Devex LP code. *Mathematical Programming* **5** (1973), 1-28
[8] P.-Q. Pan, A largest-distance pivot rule for the simplex algorithm. *European Journal of Operational Research*, **187** (2008), 393-402.
[9] P.-Q. Pan, Practical finite pivoting rules for the simplex method. *OR Spektrum*, **12** (1990), 219-225.
[10] V. Klee and GJ. Minty, How good is the simplex algorithm? In: Shisha O, editor. *Inequalities III*, Academic Press, New York, (1972), 158-172.