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Asymptotic behavior of a generalized Burgers equation solutions on a finite interval

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Abstract. The article is concerned with the study of asymptotic behavior of solutions of the Burgers equation and its generalizations with initial value — boundary problem on a finite interval, with constant boundary conditions. Since these equations take a dissipation into account, it is naturally to presuppose that any initial profile will evolve to an invariant time-independent solution with the same boundary values. Yet the answer happens to be slightly more complex. There are three possibilities: the initial profile may regularly decay to an invariant solution; or a Heaviside-type gap develops through a dispersive shock and multi-oscillations; or, exotically, an asymptotic limit is a 'frozen multi-oscillation' piecewise-differentiable solution, composed of different smooth invariant solutions.

1. Introduction

The Burgers equation

$$u_t(x,t) = \varepsilon^2 u_{xx}(x,t) - u(x,t)u_x(x,t). \tag{1}$$

is related to the viscous medium whose oscillations it describes. This viscosity dampens oscillations except for stationary solutions which are invariant for some subalgebra of the full symmetry algebra of the equation. While studying the equation on the whole line only bounded solutions are usually taken into account since only they have a physical meaning. It is not the case for a finite interval as an unbounded solution may still remain bounded within an interval. Thus we obtain a wider choices of invariant solutions and asymptotics and, consequently, some new effects. The generalized Burgers equation here is of the form

$$u_t(x,t) = \varepsilon^2 u_{xx}(x,t) - \alpha u^n(x,t)u_x(x,t). \tag{2}$$

We consider initial value - boundary problem (IVBP) for the Burgers equation on a finite interval:

$$u(x,0) = f(x), \quad u(a,t) = l(t), \quad u(b,t) = r(t), \quad x \in [a,b]. \tag{3}$$

The case of constant boundary conditions $u(a,t) = A, \quad u(b,t) = B$ and the related asymptotics are of a special interest here.

Some of our results are similar to those of Dubrovin *et al* [1, 2, 3] dealing with a formation of dispersive shocks in a class of Hamiltonian dispersive regularizations of the quasi-linear transport equation. For the Burgers equation the shocks resulting in breaks (and preceded by a multi-oscillation) do develops for some IVBPs; some other IVBPs lead to a monotonic convergence to an invariant solutions. One new possibility for the asymptotic profile is a class of frozen multi-oscillating solutions.

This paper is a continuation of [4, 5]. Numeric results are obtained via the Maple *PDETools* package.

2. Solutions and symmetries

The Burgers (1) stationary solutions are:

$$
\begin{aligned}
u(x,t) &= c, && (4)\\
u(x,t) &= -\varepsilon^2 k \tanh(kx+c), && (5)\\
u(x,t) &= -\varepsilon^2 k \coth(kx+c), && (6)\\
u(x,t) &= \varepsilon k \tan(\frac{kx+c}{\varepsilon}), && (7)\\
u(x,t) &= \frac{k\varepsilon^2}{kx+c}. && (8)
\end{aligned}
$$

For the generalization (2) stationary solutions are given by

$$
\begin{aligned}
x &= C_1 + \varepsilon^2(n+1)\int \frac{dy}{C_2+\alpha y^{n+1}},\\
y &= C.
\end{aligned}
$$

These solutions are invariant under the t-translation symmetry.

Consider a simple IVBP for (1) of the form (3)

$$u(x,0)=f(x), \quad u(0,t)=A, \quad u(1,t)=B, \quad A,B\in\mathbb{R}. \tag{9}$$

Taking the dissipation into the account it is naturally to presuppose that at $t\to\infty$ we get $u(x,t)\to y_{AB}(x))$ where $y_{AB}(x))$ is a unique stationary solution corresponding to the ordinary differential problem $y''-2yy'=0, \quad y(0)=A, \quad y(1)=B$.

Such solutions do exist and the first conjecture was that this limit does not depend on the initial profile $f(x)$.

Note that only bounded solutions (they are, incidentally, non-decreasing) are of interest if (1) is considered on the whole line $x\in\mathbb{R}$. But on $x\in[a,b]$ anyone of the above list suits, providing the singularity is not on the interval. The same is true for a more general (2).

3. Stability of invariant solutions

A solution of the (generalized) Burgers equation

$$u_t = u_{xx} - \alpha u^n u_x \tag{10}$$

with zero boundary conditions

$$u(t,a)=u(t,b)=0, \quad u(0,x)|_{[a,b]}=f(x) \tag{11}$$

monotonically tends to zero as $t\to\infty$ in $\mathcal{L}^2$ norm since

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_a^b u^2dx = \int_a^b 2uu_tdx = 2\int_a^b u(u_{xx}-\alpha u^n u_x)dx \quad &= \qquad (12)\\
2\int_a^b udu_x + \frac{-\alpha}{n+1}u^{n+1}\Big|_a^b = 2uu_x|_a^b - 2\int_a^b u_x^2dx = -2\int_a^b u_x^2dx \quad &\leqslant \quad 0
\end{aligned}
$$

The greater u_x the faster the convergence.

When the boundary conditions are non-zero but constant

$$u(0,x)|_{[a,b]} = f(x) \quad u(t,a) = f(a) = A, \quad u(t,b) = f(b) = B, \tag{13}$$

one may expect the solution to converge to the respective stationary invariant solution, ie, to $\mu(x)$,

$$\mu_{xx} - \alpha\mu^n\mu_x = 0, \quad \mu(a) = A, \quad \mu(b) = B \tag{14}$$

Such a solution exists and is of one of the above listed forms depending on the combination of A and B.

The answer to this hypothesis is complex.

Let us see how evolves the difference between u and the solution of (14). Denote $\nu(t,x) = u(t,x) - \mu(x)$, ie, $u(t,x) = \nu(t,x) + \mu(x)$. Substituting the latter into (10) we get

$$u_t = (\nu(t,x) + \mu(x))_t = \nu(t,x)_t = u_{xx} - \alpha u^n u_x \quad = \tag{15}$$
$$(\nu(t,x) + \mu(x))_{xx} - \alpha(\nu(t,x) + \mu(x))^n(\nu(t,x) + \mu(x))_x.$$

In the case $n = 1$ and $\alpha = 2$ it equals $\nu_{xx} - 2\nu\nu_x + [\mu_{xx} - 2\mu\mu_x] - 2\{\nu_x\mu + \nu\mu_x\}$ The expression in square brackets equals zero. So

$$\nu_t = \nu_{xx} - 2\nu\nu_x - 2(\nu\mu)_x. \tag{16}$$

Boundary conditions for ν are zero by definition.

We evaluate the rate of ν by analogy with (12):

$$\langle\nu_t\rangle|_{\mathcal{L}^2} = \frac{\partial}{\partial t}\int_a^b \nu^2 dx = \int_a^b 2\nu\nu_t dx = 2\int_a^b \nu(\nu_{xx} - 2\nu\nu_x - 2(\nu\mu)_x)dx \quad =$$
$$2\int_a^b \nu d\nu_x - \frac{4}{3}\nu^3\Big|_a^b - 4\int_a^b \nu d(\nu\mu) = 2\nu\nu_x|_a^b - 2\int_a^b \nu_x^2 dx - 4\nu(\nu\mu)|_a^b + 4\int_a^b \mu\nu\nu_x dx \quad =$$
$$-2\int_a^b \nu_x^2 dx + 2\int_a^b \mu d\nu^2 = -2\int_a^b \nu_x^2 dx + 2\int_a^b \mu_x\nu^2 dx - 2\,\nu^2\mu\big|_a^b = -2\int_a^b \left(\nu_x^2 + \mu_x\nu^2\right)dx. \tag{17}$$

Thus the monotony of $\mathcal{L}^2$-convergence is not automatically guaranteed; but it surely takes place, for instance in the case $\mu_x \geqslant 0$ (the case of the increasing initial profile, which agrees with the characteristics method). In the case $n > 1$ the corresponding conditions are somewhat less transparent; for instance when $n = 2$

$$\frac{\partial}{\partial t}\int_a^b \nu^2 dx = -2\int_a^b \left(\nu_x^2 - \mu_x\nu^2(\mu_x - \nu_x)\right)dx.$$

It follows that $\mu_x(\mu_x - \nu_x) > 0$ guarantees decay: if such conditions are satisfied the deviation ν decays to zero. When the inequality $\langle\nu_t\rangle|_{\mathcal{L}^2} \geqslant 0$ fails (eg, for decreasing initial profile) the difference ν doesn't necessarily tend to zero. Usually the evolution ends in catastrophe or decay, but sometimes and somehow it stabilizes half-way.

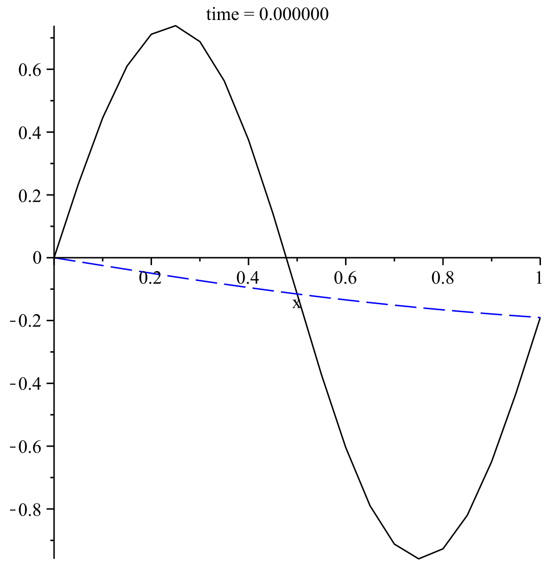

Figure 1. Initial profile $-\varepsilon^2 \tanh(x) + 1.6\varepsilon \sin(2\pi x)$, $u(0,t) = 0$, $u(1,t) = -\varepsilon^2 \tanh(1)$. Asymptotic limit (dash line) is the invariant solution $-\varepsilon^2 \tanh(x)$; n=1

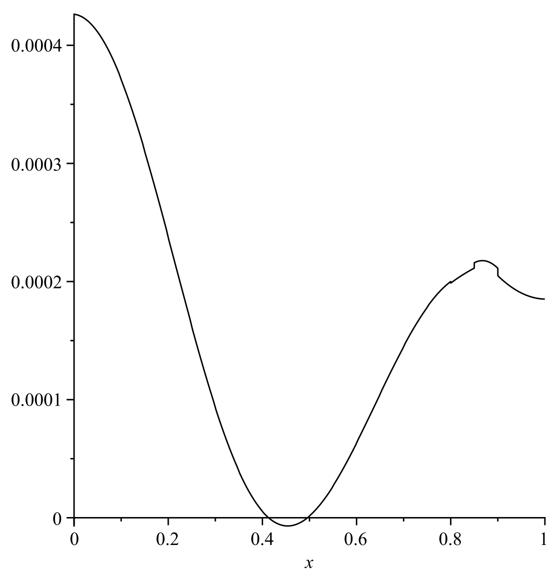

Figure 2. The graph of the integrand $\nu_x^2 + \mu_x \nu^2$ in equation (17), at $t = 2$.

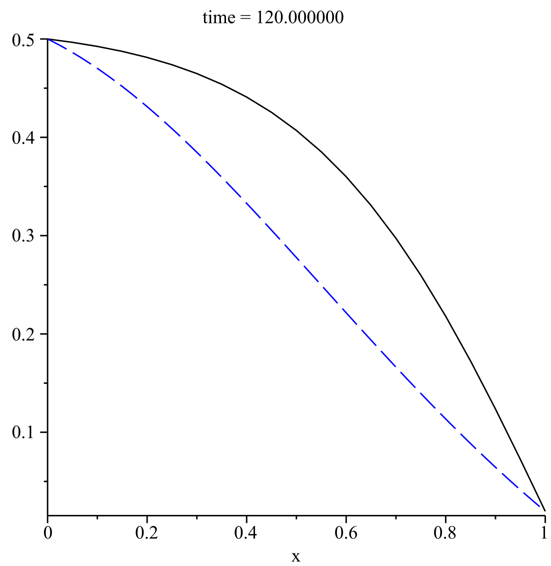

Figure 3. Initial profile $-\varepsilon^2 \tanh(x) + \varepsilon((\mathrm{sech}^2(x))$ (dashed) and asymptotic limit $2.06\varepsilon^2 \tanh(-2.06x + 2.1)$ (solid line).

3.1. Decay

Here are two examples of a decay towards a *decreasing* invariant solution. In both cases the initial profile is chosen in a vicinity of this solution and the right-hand side of (17) is negative.

Consider the equation $u_t = \varepsilon^2 u_{xx} - 2uu_x$.

(i) Choose IVBP: $u(x,0) = -\varepsilon^2 \tanh(x) + 1.6\varepsilon \sin(2\pi x), u(0,t) = 0, u(1,t) = -\varepsilon^2 \tanh(1)$; $\varepsilon = 0.05$. Here $\mu = -\varepsilon^2 \tanh(x)$ is a decreasing invariant solution, $\nu = 1.6\varepsilon \sin(2\pi x)$ — the perturbation. Asymptotics at $t \to \infty$ coincides with μ, see fig. 1. The dissipation reigns in and no catastrophe develops. The explanation can be seen in fig. 2 where the typical graph of integrand $\nu_x^2 + \mu_x \nu^2$ in (17) is given at $t = 2$; clearly $\langle \nu_t \rangle|_{\mathcal{L}^2} < 0$.

(ii) For the same equation choose another IVBP: $u(1,t) = -\varepsilon^2 \tanh(1) + \varepsilon((\mathrm{sech}^2(1))$, $u(0,t) = \varepsilon, u(x,0) = -\varepsilon^2 \tanh(x) + \varepsilon((\mathrm{sech}^2(x))$. The initial profile $u(x,0)$ gives an impression of being in vicinity of the invariant solution $-\varepsilon^2 \tanh(x)$ as it is modestly perturbed by $\varepsilon((\mathrm{sech}^2(x))$. In fact it tends to another (decreasing) invariant solution $2.06\varepsilon^2 \tanh(-2.06x + 2.1)$, see fig. 3.

3.2. Catastrophe

As it is known, for a general quasilinear transport equation ($x \in \mathbb{R}$)

$$w_t + f(w)w_x = 0 \tag{18}$$

the moment of gradient catastrophe can be defined as follows. Let $w = \varphi(x)$ be an initial profile. The solution of this problem may be given in a parametric form $w = \varphi(\xi)$, $x = \xi + \mathcal{F}(\xi)t$ where $\mathcal{F} = f(\varphi(\xi))$. The characteristics of the form $x = \xi + \mathcal{F}(\xi)t$ intersect in the case $\varphi'(\xi) < 0$ thus resulting in many-valued w (the tilting of a wave or a gradient catastrophe). If the inequality holds on a finite interval there exist a minimal value of time, t_c, when this problem arises. One may determine t_c by the formula $t_c = -1/\mathcal{F}'(\xi_c)$ where $|\mathcal{F}(\xi_c)| = \max|\mathcal{F}'(\xi)|$ on the interval $[\alpha, \beta]$ while $\mathcal{F}'(\xi) < 0$.

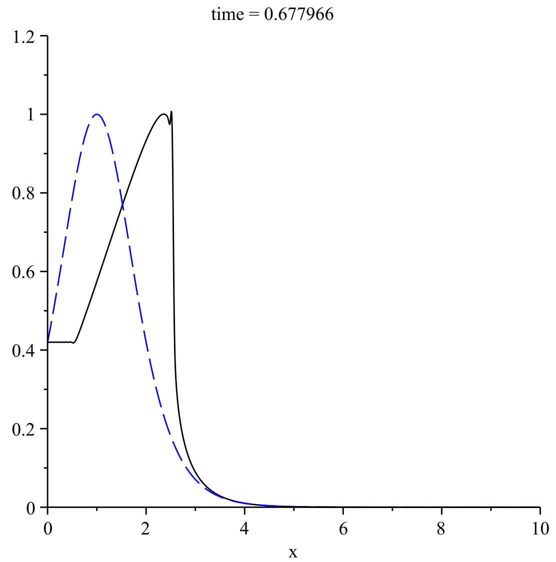

Figure 4. Start of gradient catastrophe at $t_c \approx 0.67$. Dash line is the initial profile $\mathrm{sech}^2(x-1)$. n=1

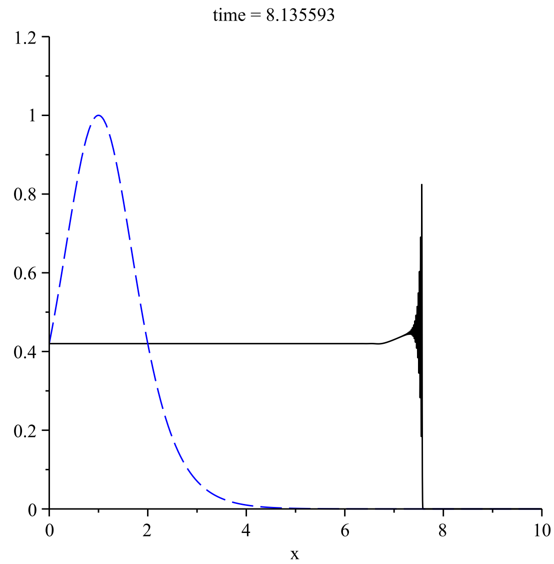

Figure 5. Multi-oscillations move to a Heaviside-type break $\tanh^2(1) - \tanh^2(9)$ at $x = 10$; $t \approx 8$.

We demonstrate this gradient catastrophe to be inherited by Burgers-like equations for some initial profiles, with modest dissipative effects added to a model (18); (cf [2, 1] dealing with a formation of dispersive shocks in a different class of extension of (18), namely Hamiltonian dispersive regularizations of (18) including KdV-likes and Kawahara equations.

In a complex environment of a finite interval combined with an added dissipation for the Burgers-like equation the catastrophe may be delayed or occur earlier, still the main features remain. We begin with the Burgers equation $u_t(x,t) = \varepsilon^2 u_{xx}(x,t) - 2u(x,t)u_x(x,t)$, with IVBP $\{u(x,0) = \mathrm{sech}^2(x-1), u(0,t) = \mathrm{sech}^2(1), u(10,t) = \mathrm{sech}^2(9)\}$ and $\varepsilon = 0.02$. The initial peak

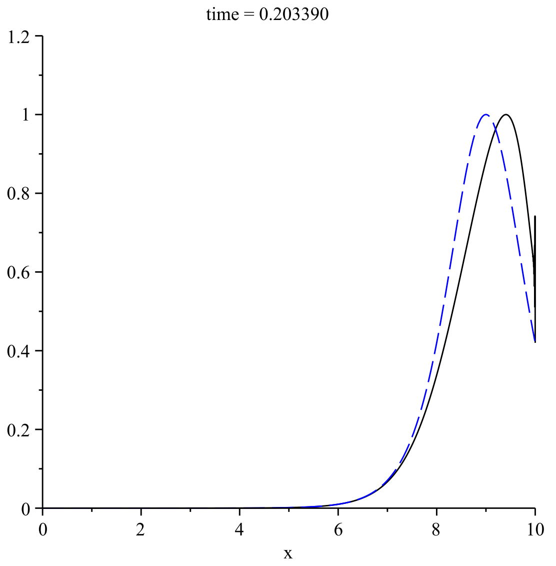

Figure 6. Earlier stage of multi-oscillations at $t_c \approx 0.2$. Dash line is the initial profile $\operatorname{sech}^2(x-9)$. n=1

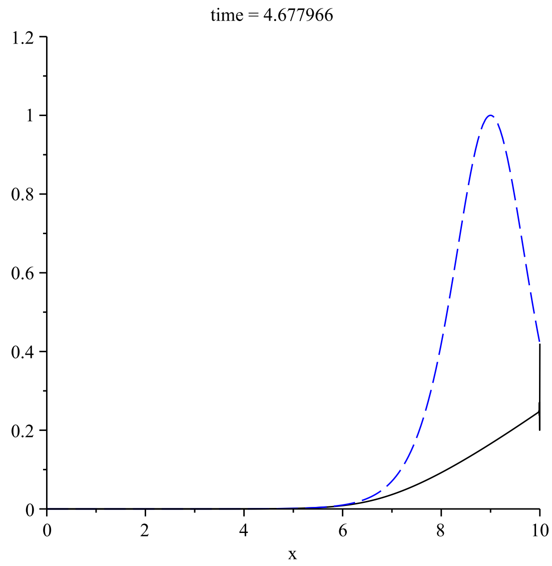

Figure 7. Multi-oscillations move to a Heaviside-type break $-\tanh^2(1)+\tanh^2(9)$ at $x=10$; $t \approx 4.7$.

moves to the right from the far left of the interval, so the moment t_c nearly coincides with that for a whole line. The ensuing multi-oscillating process results in a Heaviside-type break between boundary values at the right end of the interval, fig. 4, 5. Note that constants *are* invariant solutions.

In another example we change IVBP of the previous problem for $\{u(x,0) = \operatorname{sech}^2(x-9), u(0,t) = \operatorname{sech}^2(9), u(10,t) = \operatorname{sech}^2(1)\}$. Here, the right end of the interval being nearer, the catastrophe begins earlier, at $t \approx 0.1$. At $t = 0.2$ we see multi-oscillations fig. 6, gradually developing into a breakup, fig. 7.

Now take the generalized Burgers equation $u_t(x,t) = \varepsilon^2 u_{xx}(x,t) - 2u^2(x,t)u_x(x,t)$ and IVBP $\{u(x,0) = \operatorname{sech}^2(x-1), u(0,t) = \operatorname{sech}^2(1), u(10,t) = \operatorname{sech}^2(9)\}$ and $\varepsilon = 0.02$ as in the first example. The gradient catastrophe starts at $t = 0.45$ in agreement with characteristics method) and likewise develops into a breakup, fig. 8 and 9.

This is not a behavior specific for the sech^2-type initial data. In yet one more example change the IVBP of the previous example for $u(x,0) = -0.01x^2 + 0.9, u(0,t) = 0.9, u(10,t) = -0.1$. The overall picture changes only slightly, fig. 10, though the catastrophe starts at $t = 3.9$, much later than $t = 1,9$ predicted by the characteristics method.

3.3. Stabilization and/or frozen multi-oscillation

In some cases the evolution of the initial profile results early and clearly *not* in an invariant solution from the list (4), see for example fig. 11 with IVBP $\{u(x,0) = -k\varepsilon^2 \tanh(k)(2x^4 - x^2), u(0,t) = 0, u(1,t) = -k\varepsilon^2 \tanh(k)\}$, for $\varepsilon = 0.05$; $k = 50$.

The effect is stable, as the final profile (solid line) in this example does not change if the initial one is perturbated, provided boundary data is the same: identical asymptotics are obtained for $u(x,0) = -k\varepsilon^2 \tanh(k)x$ or $-k\varepsilon^2 \tanh(k)x^2$ (note that the invariant solution with the same boundary values is $\mu(x) = -k\varepsilon^2 \tanh(kx)$). The stabilization may be rather quick. The graph of $\mathcal{L}^2$-estimate for the difference ν, $\int_0^1 (u(s,t) - \mu(s))^2\, ds$ is presented in fig. 13.

The reason for his effect is not entirely clear. Calculus of variations suggests to seek such a

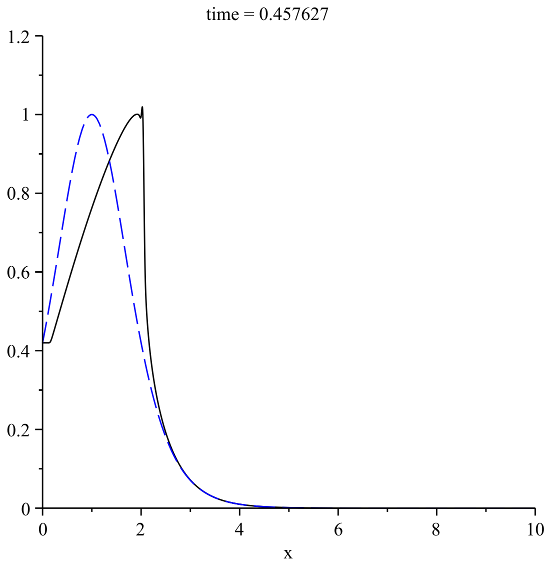

Figure 8. Start of gradient catastrophe at $t_c \approx 0.45$. Dash line is the initial profile $\operatorname{sech}^2(x-1)$. n=2

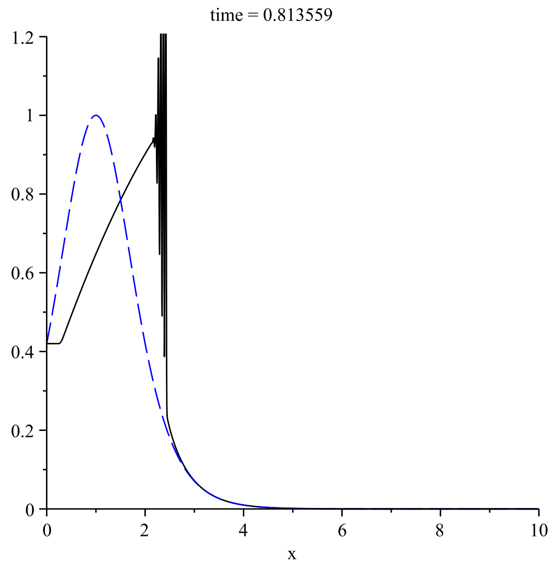

Figure 9. Multi-oscillations move to a Heaviside-type break $-\tanh^2(1)+\tanh^2(9)$ at $x = 10$; $t \approx 0.8$.

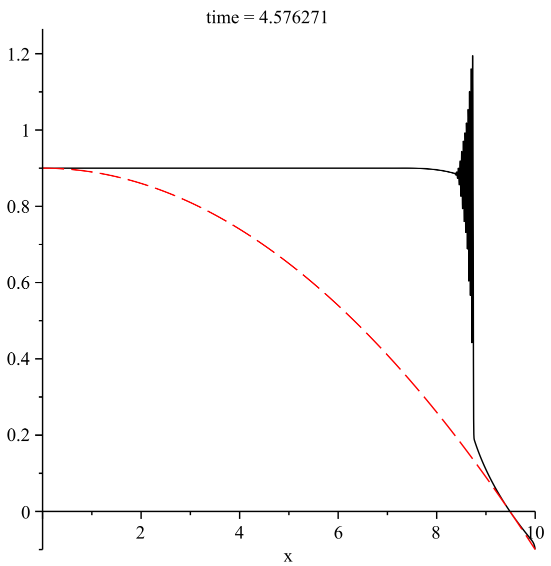

Figure 10. Multi-oscillations move to a Heaviside-type break at $x = 10$; $t \approx 4.6$.Dash line is the initial profile $-0.01x^2 + 0.9$. n=2

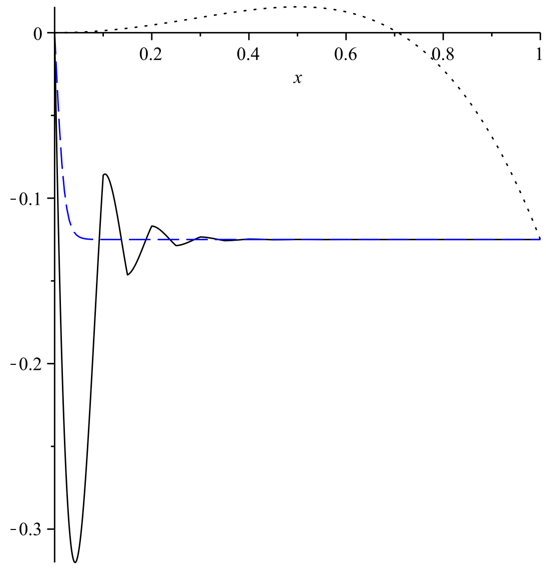

Figure 11. Initial profile (dots line) $-k\varepsilon^2 \tanh(k)(2x^4 - x^2)$, $u(0,t) = 0$, $u(1,t) = -k\varepsilon^2 \tanh(k)$. Asymptotic limit (solid) and the invariant solution (dash) $-k\varepsilon^2 \tanh(kx)$; $\varepsilon = 0.05$, $k = 50$. n=1

stationary point as an extremal of the functional (17).

$$\frac{\partial}{\partial \tau}\bigg|_{\tau=0} \int_a^b \left((\nu + \tau h)_x^2 - \mu_x(\nu + \tau h)^2\right) dx = 0. \tag{19}$$

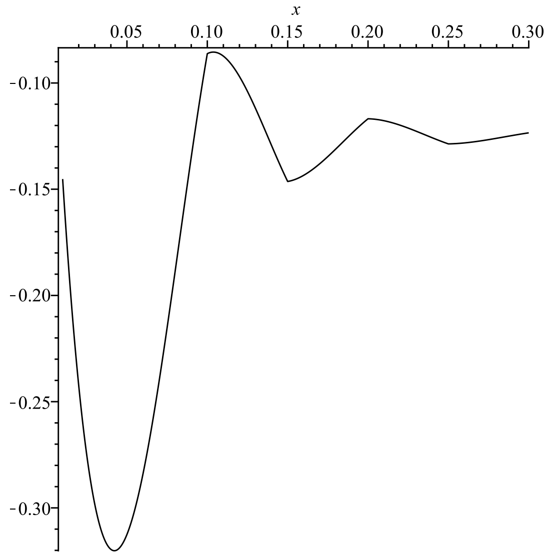

Figure 12. Enlarged part of fig. 11. Piecewise-smooth difference $\nu(x)$; $t = 20$.

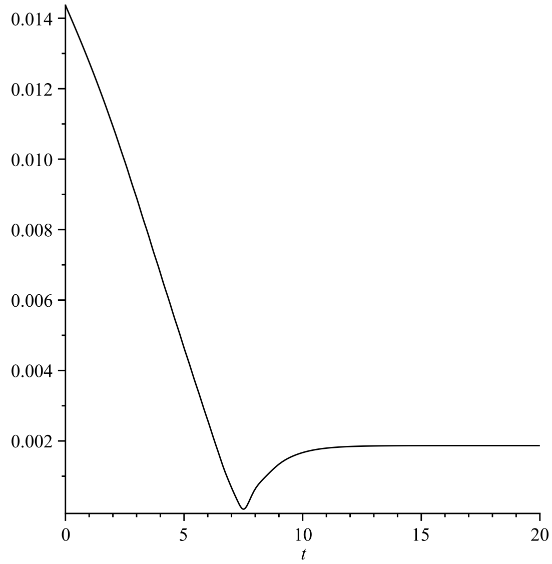

Figure 13. $\mathcal{L}^2$-estimate of difference between invariant $\mu(x) = -\alpha\varepsilon^2 \tanh(\alpha x)$ and stabilizing multi-oscillating solution of fig. 11, 12; n=1.

It follows

$$\nu_{xx} + \mu_x \nu = 0. \tag{20}$$

It is hard to compare the numeric extremal presented on figures 11 to solutions of (20) since these solutions are hard to obtain by numeric methods. The obstacle is that the decreasing solutions of the Burgers equation are of the form $\mu(x) = -k \tanh(kx + c)$ and the potential of the linear problem (20), μ_x, is numerically finite. Some of solutions of (20) are discontinuous (eg, the real part of complex solution of (20) is both discontinuous and multi-oscillating in some cases).

This discontinuity can generate a chaos on the numeric graph and may be a possible reason of a failed smoothness of $\nu(x)$, as presented on figure 12. The graph is composed with parts of different invariant solutions.

The exact mechanism for formations of such exotic solutions is yet to be described in detail; it will be published elsewhere.

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