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On Paramagnetic Phases of the Potts Model on a Bethe Lattice in the Presence of Competing Interactions

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Abstract. We consider a Potts model on a Bethe lattice with competing nearest-neighbour J and next-nearest-neighbour interactions J_p. The phase diagram of this model was studied by Ganikhodjaev *et al.* In this paper we investigate the problem of phase transition for the considered model and show that for some parameter values of the model there is phase transition.

1. Introduction

The Ising model with competing interactions on the Bethe lattice (or the Cayley tree; see [1] for terminology) has received widespread attention by many authors recently since the appearance of the Vannimenus model [2], [3],[4],[5] etc. A Potts model just as an Ising model with competing interactions has recently been studied extensively because of the appearance of nontrivial magnetic ordering (see [6],[7], [8] and reference therein). The Bethe lattice, that is, an infinite connected tree whose sites have the same coordination number, has a thin structure without closed paths but with infinite dimensionality. By introducing competing interactions between Ising or Potts spin variables, assigned to each site, we enable the system to present a very rich phase diagram with many modulated phases. The Vannimenus model [2], that is, the Ising model on a Bethe lattice of coordination number $q = 3$, with ferromagnetic nearest-neighbor interaction and with an antiferromagnetic next-nearest-neighbor interaction (then latter restricted to the sites belonging to the same branch) is the counterpart of the anisotropic next-nearest-neighbor Ising model (ANNNI) model defined on regular lattices [9]. The three states Potts model on the Bethe lattice tree of coordination number $q = 3$ with nearest-neighbor and next-nearest-neighbor interactions (latter restricted to the sites belonging to the same branch) was considered by Ganikhodjaev *et al* [6]. The Hamiltonian of this model is

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)} - J_p \sum_{\rangle \widetilde{x,y} \langle} \delta_{\sigma(x)\sigma(y)}, \tag{1}$$

where $J, J_p \in R$ are coupling constants with $J > 0, J_p < 0$ and $\langle x, y\rangle$ stands for nearest-neighbours vertices and $\rangle \widetilde{x,y} \langle$ stands for next-nearest-neighbor interaction restricted to the sites belonging to the same branch.

As usual, one can introduce the notions of the conditional Gibbs measure, translation-invariant and periodic limiting Gibbs distributions (see [10],[11],[12]). In order to produce the recurrent equations, we consider the relation of the partition function on V_n to the partition function on subsets of V_{n-1} (see details in [6].) Let $Z^{(n)}(i_1, i_0, i_2)$ be a partition function on V_n where the spin in the root x^0 is i_0 and the two spins in the proceeding x^1 and x^2 are i_1 and i_2, respectively. As shown in [6] one can select only five independent variables $Z^{(n)}(1,1,1)$, $Z^{(n)}(2,1,2)$, $Z^{(n)}(1,2,1)$, $Z^{(n)}(2,2,2)$, $Z^{(n)}(3,2,3)$ and with the introduction of new variables

$$\begin{aligned} u_1^{(n)} &= \sqrt{Z^{(n)}(1,1,1)},\ u_2^{(n)} = \sqrt{Z^{(n)}(2,1,2)},\\ u_3^{(n)} &= \sqrt{Z^{(n)}(1,2,1)},\ u_4^{(n)} = \sqrt{Z^{(n)}(2,2,2)},\\ & u_5^{(n)} = \sqrt{Z^{(n)}(3,2,3)}, \end{aligned}$$

straightforward calculations (see more detail [6]) show that

$$\begin{aligned} u_1^{(n+1)} &= a(bu_1^{(n)} + 2u_2^{(n)})^2,\\ u_2^{(n+1)} &= (bu_3^{(n)} + u_4^{(n)} + u_5^{(n)})^2,\\ u_3^{(n+1)} &= (u_1^{(n)} + (b+1)u_2^{(n)})^2,\\ u_4^{(n+1)} &= a(u_3^{(n)} + bu_4^{(n)} + u_5^{(n)})^2,\\ u_5^{(n+1)} &= (u_3^{(n)} + u_4^{(n)} + u_5^{(n)})^2, \end{aligned} \tag{2}$$

and the total partition function is given in terms of (u_i) by

$$Z^{(n)} = (u_1^{(n)} + 2u_2^{(n)})^2 + 2(u_3^{(n)} + u_4^{(n)} + u_5^{(n)})^2, \tag{3}$$

where β is the inverse temperature and $a = \exp(\beta J), b = \exp(\beta J_p)$. We note that, in the paramagnetic phase (high symmetry phase), $u_1 = u_4$ and $u_2 = u_3 = u_5$. For discussing the phase diagram, the following choice of reduced variables is convenient:

$$\begin{aligned} x &= \tfrac{2u_2+u_3+u_5}{u_1+u_4}, \qquad y_1 = \tfrac{u_1-u_4}{u_1+u_4},\\ y_2 &= \tfrac{u_2-u_3}{u_1+u_4}, \qquad y_3 = \tfrac{u_2-u_5}{u_1+u_4}. \end{aligned} \tag{4}$$

The variable x is just a measure of the frustration of the nearest-neighbour bonds and is not an order parameter like y_1, y_2 and y_3. In this case the basic equations have following form:

$$\begin{aligned} x' &= \tfrac{1}{2aD}[P(y_1,y_2,y_3) + ((b+1)x + 2 - y_1 - by_2 - y_3)^2],\\ y_1' &= \tfrac{2}{D}(b+x)(by_1 + y_2 + y_3),\\ y_2' &= -\tfrac{1}{aD}[y_1 + by_2 - y_3][2 + (b+1)x - (b-1)(y_2 - y_3)],\\ y_3' &= \tfrac{1}{aD}(b-1)(y_3 - y_2)[2 + (b+1)x - 2y_1 - (b+1)(y_2 + y_3)], \end{aligned} \tag{5}$$

where

$$\begin{aligned} D &= (b+x)^2 + (by_1 + y_2 + y_3)^2,\\ P(y_1,y_2,y_3) &= 3y_1^2 + (4b^2 - 4b + 3)y_2^2 + (3b^2 - 4b + 4)y_3^2 + 2(2b+1)y_1y_2 + 2(b+2)y_1y_3\\ &- (2b^2 - 7b + 2)y_2y_3. \end{aligned}$$

In the next section we consider the problem of phase transition for considered model (1).

2. Translation-invariant Gibbs Measures:Phase Transitions

To investigate the phase transitions of considered model we consider the dynamic system (2) and study its limiting behaviour. Let $\mathbf{u} = (u_1, u_2, u_3, u_4, u_5) \in R^5_+$ and the dynamic system $F : R^5_+ \to R^5_+$ is defined by

$$\begin{aligned} u_1' &= a\left(bu_1 + 2u_2\right)^2 \\ u_2' &= \left(bu_3 + u_4 + u_5\right)^2 \\ u_3' &= \left(u_1 + (b+1)u_2\right)^2 \\ u_4' &= a\left(u_3 + bu_4 + u_5\right)^2 \\ u_5' &= \left(u_3 + u_4 + bu_5\right)^2, \end{aligned} \tag{6}$$

Then the recurrent equations (2) one can rewrite as $\mathbf{u}^{(n+1)} = F(\mathbf{u}^{(n)})$, $n \geq 0$. Let us describe fixed points of this dynamical system, i.e., solutions of equation $F(\mathbf{u}) = \mathbf{u}$. Denote $\mathrm{Fix}(F) = \{\mathbf{u} : F(\mathbf{u}) = \mathbf{u}\}$.

As noted in [6], in the paramagnetic phase (high symmetry phase), we have $u_1 = u_4$ and $u_2 = u_3 = u_5$. One can see that a set $M = \{\mathbf{u} = (u_1, u_2, u_3, u_4, u_5) \in R^5_+ :\ u_1 = u_4,\ u_2 = u_3 = u_5\}$ is an invariant set of the transformation F. On this set the reduced variables x, y_1, y_2, y_3 (4) have the following form $x = \frac{2u_2}{u_1}, y_1 \equiv 0, y_2 \equiv 0, y_3 \equiv 0$ and the system of equations (4) is reduced to single equation

$$x' = \frac{1}{2a}\left(\frac{(b+1)x+2}{b+x}\right)^2. \tag{7}$$

In this case the measure of the frustration $x^{(n)}$ one can consider as

$$x^{(n)} = 2\sqrt{\frac{Z^{(n)}(1,1,1)}{Z^{(n)}(2,1,2)}}$$

and since a conditional Cibbs measure of the cylinder set $\{x^0 = i_0, x^1 = i_1, x^2 = i_2\}$ is defined as

$$\mu^{(n)}(\{x^0 = i_0, x^1 = i_1, x^2 = i_2\}) = \frac{Z^{(n)}(i_0, i_1, i_2)}{Z^{(n)}}$$

one can conclude that corresponding limit Gibbs measure is fully determined by fixed points of recurrent equation (7). If we define $f : R^+ \to R^+$ by

$$f(x) = \left(\frac{(b+1)x+2}{b+x}\right)^2. \tag{8}$$

then f is bounded and thus the curve $y = f(x)$ must intersect the line $y = 2ax$, i.e., the recurrent equation (7) has a fixed points (see [13]). Note that if there is more than one positive fixed point, then there is more than one translation-invariant paramagnetic Cibbs measure corresponding to these fixed points. It is thus worthwhile to examine how many solutions the equation $f(x) = 2ax$ has.

Lemma 1 The equation

$$\left(\frac{(b+1)x+2}{b+x}\right)^2 = 2ax \tag{9}$$

(with $x \geq 0, a > 0, b > 0$) has one solution if $b < \frac{\sqrt{73}-1}{2}$. If $b > \frac{\sqrt{73}-1}{2}$, then there exist $\nu_1(b)$ and $\nu_2(b)$ with $0 < \nu_1(b) < \nu_2(b)$ such that the equation (9) has three solutions if $\nu_1(b) < 2a < \nu_2(b)$ and has two solutions if either $2a = \nu_1(b)$ or $2a = \nu_2(b)$. In fact

$$\nu_i(b) = \frac{1}{x_i}\left(\frac{(b+1)x_i+2}{b+x_i}\right)^2,$$

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where x_1 and x_2 are the solutions of the equation

$$(b+1)x^2 - (b^2+b-6)x + 2b = 0.$$

Proof As before let

$$f(x) = \left(\frac{(b+1)x+2}{b+x}\right)^2;$$

thus we have

$$f'(x) = 2(b-1)(b+2)\frac{[(b+1)x+2]}{(x+b)^3},$$

$$f''(x) = 2(b-1)(b+2)\frac{[b^2+b-6-2(b+1)x]}{(x+b)^4}.$$

In particular if $b \le 1$ then f is decreasing and there can only be one solution of $f(x) = 2ax$; thus we can restrict ourselves to the case $b > 1$. We have that f is convex for $x < \frac{b^2+b-6}{2(b+1)}$ and is concave for $x > \frac{b^2+b-6}{2(b+1)}$. If $\frac{b^2+b-6}{2(b+1)} < 0$ there can be only be one solution of $f(x) = 2ax$, i.e., $b < 2$. Thus there are at most three solutions to $f(x) = 2ax$ if $\frac{b^2+b-6}{2(b+1)} > 0$, i.e., $b > 2$. In fact it is quite easy to see that there is more than one solution if and only if there is more than one solution to the equation $xf'(x) = f(x)$, which is the same as

$$(b+1)x^2 - (b^2+b-6)x + 2b = 0.$$

The discriminant of this equation is equal to $(b-1)(b+2)(b^2+b-18)$ and it has two positive roots x_1 and x_2 if $b > \frac{\sqrt{73}-1}{2}$. This completes the proof.
Denote

$$T_c = \frac{J_p}{\ln\frac{\sqrt{73}-1}{2}},$$

where $J_p > 0$. Using this Lemma 1 we obtain following
Theorem 1 Let $J_p > 0$. Then if $T \ge T_c$ the model has unique paramagnetic phase and if $T < T_c$ and the conditions of the Lemma 1 are satisfied then for the model there are three paramagnetic translation-invariant Gibbs measures, i.e., there is phase transition.

3. Periodic Gibbs Measures: Phase Transitions

To describe all phases of considered model, one iterates the recurrence relations (5) and observes their behavior after a large number of iterations. In the simplest situation a fixed point $(x^*, y_1^*, y_2^*, y_3^*)$ is reached. It corresponds to a paramagnetic phase if $y_1^* = 0, y_2^* = 0, y_3^* = 0$ or to a ferromagnetic phase if $y_1^*, y_2^*, y_3^* \neq 0$. Secondary, the system may be periodic with period p, where case $p = 2$ corresponds to antiferromagnetic phase and case $p = 4$ corresponds to so-called antiphase, that denoted $< 2 >$ for compactness. Finally, the system may remain aperiodic. The distinction between a truly aperiodic case and one with a very long period is difficult to make numerically. Below we discuss existing of paramagnetic phases with period $p = 2$. Using numerical method in [6] presented new phase, namely, paramodulated phase. In this section we prove existence of such phases and study the phase transition problem. Let us first describe periodic points of the equation $F(F(u)) = u$ with $p = 2$ on the set M. In this case this equation can be reduced to following equation

$$f(f(x)) = x. \qquad (10)$$

To find periodic solutions we consider the equation

$$\frac{f(f(x))-x}{f(x)-x}=0.$$

Simple but tedious algebra gives that the last equation is equivalent to the following

$$[(b+1)^2+2ab]^2x^2+4a[4a^2b^3+a(b^4+2b^3+9b^2+8b-4)+2(b+1)^3]x+4(ab^2+b+1)^2=0 \quad (11)$$

It is evident that the equation $Ax^2+Bx+C=0$ has two positive solutions if $b^4+2b^3+9b^2+8b-4<0$, i.e., $0<b<0.3498$, i.e., $J_p<0$. To prove the existing of two positive roots we need to show that for some values of parameters a and b we have $B<0$ and discriminant $D=B^2-4AC>0$, where

$$\begin{aligned} B &= 4a[4a^2b^3+a(b^4+2b^3+9b^2+8b-4)+2(b+1)^3], \\ D &= 16[4a^3b^2+(b^4+4b^3+9b^2+8b-4)a^2+(b^3+2b^2+4b+1)a+(b+1)^2] \\ &\quad [4a^3b^2+(b^4+9b^2+8b-4)a^2-(b^3+3b^2-2)a-(b+1)^2]. \end{aligned}$$

Consider the quadratic equation (with respect to a) $B=0$, i.e.,

$$4a^2b^3+a(b^4+2b^3+9b^2+8b-4)+2(b+1)^3=0.$$

with discriminant $\Delta_B=(b+1)^2(b-1)^2(b^4+2b^3-11b^2-12b+4)$. For $0<b<0.2702$ we have $\Delta_B>0$ and two positive roots $a_1(b)$ and $a_2(b)$. Then for any $0<b<0.2702$ (since $0<b<0.3498$) and a with $a_1<a<a_2$ we have $B<0$. Similarly one can prove that for any $0<b<0.2702$ there are $a_1^*(b)$ and $a_2^*(b)$ such that for a with $a_1^*(b)<a<a_2^*(b)$ discriminant $D>0$. In fact, equalizing to zero each factor of discriminant D, we consider two equations (with respect to a)

$$4a^3b^2+(b^4+4b^3+9b^2+8b-4)a^2+(b^3+2b^2+4b+1)a+(b+1)^2=0 \quad (12)$$

and

$$4a^3b^2+(b^4+9b^2+8b-4)a^2-(b^3+3b^2-2)a-(b+1)^2=0. \quad (13)$$

Since $0<b<0.2702$ first equation has two positive roots $\bar{a}_1(b)$ and $\bar{a}_2(b)$, such that for any a with $\bar{a}_1(b)<a<\bar{a}_2(b)$, first factor is negative. Second equation has single positive root $a(b)$ such that for any a with $0<a<a(b)$ second factor also is negative. Simple but tedious calculus shows that for any b in $(0,0.2702)$ there are $\tilde{a}_1$ and $\tilde{a}_2$ such that for any a in $(\tilde{a}_1,\tilde{a}_2)$ we have $B<0$ and $D>0$. Thus we have proved the following

Lemma 2 The equation (10) has no solution if $b\geq 0.2702$ and if $b<0.2702$ then there exist $\tilde{a}_1,\tilde{a}_2$ such that the equation has two positive solutions if $\tilde{a}_1<a<\tilde{a}_2$.

Let

$$T_c^{(2)}=\frac{J_p}{\ln 0.2702},$$

where $J_p<0$. From Lemma 2 we get the following

Theorem 2 Let $J_p<0$. Then if $T\geq T_c^{(2)}$ the model has no paramagnetic phase with period 2 and if $T<T_c^{(2)}$ and the conditions of the Lemma 2 are satisfied then for the model there are two paramagnetic Gibbs measures with period 2, i.e., there is phase transition.

Remark The paramagnetic Gibbs measure with period 2 was found in [6] numerically and called paramodulated phase.

4. Conclusion

It is proved the existence of phase transition for translation-invariant paramagnetic Gibbs measures when $J_p > 0$ and for periodic with period 2 paramagnetic Gibbs measures when $J_p < 0$. These results fully consistent with numerical results in [6].

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