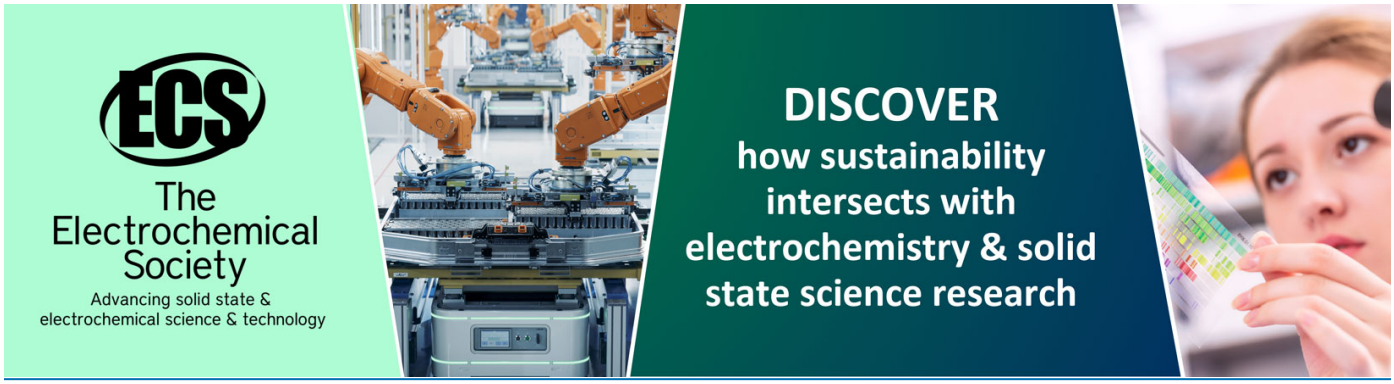
Two Potts Models on Cayley Tree of Arbitrary Order



Journal of Physics: Conference Series **435** (2013) 012033 doi:10.1088/1742-6596/435/1/012033

Two Potts Models on Cayley Tree of Arbitrary Order

Nasir Ganikhodjaev[1], Ashraf Mohamed Nawi, and Mohd Hirzie Mohd Rodzhan

[1]Department of Computational and Theoretical Sciences, Faculty of Science, IIUM, 25200 Kuantan, Malaysia.

E-mail: gnasir@iium.edu.my

Abstract. On a Cayley tree of arbitrary order k we consider two different Potts models with competing nearest-neighbour interactions J_1 and next-nearest-neighbour interactions J_p and J_o, where coupling J_p corresponds to interaction of spins belonging to the same branch of the tree (prolonged) and coupling J_o corresponds to interaction of spins belonging to the same shell of the tree (one-level) and find for each model in addition to the expected paramagnetic, ferromagnetic and antiferromagnetic phases, an intermediate range of coupling values where the local magnetization has chaotic oscillatory glass-like behaviour. We also show that the ranges corresponding two models are different.

1. Introduction

Consideration of spin models with multispin interactions has proved to be fruitful in many fields of physics, ranging from the determination of phase diagrams in metallic alloys and exhibition of new types of phase transition, to site percolation. Systems exhibiting spatially modulated structures, commensurate or incommensurate with the underlying lattice, are of current interest in condensed matter physics [1]. Among the idealized systems for modulated ordering, the axial next-nearest-neighbour Ising (ANNNI) model, originally introduced by Elliot [2] to describe the sinusoidal magnetic structure of Erbium, and the chiral Potts model, introduced by Ostlund [3] and Huse [4] in connection with monolayers adsorbed on rectangular substrates, have been studied extensively by a variety of techniques. A particularly interesting and powerful method is the study of modulated phases through the measure-preserving map generated by the mean-field equations, as applied by Bak [5] and Jensen and Bak [6] to the ANNNI model. The main drawback of the method lies in the fact that thermodynamic solutions correspond to stationary but unstable orbits. However, when these models are defined on a Cayley tree (or on a the Bethe lattice; see [7] for terminology), as in the case of the Ising model with competing interactions examined by Vannimenus [8], it turns out that physically interesting solutions correspond to the attractors of the mapping. This simplifies the numerical work considerably, and detailed study of the whole phase diagram becomes feasible. Apart from the intrinsic interest attached to the study of models on trees, it is possible to argue that the results obtained on trees provide a useful guide to the more involved study of their counterparts on crystal lattices.

The ANNNI model, which consists of an Ising model with nearest-neighbour interactions augmented by competing next-nearest-neighbour couplings acting parallel to a single axis direction, is perhaps the simplest nontrivial model displaying a rich phase diagram with a Lifshitz point and many spatially modulated phases. There has been a considerable theoretical effort to

obtain the structure of the global phase diagram of the ANNNI model in the $T-p$ space, where T is temperature and $p = -J_2/J_1$ is the ratio between the competing exchange interactions. On the basis of numerical mean-field calculations, Bak and von Boehm [9] suggested the existence of an infinite succession of commensurate phases, the so-called devil's staircase, at low temperatures. This mean-field picture has been supported by low-temperature series expansions performed by Fisher and Selke [11]. At the paramagnetic-modulated boundary analytic mean-field calculations show that the critical wave number varies continuously and vanishes at the Lifshitz point.

A phase diagram of a model describes a morphology of phases, stability of phases, transitions from one phase to another and corresponding transition lines. The Cayley tree is not a realistic lattice; however, its amazing topology makes the exact calculation of various quantities possible. For many problems the solution on a tree is much simpler than on a regular lattice and is equivalent to the standard Bethe-Peierls theory [20]. On the Cayley tree one can consider two type of next-nearest-neighbours: prolonged and one-level next-nearest-neighbours (definitions see below). In the case of the Ising model with competing nearest-neighbor interactions J and prolonged next-nearest-neighbour interactions J_p Vannimenus [8] was able to find new modulated phases, in addition to the expected paramagnetic and ferromagnetic ones. From this result follows that Ising model with competing interactions on a Cayley tree is real interest since it has many similarities with models on periodic lattices. In fact, it has many common features with them, in particular the existence of a modulated phase, and shows no sign of pathological behavior - at least no more than mean-field theories of similar systems [8]. Moreover a detailed study of its properties was carried out with essentially exact results, using rather simple numerical methods.

This suggest that more complicated models should be studied on trees, with the hope to discover new phases or unusual types of behavior. The important point is that statistical mechanics on trees involve nonlinear recursion equations and are naturally connected to the rich world of dynamical systems, a world presently under intense investigation [8].

The Potts model (with $q \geq 2$ spin values) was introduced as a generalization of the Ising model [26]. At present the Potts model encompasses a number of problems in statistical physics and lattice theory (see for example [27]). A Potts model just as an Ising model with competing interactions has recently been studied extensively because of the appearance of nontrivial magnetic orderings (see [8], [12],[13],[15],[17],[18],[19],[16],[21],[22],[23],
[28],[29] and references therein). The Potts model with competing interactions J_1 and prolonged next-nearest neighbours interactions J_p on Cayley tree of second order have been studied in detail in [19], and Potts model that include also interaction of one-level next-nearest-neighbour interaction J_o have been studied in [28].
In this paper we define a single-trunk Cayley tree, produce the recursion equations for model with competing interactions on the Cayley tree and for the same model on the single-trunk Cayley tree, and show how to reduce the recursion equations on Cayley tree to the simpler recursion equations on the single-trunk Cayley tree. Note that this approach is mathematical justification only of the method developed in [8] and later generalized in [12]. We will consider also the symmetry group of the corresponding model [10].
The aim of this paper is to extend the results of [28] to the Potts model with competing interactions on a Cayley tree of arbitrary order k and to clarify the role of the order k. The paper has been organized in the following way. In Section 2, the model Hamiltonian is discussed. In Section 3, the recursion equations are defined. Section 4 is devoted to the discussion of the phase diagram features. Finally, the conclusions are given in Section 5.

2. The Model Hamiltonian

2.1. Cayley Tree

A Cayley tree Γ^k of order $k \geq 1$ (or the Bethe lattice; see [7] for terminology) is an infinite tree, i.e., a graph without cycles with exactly $k+1$ edges issuing from each vertex. Let denote the Cayley tree as $\Gamma^k = (V, \Lambda)$, where V is the set of vertices of Γ^k, Λ is the set of edges of Γ^k. Two vertices x and y, $x, y \in V$ are called *nearest-neighbors* if there exists an edge $l \in \Lambda$ connecting them, which is denoted by $l =< x, y >$. The distance $d(x, y), x, y \in V$, on the Cayley tree Γ^k, is the number of edges in the shortest path from x to y. For a fixed $x^0 \in V$ we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \{x \in V | d(x, x^0) \leq n\}$$

and L_n denotes the set of edges in V_n. The fixed vertex x^0 is called the 0-th level and the vertices in W_n are called the n-th level. For the sake of simplicity we put $|x| = d(x, x^0)$, $x \in V$. Two vertices $x, y \in V$ are called *the next-nearest-neighbours* if $d(x, y) = 2$. The next-nearest-neighbour vertices x and y are called *prolonged next-nearest-neighbours* if $|x| \neq |y|$ and is denoted by $> \widetilde{x, y} <$. The next-nearest-neighbour vertices $x, y \in V$ that are not prolonged are called *one-level next-nearest-neighbours* since $|x| = |y|$ and are denoted by $> \bar{x, y} <$.

We write $x \prec y$ if the path from x^0 to y goes through x. We call the vertex y a direct successor of x, if $y \succ x$ and x, y are nearest neighbours. The set of the direct successors of x is denoted by $S(x)$, i.e., if $x \in W_n$, then

$$S(x) = \{y_i \in W_{n+1} | d(x, y_i) = 1, i = 1, \cdots, k\}.$$

We observe that for any vertex $x \neq x^0, x$ has k direct successors and x^0 has $k+1$.

The collection $S(x) = \{y_1, \cdots, y_k\}$ we will call one-level k tuple of neighbours. Note that if $k = 2$ then $S(x)$ is a one-level next-nearest-neighbours.

Below we will consider a semi-infinite Cayley tree Γ^k_+ of k-th order, i.e. an infinite graph without cycles with $k+1$ edges issuing from each vertex except for x^0 which has only k edges. In this case $|S(x)| = k$ for any $x \in V$. Let $< x^0, x >= l \in L$ be an edge of semi-infinite Cayley tree Γ^k_+.

Definition 2.1 The infinite subtree $\Gamma^k_+(l) = (V^l, L^l)$ is called a *single-trunk Cayley tree*, if from vertex x^0 a single edge l emanates and from any other vertex $x \in V^l, x \neq x^0$ exactly $k+1$ edges emanate.

Let $W_1 = \{x_1, x_2, \cdots x_k\}$ and $< x^0, x^1 >= l_1, < x^0, x^2 >= l_2, \cdots < x^0, x^k >= l_k$ be k edges emanating from x^0. It is evident that semi-infinite Cayley tree Γ^k_+ splits into k components - k single-trunk Cayley trees $\Gamma^k_+(l_i), i = 1, 2 \cdots, k$. Let V^{l_i} is the set of vertices of single-trunk Cayley tree $\Gamma^k_+(l_i)$ and $V^{l_i}_n = V_n \cap V^{l_i}$ is the set of vertices $x \in V^{l_i}$ with $d(x^o, x) \leq n$.

2.2. The Models

Let Γ^k_+ be a semi-finite Cayley tree of order k. In addition to nearest-neighbours interaction J_1 and prolonged next-nearest -neighbours interaction J_p we consider also interactions of sites in the set $S(x) = \{y_1, \cdots, y_k\}$ of direct successors for any $x \in V$. Here one can introduce two type interactions:

1) jointly interaction of all sites in $S(x)$, called one-level k-tuple interaction ;

2) binary interaction of any two sites in $S(x) = \{y_1, \cdots, y_k\}$, called one-level next-nearest neighbours interaction.

For the Potts model with spin values in $\Phi = \{1, 2, 3\}$, i.e., the spin variables $\sigma(x), x \in V$ assume the values $\{1, 2, 3\}$, the relevant Hamiltonians with competing nearest-neighbor interactions J_1, prolonged next-nearest-neighbor interactions J_p and one-level k-tuple neighbours or one-level next-nearest neighbors interaction J_o have the forms:

1) Model with one-level k-tuple neighbours interactions

$$H(\sigma) = -J_o \sum_{y_i \in S(x)} \delta_{\sigma(y_1)\sigma(y_2)\cdots\sigma(y_k)} - J_p \sum_{\widetilde{>x,y<}} \delta_{\sigma(x)\sigma(y)} - J_1 \sum_{<x,y>} \delta_{\sigma(x)\sigma(y)}, \tag{1}$$

2) Model with one-level next-nearest neighbor interactions

$$H(\sigma) = -J_o \sum_{y_i, y_j \in S(x): i \neq j} \delta_{\sigma(y_i)\sigma(y_j)} - J_p \sum_{\widetilde{>x,y<}} \delta_{\sigma(x)\sigma(y)} - J_1 \sum_{<x,y>} \delta_{\sigma(x)\sigma(y)}, \tag{2}$$

where $J_0, J_p, J_1 \in R$ are coupling constant, δ in second and third sum in (1) and all sums in (2) is the usual Kronecker symbol and δ in first sum in (1) is the generalized Kronecker symbol, that is defined as follow:

$$\delta_{\sigma(y_1)\sigma(y_2)\cdots\sigma(y_q)} = \begin{cases} 1 & \text{if } \sigma(y_1) = \sigma(y_2) = \cdots = \sigma(y_k) \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $k = 2$ generalized Kroneker's symbol coincides with usual one and one-level 2-tuple of neighbours coincide with one-level next-nearest neighbours. These models recover that in [19] for $k = 2$ with $J_o = 0$ and in [28] for $k = 2$ with $J_o \neq 0$ and they coincide if $J_o = 0$. The generalized Kroneker's symbol was introduced in [24]. The Ising model with one-level k-tuple interactions was considered in [23] and Ising model with one-level next-nearest-neighbours interactions was considered in [30].

2.3. Conditional Gibbs measures with symmetries

Recall that in the case of the Ising model with competing interactions examined by Vannimenus [8], it turns out that physically interesting solutions correspond to the attractors of the recurrence relations, where initial conditions chosen randomly. The main drawback of the method lies in the fact that thermodynamic solutions correspond to stationary but unstable orbits. In this paper we produce recurrence relations fixing boundary configuration $\bar{\sigma}(V \setminus \Lambda)$.

Let Λ be a finite subset of V. We will denote by $\sigma(\Lambda)$ the restriction of a configuration $\sigma : V \to \Phi = \{1, 2, 3\}$ to Λ. Let $\bar{\sigma}(V \setminus \Lambda)$ be a fixed boundary configuration. As usual, one can introduce the notions of total energy $H(\sigma(\Lambda)|\bar{\sigma}(V \setminus \Lambda))$ of configuration $\sigma(\Lambda)$ under boundary condition $\bar{\sigma}(V \setminus \Lambda)$ and partition function $Z_\Lambda(\bar{\sigma}(V \setminus \Lambda))$ in volume Λ under boundary condition $\bar{\sigma}(V \setminus \Lambda)$ that is defined as

$$Z_\Lambda(\bar{\sigma}(V \setminus \Lambda)) = \sum_{\sigma(\Lambda) \in \Omega(\Lambda)} \exp(-\frac{1}{k_B T} H(\sigma(\Lambda)| \ \ \bar{\sigma}(V \setminus \Lambda))), \tag{3}$$

where $\Omega(\Lambda)$ is the set of all configurations in volume Λ, k_B is Boltzmann constant and T is the absolute temperature. Then conditional Gibbs measure μ_Λ of a configuration $\sigma(\Lambda)$ is defined as

$$\mu_\Lambda(\sigma(\Lambda)| \ \ \bar{\sigma}(V \setminus \Lambda)) = \frac{\exp(-\beta H(\sigma(\Lambda)| \ \ \bar{\sigma}(V \setminus \Lambda))}{Z_\Lambda(\bar{\sigma}(V \setminus \Lambda))}$$

We consider the configuration $\sigma(V_n)$, the partition function $Z_{V_n}(\bar{\sigma}(V \setminus V_n))$ and conditional Gibbs measure $\mu_{V_n}(\sigma(V_n)|\bar{\sigma}(V \setminus V_n))$ in volume V_n and for brevity denote them as $\sigma_n, Z^{(n)}$ and μ_n respectively. Let S_3 be a group of all permutations of a set $\Phi = \{1, 2, 3\}$. **Definition 2.2** For any permutation $\pi \in S_3$ let us define transformation $T_\pi : \Omega \to \Omega$ by the following way: for any $\sigma \in \Omega$ assume $T_\pi \sigma(x) = T_\pi(\sigma(x))$ for any $x \in V$.

It is evident the following Proposition.

Proposition 2.1 The Hamiltonians (1) and (2) are S_3 invariant, i.e., for any $\pi \in S_3$ we have $H(T_\pi \sigma) = H(\sigma)$ for any $\sigma \in \Omega$.
In order to produce the recurrence equations, we consider the relation of the partition function on V_{n+1} to the partition function on subsets of V_n. Given the initial condition $\sigma(x^0) = i_0$ on $V_0 = \{x^0\}$, where x^0 is the root of the Cayley tree, the recurrence equations indicate how their influence propagates down the tree. Below we consider following partition functions.

Let $Z_i^{(n)}$ be a partition function on V_n with the spin i in the root x^0, $i = 1, 2, 3$, i.e.,

$$Z_i^{(n)} = \sum_{\sigma_n \in \Omega_n: \ \sigma_n(x^0)=i} \exp(-\beta H(\sigma_n| \ \bar{\sigma}(V \setminus V_n))), \tag{4}$$

with

$$Z^{(n)} = Z_1^{(n)} + Z_2^{(n)} + Z_3^{(n)},$$

where $n = 0, 1, 2, \cdots$.

Let

$$\sigma_S = \begin{pmatrix} \sigma(y_1), \sigma(y_2), \cdots, \sigma(y_k) \\ \sigma(x^0) \end{pmatrix}$$

be a configuration on the set $S = x^0 \cup S(x^0)$ and $\Omega(S)$ be the set of all such configurations. Assume

$$Z^{(n)}(\sigma_S) = Z^{(n)} \begin{pmatrix} \sigma(y_1), \sigma(y_2), \cdots, \sigma(y_k) \\ \sigma(x^0) \end{pmatrix}$$

be the partition function on V_n with fixed configuration σ_S on V_1. There are a priori 3^{k+1} different partition functions $Z^{(n)}(\sigma_S)$ and the partition function $Z^{(n)}$ in volume V_n can be written as follows:

$$Z^{(n)} = \sum_{\sigma_S \in \Omega(S)} Z^{(n)}(\sigma_S). \tag{5}$$

Lastly let $Z^{(n)}(i, j)$ be a partition function on the single-trunk Cayley tree V_n^l with the configuration (i, j) on an edge $l =< x^0, y >$, where $y \in W_1$ and $i, j = 1, 2, 3$.

For each model there exists corresponding correlations between the partition functions $Z^{(n)}(i, j)$ and $Z^{(n)}(\sigma_S)$.
1) For first model with one-level k-tuple interactions

$$Z^{(n)}(\sigma_S) = \exp\left(\frac{J_0}{k_B T} \delta_{\sigma(y_1)\sigma(y_2)\cdots\sigma(y_k)}\right) \cdot \prod_{i=1}^{k} Z^{(n)}(\sigma(x^0), \sigma(y_i)); \tag{6}$$

2) For second model with one-level next-nearest neighbor interactions

$$Z^{(n)}(\sigma_S) = \exp\left(\frac{J_0}{k_B T} \sum_{i<j} \delta_{\sigma(y_i)\sigma(y_j)}\right) \cdot \prod_{i=1}^{k} Z^{(n)}(\sigma(x^0), \sigma(y_i)). \tag{7}$$

Assume that

$$a = \exp\left(\frac{J_o}{k_B k T}\right); b = \exp\left(\frac{J_p}{k_B T}\right); c = \exp\left(\frac{J_1}{k_B T}\right);$$

where k_B is Boltzmann constant and T is the absolute temperature.
We may assume that the different branches are equivalent, as is usual done for models on tree. Let us fix boundary configuration $\bar{\sigma}_n$ on $V \setminus V_n$ as follows: $\bar{\sigma}_n(x) \equiv i$ for any $x \in V \setminus V_n$, where $i = 1, 2, 3$. It is evident the following Proposition.
Proposition 2.2 For fixed boundary configuration $\sigma(x) \equiv 1$ for any $x \in V \setminus V_n$, the Hamiltonian

$H(\sigma(\Lambda)|\bar{\sigma}(V \setminus \Lambda))$ is S_3^* invariant, where $S_3^* = \{\pi \in S_3 : \pi(1) = 1\}$ is subgroup of group S_3.

Corollary 2.1 The partition functions $\{Z^{(n)}(i,j) : i,j = 1,2,3\}$ satisfy following relations:

$$\begin{array}{l} Z^{(n)}(1,2) = Z^{(n)}(1,3); \\ Z^{(n)}(2,1) = Z^{(n)}(3,1); \\ Z^{(n)}(2,2) = Z^{(n)}(3,3); \\ Z^{(n)}(2,3) = Z^{(n)}(3,2); \end{array} \tag{8}$$

i.e., the following 5 variables

$$Z^{(n)}(1,1); Z^{(n)}(1,2); Z^{(n)}(2,1); Z^{(n)}(2,2); Z^{(n)}(2,3)$$

are independent. Then from (6) and (7) follow that we can select only 5 independent variables from 3^{k+1} partition functions, the same for both models (1) and (2), namely:

$$\begin{array}{c} Z^{(n)}(1,1,1,\cdots,1),\ Z^{(n)}(1,2,2,\cdots,2),\ Z^{(n)}(2,1,1,\cdots,1), \\ Z^{(n)}(2,2,2,\cdots,2),\ Z^{(n)}(2,3,3,\cdots,3). \end{array}$$

We note that, in the paramagnetic phase (high symmetry phase, i,e., symmetry with respect to S_3) we have

$$Z^{(n)}(1,1) = Z^{(n)}(2,2);$$

and

$$Z^{(n)}(1,2) = Z^{(n)}(2,1) = Z^{(n)}(2,3)$$

and respectively

$$Z^{(n)}(1,1,1,\cdots,1) = Z^{(n)}(2,2,2,\cdots,2)$$

and

$$Z^{(n)}(1,2,2,\cdots,2) = Z^{(n)}(2,1,1,\cdots,1) = Z^{(n)}(2,3,3,\cdots,3).$$

3. Basic Equations

In order to set up our basic equations in a recurrence scheme relating the partition function of an n- generation tree to the partition functions of its subsystems, we should take into account the partial partition functions for all the possible configurations of the spins in two successive generations. Below we produce the recurrent equations relating the partition functions $Z^{(n)}(i,j)$ for Hamiltonian considered on the single-trunk Cayley tree.

3.1. First Model

Firstly we consider Potts model (1) with competing prolong next-nearest-neighbour interactions, nearest-neighbour interactions and one level k-tuple neighbour interactions. We introduce 5 new variables

$$\begin{array}{l} u_1^{(n)} = \sqrt[q]{Z^{(n)}(1,1,1,\cdots,1)} = a\left[Z^{(n)}(1,1)\right] \\ u_2^{(n)} = \sqrt[k]{Z^{(n)}(1,2,2,\cdots,2)} = a\left[Z^{(n)}(1,2)\right] \\ u_3^{(n)} = \sqrt[k]{Z^{(n)}(2,1,1,\cdots,1)} = a\left[Z^{(n)}(2,1)\right] \\ u_4^{(n)} = \sqrt[k]{Z^{(n)}(2,2,2,\ldots,2)} = a\left[Z^{(n)}(2,2)\right] \\ u_5^{(n)} = \sqrt[k]{Z^{(n)}(2,3,3,\ldots,3)} = a\left[Z^{(n)}(2,3)\right] \end{array}$$

where k is the order of the Cayley tree.
Let $Z^{(n)}(i_0, i_1, i_2, \cdots, i_k)$ be a partition function and let m_1 be the number of spins $\{1\}$ and m_2 be the number of spins $\{2\}$ on first level W_1. Then we have

$$Z^{(n)}\left(1, i_1, i_2, \cdots, i_k\right) = a^{-k}\left(u_1^{m_1} u_2^{k-m_1}\right)$$
$$Z^{(n)}\left(2, i_1, i_2, \cdots, i_k\right) = a^{-k}\left(u_3^{m_1} u_4^{m_2} u_5^{q-(m_1+m_2)}\right)$$

One gets the following system of recurrent equations through direct calculation:

$$\begin{aligned}
u_1^{(n+1)} &= a^{1-k} c\left[\left(a^k-1\right)\left(b^k u_1^k + 2u_2^k\right) + \left(bu_1 + 2u_2\right)^k\right], \\
u_2^{(n+1)} &= a^{1-k}\left[\left(a^k-1\right)\left(b^k u_3^k + u_4^k + u_5^k\right) + \left(bu_3 + u_4 + u_5\right)^k\right], \\
u_3^{(n+1)} &= a^{1-k}\left[\left(a^k-1\right)\left(u_1^k + \left(b^k+1\right)u_2^k\right) + \left(u_1 + (b+1)\,u_2\right)^k\right], \\
u_4^{(n+1)} &= a^{1-k} c\left[\left(a^k-1\right)\left(u_3^k + b^k u_4^k + u_5^k\right) + \left(u_3 + bu_4 + u_5\right)^k\right], \\
u_5^{(n+1)} &= a^{1-k}\left[\left(a^k-1\right)\left(u_3^k + u_4^k + b^k u_5^k\right) + \left(u_3 + u_4 + bu_5\right)^k\right].
\end{aligned} \tag{9}$$

The total partition function is given in terms of (u_i) by
$$Z^{(n)} = \left(a^k-1\right)\left[u_1^k + 2\left(u_2^k + u_3^k + u_4^k + u_5^k\right)\right] + \left(u_1 + 2u_2\right)^k + 2(u_3 + u_4 + u_5)^k.$$

For discussing the phase diagram, the following choice of reduced variables is convenient (see (8)):

$$\begin{aligned}
x^{(n)} &= \frac{2u_2^{(n)} + u_3^{(n)} + u_5^{(n)}}{u_1^{(n)} + u_4^{(n)}} = \frac{Z^{(n)}(1,2) + Z^{(n)}(1,3) + Z^{(n)}(2,1) + Z^{(n)}(2,3)}{Z^{(n)}(1,1) + Z^{(n)}(2,2)} \\
y_1^{(n)} &= \frac{u_1^{(n)} - u_4^{(n)}}{u_1^{(n)} + u_4^{(n)}} = \frac{Z^{(n)}(1,1) - Z^{(n)}(3,3)}{Z^{(n)}(1,1) + Z^{(n)}(2,2)} \\
y_2^{(n)} &= \frac{u_2^{(n)} - u_3^{(n)}}{u_1^{(n)} + u_4^{(n)}} = \frac{Z^{(n)}(1,3) - Z^{(n)}(3,1)}{Z^{(n)}(1,1) + Z^{(n)}(2,2)} \\
y_3^{(n)} &= \frac{u_2^{(n)} - u_5^{(n)}}{u_1^{(n)} + u_4^{(n)}} = \frac{Z^{(n)}(1,2) - Z^{(n)}(3,2)}{Z^{(n)}(1,1) + Z^{(n)}(2,2)}
\end{aligned} \tag{10}$$

As shown above, in the paramagnetic phase (high symmetry phase, i,e., symmetry with respect to S_3) we have

$$u_1^{(n)} = u_4^{(n)};$$

and

$$u_2^{(n)} = u_3^{(n)} = u_5^{(n)}.$$

The variable x is just a measure of the frustration of the nearest-neighbour bonds and is not an

order parameter like y_1, y_2 and y_3. The relations (6) now have following form:

$$\begin{aligned}
x' &= 1cD[2\{(a^k-1)[b^k(x-3y_2+y_3)^k+2^k(1-y_1)^k+(x+y_2+y_3)^k] \\
&\quad +[(b+1)x+(1-3b)y_2+(b-3)y_3+2(1-y_1)]^k\} \\
&\quad +\{(a^k-1)[(2(1+y_1))^k+(b^k+1)(x+y_2+y_3)^k] \\
&\quad +[2(1+y_1)+(b+1)(x+y_2+y_3)]^k\} \\
&\quad +\{(a^k-1)[(x-3y_2+y_3)^k+(2(1-y_1))^k+(b(x+y_2-3y_3))^k] \\
&\quad +[(1+b)x+(b-3)y_2+(1-3b)y_3+2(1-y_1)]^k\}]; \\
y_1' &= 1D[\{(a^k-1)[b^k(1+y_1)^k+2^{k+1}(x+y_2+y_3)^k]+[b(1+y_1)+x+y_2+y_3]^k\} \\
&\quad -\{(a^k-1)[(x-3y_2+y_3)^k+2^kb^k(1-y_1)^k+(x+y_2-3y_3)^k] \\
&\quad +[2(x-y_2-y_3)+b(1-y_1)]^k\}]; \\
y_2' &= 1cD[\{(a^k-1)[b^k(x-3y_2+y_3)^k+2^k(1-y_1)^k+(x+y_2+y_3)^k] \\
&\quad +[(b+1)x+(1-3b)y_2+(b-3)y_3+2(1-y_1)]^k\} \\
&\quad -\{(a^k-1)[2^k(1+y_1)^k+(b^k+1)(x+y_2+y_3)^] \\
&\quad +[2(1+y_1)+(b+1)(x+y_2+y_3)]^k\}] \\
y_3' &= 1cD[\{(a^k-1)[b^k(x-3y_2+y_3)^k+2^k(1-y_1)^k+(x+y_2+y_3)^k] \\
&\quad +[(b+1)x+(1-3b)y_2+(b-3)y_3+2(1-y_1)]^k\} \\
&\quad -\{(a^k-1)[(x-3y_2+y_3)^k+2^k(1-y_1)^k+b^k(x+y_2+y_3)^k] \\
&\quad +[(b+1)x+(b-3)y_2+(1-3b)y_3+2(1-y_1)]^k\}]
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
D &= \{(a^k-1)[b^k(1+y_1)^k+2^{k+1}(x+y_2+y_3)^k]+[b(1+y_1)+x+y_2+y_3]^k\} \\
&\quad +\{(a^k-1)[(x-3y_2+y_3)^k+(2b(1-y_1))^k+(x+y_2-3y_3)^k] \\
&\quad +[2(x-y_2-y_3)+b(1-y_1)]^k\}]
\end{aligned}$$

3.2. Second Model

Now we consider Potts model (2) with competing prolong next-nearest-neighbour interactions, nearest-neighbour interactions and one level next nearest neighbour interactions. Over again, we select only five variables with introducing new variables, then we have

$$u_1^{(n)} = \sqrt[k]{Z^{(n)}(1,1,1,\ldots,1)} = a^{\frac{k-1}{2}}\left[Z^{(n)}(1,1)\right],$$

$$u_2^{(n)} = \sqrt[k]{Z^{(n)}(1,2,2,\ldots,2)} = a^{\frac{k-1}{2}}\left[Z^{(n)}(1,2)\right],$$

$$u_3^{(n)} = \sqrt[k]{Z^{(n)}(2,1,1,\ldots,1)} = a^{\frac{k-1}{2}}\left[Z^{(n)}(2,1)\right],$$

$$u_4^{(n)} = \sqrt[k]{Z^{(n)}(2,2,2,\ldots,2)} = a^{\frac{k-1}{2}}\left[Z^{(n)}(2,2)\right],$$

$$u_5^{(n)} = \sqrt[k]{Z^{(n)}(2,3,3,\ldots,3)} = a^{\frac{k-1}{2}}\left[Z^{(n)}(2,3)\right].$$

In this case assume

$$a = \exp\left(\frac{J_0}{k_BT}\right), b = \exp\left(\frac{J_p}{k_BT}\right), c = \exp\left(\frac{J_1}{k_BT}\right).$$

As above let m_1 be the number of spins $\{1\}$ and m_2 be the number of spins $\{2\}$ on first level W_1, then we we have

$$Z^{(n)}(1,i_1,i_2,\ldots,i_k) = A\left[u_1^{m_1}u_2^{k-m_1}\right]$$

$$Z^{(n)}(2,i_1,i_2,\ldots,i_k) = A\left[u_3^{m_1}u_4^{m_2}u_5^{k-(m_1+m_2)}\right]$$

$$Z^{(n)}\left(3,i_1,i_2,\ldots,i_k\right)=A\left[u_3^{m_1}u_4^{k-(m_1+m_2)}u_5^{m_2}\right]$$

where $A=a^{{m_1}^2+{m_2}^2+m_1m_2-k(m_1+m_2)}$.
Next, we produce the following recurrence system through a direct calculation:

$$u_1^{'}=a^{\frac{k-1}{2}}c\left[\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-i}bu_1\right)^{k-i}u_2^i\left(\sum_{j=0}^{i}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-j}\right)^{i-j}\right)\right]$$

$$u_2^{'}=a^{\frac{k-1}{2}}\left[\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-i}bu_3\right)^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)\left(a^{-j}u_4\right)^{i-j}u_5^j\right)\right]$$

$$u_3^{'}=a^{\frac{k-1}{2}}\left[\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-i}u_1\right)^{k-i}u_2^i\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)\left(a^{-j}\right)^{i-j}b^j\right)\right]$$

$$u_4^{'}=a^{\frac{k-1}{2}}c\left[\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-i}u_3\right)^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)\left(a^{-j}u_5\right)^{i-j}(bu_4)^j\right)\right]$$

$$u_5^{'}=a^{\frac{k-1}{2}}\left[\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-i}u_3\right)^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)\left(a^{-j}u_4\right)^{i-j}(bu_5)^j\right)\right]$$

Then, total partition function is given in terms of (u_i) by

$$Z^{(n)}=\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-i}\right)^{k-i}\left[\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)\left(a^{-j}\right)^{i-j}\left[u_1^{k-i}u_2^i+u_3^{k-i}\left(u_4^ju_5^{i-j}+u_4^{i-j}u_5^j\right)\right]\right]$$

Choosing the same reduced variables as above we produce following recurrent equations (11′):

$$\begin{aligned}
x^{'}&=\frac{\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-i}\right)^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)\left(a^{-j}\right)^{i-j}A_1(i,j,x,y_1,y_2,y_3)\right)}{c\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)(a^{-i})^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)(a^{-j})^{i-j}D(i,j,x,y_1,y_2,y_3)\right)}\\
y_1^{'}&=\frac{\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-i}\right)^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)\left(a^{-j}\right)^{i-j}A_2(i,j,x,y_1,y_2,y_3)\right)}{\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)(a^{-i})^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)(a^{-j})^{i-j}D(i,j,x,y_1,y_2,y_3)\right)}\\
y_2^{'}&=\frac{\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-i}\right)^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)\left(a^{-j}\right)^{i-j}A_3(i,j,x,y_1,y_2,y_3)\right)}{c\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)(a^{-i})^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)(a^{-j})^{i-j}D(i,j,x,y_1,y_2,y_3)\right)}\\
y_3^{'}&=\frac{\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)\left(a^{-i}\right)^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)\left(a^{-j}\right)^{i-j}A_4(i,j,x,y_1,y_2,y_3)\right)}{c\sum_{i=0}^{k}\left(\begin{array}{c}k\\i\end{array}\right)(a^{-i})^{k-i}\left(\sum_{j=0}^{i}\left(\begin{array}{c}i\\j\end{array}\right)(a^{-j})^{i-j}D(i,j,x,y_1,y_2,y_3)\right)}
\end{aligned}\tag{12}$$

where

$$
\begin{aligned}
A_1(i,j,x,y_1,y_2,y_3) &= 2[b(x-3y_2+y_3)]^{k-i}[2(1-y_1)]^{i-j}(x+y_2-3y_3)^j \\
&\quad +b^j[2(1+y_1)]^{k-i}(x+y_2+y_3)^i \\
&\quad +(x-3y_2+y_3)^{k-i}[2(1-y_1)]^{i-j}[b(x+y_2-3y_3)]^j; \\
A_2(i,j,x,y_1,y_2,y_3) &= [2b(1+y_1)]^{k-i}(x+y_2+y_3)^i \\
&\quad -[2b(1-y_1)]^j(x-3y_2+y_3)^{k-i}(x+y_2-3y_3)^{i-j}; \\
A_3(i,j,x,y_1,y_2,y_3) &= [b(x-3y_2+y_3)]^{k-i}[2(1-y_1)]^{i-j}(x+y_2-3y_3)^j \\
&\quad -b^j[2(1+y_1)]^{k-i}(x+y_2+y_3)^i; \\
A_4(i,j,x,y_1,y_2,y_3) &= [b(x-3y_2+y_3)]^{k-i}[2(1-y_1)]^{i-j}(x+y_2-3y_3)^j \\
&\quad -(x-3y_2+y_3)^{k-i}[2(1-y_1)]^{i-j}[b(x+y_2-3y_3)]^j
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
D(i,j,x,y_1,y_2,y_3) &= [2b(1+y_1)]^{k-i}(x+y_2+y_3)^i \\
&\quad +[2b(1-y_1)]^j(x-3y_2+y_3)^{k-i}(x+y_2-3y_3)^{i-j}.
\end{aligned}
$$

3.3. The Average Magnetization

In order to investigate the phase diagram we have to look at local properties, namely the local magnetization or the magnetization of the root x^0. The average magnetization m for the n-th generation is given by,

$$
m = \left\langle \sigma(x^0) \right\rangle_n = \frac{1\cdot Z_1^{(n)} + 2\cdot Z_2^{(n)} + 3\cdot Z_3^{(n)}}{Z^{(n)}} = 2 - \left(\frac{Z_1^{(n)} - Z_3^{(n)}}{Z^{(n)}} \right)
$$

Since the form of spins of the Potts model is unessential one can replace the set of spin values {1,2,3} by the centered set {-1,0,1} [31] and then we have

$$
\tilde{m} = \left\langle \sigma(x^0) \right\rangle_n = - \left(\frac{Z_1^{(n)} - Z_3^{(n)}}{Z^{(n)}} \right)
$$

Below we will consider the average magnetization $\tilde{m}$ to study the wavevectors.
Average magnetization for first model is given by

$$
\begin{aligned}
\tilde{m} &= \frac{1}{E}[(a^k-1)\{2^{-k}((x-3y_2+y_3)^k + (x+y_2-3y_3)^k - 2(x+y_2+y_3)^k)\} \\
&\quad +(1-y_1)^k - (1+y_1)^k + [1+x-(y_1+y_2+y_3)]^k - [1+x+(y_1+y_2+y_3)]^k],
\end{aligned}
$$

where

$$
\begin{aligned}
E &= [(a^k-1)\{(1+y_1)^k + 2(1-y_1)^k + 2^{1-k}[((x+y_2+y_3)^k + (x-3y_2+y_3)^k \\
&\quad +(x+y_2-3y_3)^k]\} + (1+x+y_1+y_2+y_3)^k + 2(1+x-y_1-y_2-y_3)^k]
\end{aligned}
$$

and average magnetization for second model is given by

$$
\tilde{m} = -\frac{2^k \sum_{i=0}^{k} \left(\begin{array}{c} k \\ i \end{array} \right) \left(2a^{k-i}\right)^{k-i} \left[\sum_{j=0}^{i} \left(\begin{array}{c} i \\ j \end{array} \right) \left(a^{-j}\right)^{i-j} M\left(i,j,x,y_1,y_2,y_3\right) \right]}{\sum_{i=0}^{k} \left(\begin{array}{c} k \\ i \end{array} \right) \left(a^{-i}\right)^{k-i} \left[\sum_{j=0}^{i} \left(\begin{array}{c} i \\ j \end{array} \right) \left(a^{-j}\right)^{i-j} N\left(i,j,x,y_1,y_2,y_3\right) \right]}
$$

where

$$\begin{aligned} M(i,j,x,y_1,y_2,y_3) &= (1+y_1)^{k-i}(x+y_2+y_3)^i \\ &\quad -2^{2i-k-j}(x-3y_2+y_3)^{k-i}(1-y_1)^{i-j}(x+y_2-3y_3)^j \end{aligned}$$

and

$$\begin{aligned} N(i,j,x,y_1,y_2,y_3) &= 2^{k-i}(1+y_1)^{k-i}(x+y_2+y_3)^i \\ &\quad +2^j(x-3y_2+y_3)^{k-i}(1-y_1)^j(x+y_2-3y_3)^{i-j} \\ &\quad +2^{i-j}(x-3y_2+y_3)^{k-i}(1-y_1)^{i-j}(x+y_2-3y_3)^j \end{aligned}$$

4. Morphology of phase diagrams

It is convenient to know the broad features of the phase diagram before discussing the different transitions in more detail. Below we will consider phase diagram for $k = 3$. This can be achieved numerically in a straightforward fashion. The recursion relations (11) for the first model and (12) for the second model provide us the numerically exact phase diagram in $(-J_p/J_1, J_0/J_1, T/J_1)$ space. Let $\beta = -J_p/J_1$, $\alpha = T/J_1$ and $\gamma = J_0/J_1$, by first model and second model, for some fixed values of , starting from initial conditions

$$x^{(1)} = \frac{2b^k + c^k + 1}{c\left(c^k b^k + 1\right)}, \qquad y_1^{(1)} = \frac{c^k b^k - 1}{c^k b^k + 1}, \qquad y_2^{(1)} = \frac{b^k - c^k}{c\left(c^k b^k + 1\right)}, \qquad y_3^{(1)} = \frac{b^k - 1}{c\left(c^k b^k + 1\right)}$$

with the parameters $a = \exp\left((k\alpha)^{-1}\gamma\right)$, $b = \exp\left(-\alpha^{-1}\beta\right)$, $c = \exp\left(\alpha^{-1}\right)$ and for second model's as with the parameters $a = \exp\left(\alpha^{-1}\gamma\right)$, $b = \exp\left(-\alpha^{-1}\beta\right)$, $c = \exp\left(\alpha^{-1}\right)$ corresponding to boundary condition $\bar{\sigma} \equiv 1$, one iterates the recurrence relations (11) and (12), we observe their behaviour after a large number of iterations. In the simplest situation a fixed point $(x^*, y_1^*, y_2^*, y_3^*)$ is reached. It corresponds to paramagnetic phase if $y_1^* = 0$, $y_2^* = 0$, $y_3^* = 0$ to a ferromagnetic phase if $y_1^*, y_2^*, y_3^* \neq 0$.

The system may be periodic with period p, where case $p = 2$ correspond to antiferromagnetic phase and case $p = 4$ correspond to so-called antiphase that denoted $< 2 >$ for compactness. Finally, the system may remain aperiodic. The distinction between a truly aperiodic case and one with a very long period is difficult to make numerically. Below we consider periodic phase with period p where $p \leq 4$. All periodic phase with period $p > 4$ and aperiodic phase we will consider as modulated phase.

The resultant phase diagram on the plane (β, α) for both models with some fixed values of γ and k are shown in Figs. 1 and 3. (Here: P – paramagnetic phase, F – ferromagnetic phase, AF – antiferromagnetic phase, P3 – phase with period 3, <2> – antiphase, and M – modulated phase)

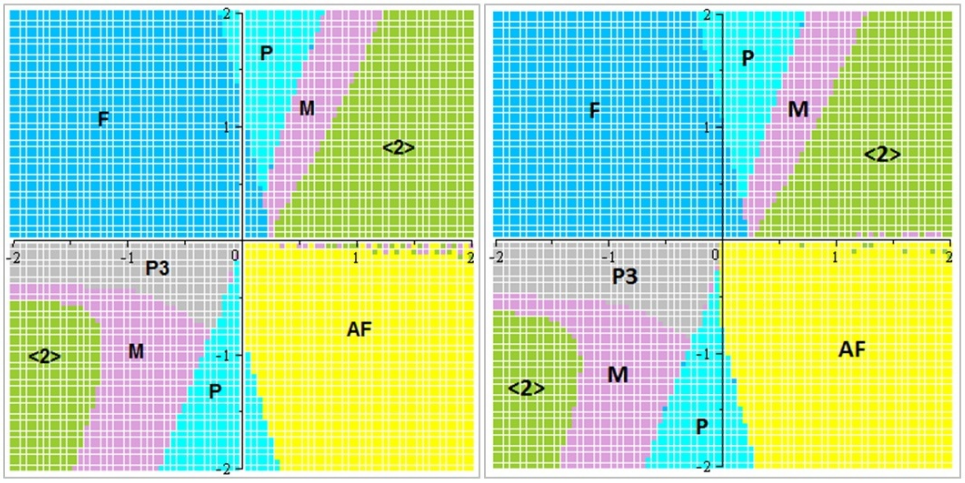


Fig. 1. Phase diagram of the (a) first model and (b) second model for $\gamma = \pm 0.1$

One can observe the significance of order k with nonzero J_0 of Cayley tree in Figs. 1 – 6 where the models with competing interactions' (1) and (2) phase diagram is illustrated. Here, the systems for both models (11) and (12) was iterated for order $k = 3$ and 4 with $\gamma = \pm 0.1$, $\gamma = \pm 1$ and $\gamma = \pm 10$. In Fig. 1, consider both models with $k = 3$ and $\gamma = \pm 0.1$, contain in these diagrams are six phases which are ferromagnetic, paramagnetic, antiferromagnetic, phase of period 3, antiphase and modulated phases.

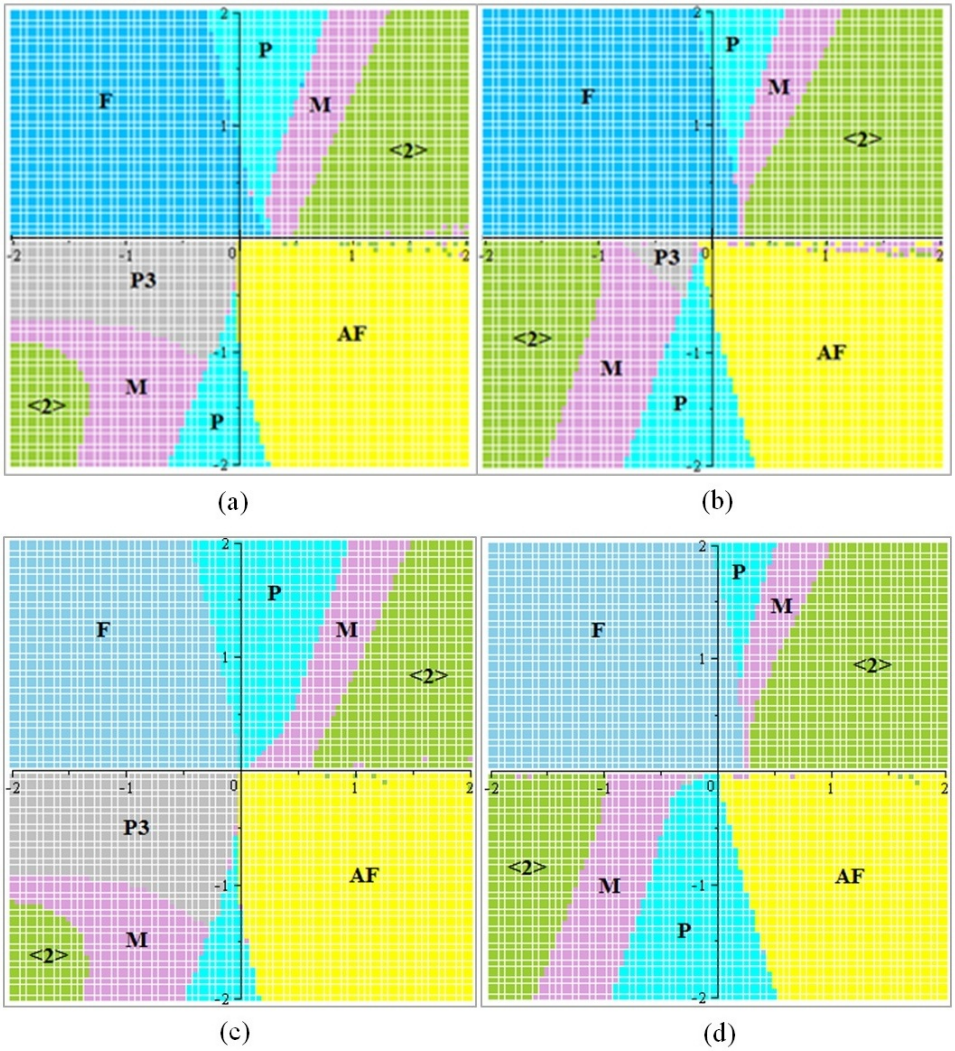


Fig. 2. Phase diagram of the first model for (a) $k = 3$ and $\gamma = -1$ and (b) $k = 3$ and $\gamma = 1$, and for second model with (c) $k = 3$ and $\gamma = -1$ and (d) $k = 3$ and $\gamma = 1$.

The phase diagrams are described in terms of its quadrant. When these phase diagrams are observed, it can be seen that ferromagnetic, paramagnetic, modulated and antiphase phases are situated in the first quadrant ($-J_p/J_1 > 0, T/J_1 > 0$). On the other hand, the second quadrant ($-J_p/J_1 < 0, T/J_1 > 0$) holds phase diagrams consisting of ferromagnetic and paramagnetic phases. Moving on, the third quadrant ($-J_p/J_1 < 0, T/J_1 < 0$) is observed to have four phases namely paramagnetic, phase of period 3, modulated and antiphase phases, except for the case in Fig. 2(d), the phase of period 3 vanish in the this quadrant. Finally in quadrant four ($-J_p/J_1 > 0, T/J_1 < 0$), the diagrams contain paramagnetic and antiferromagnetic with a small island consists of different phases.

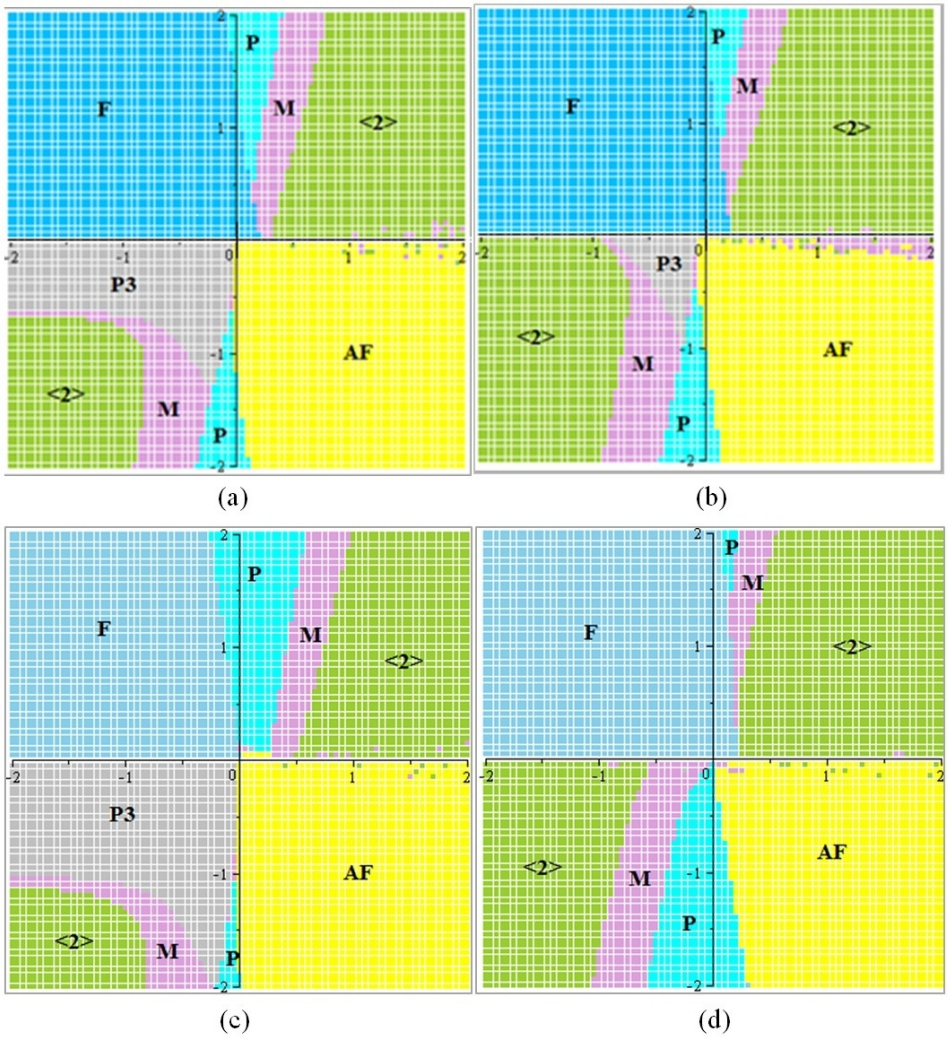


Fig. 3. Phase diagram of the first model for (a) $k = 4$ and $\gamma = -1$ and (b) $k = 4$ and $\gamma = 1$, and for second model with (c) $k = 4$ and $\gamma = -1$ and (d) $k = 4$ and $\gamma = 1$.

Order of Cayley tree with nonzero one-level next nearest neighbour interactions has played a vital role in the phase diagram on the both models (1) and (2) with competing interactions' result. Since the phase diagram was produced in the case $k > 2$, difference on the non-existing of paramodulated phase is observed. This is because this said phase exists in the case of Cayley tree order 2 (Ganikhodjaev et. al [19]). Also, the phase of antiphase exists in the third quadrant when a comparison is done with the case of Cayley tree order 2.

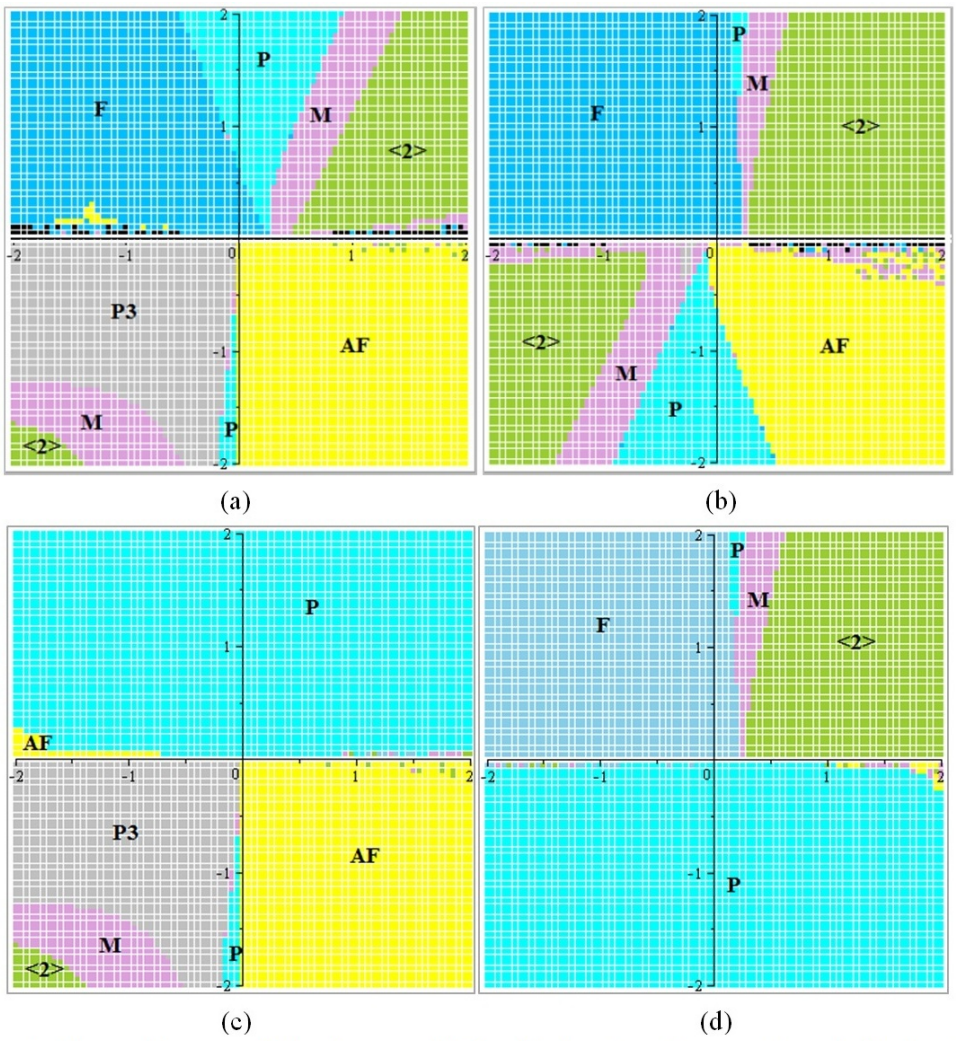


Fig. 4. Phase diagram of the first model for (a) $k = 3$ and $\gamma = -10$ and (b) $k = 3$ and $\gamma = 10$, and for second model with (c) $k = 3$ and $\gamma = -10$ and (d) $k = 3$ and $\gamma = 10$.

The phase diagrams in Figs. (4) – (5) were produced with different order k and $\gamma = \pm 10$ give difference in sizes of some phases. In Fig. 4(c), we found that the paramagnetic phase cover all in first and second quadrant, as we consider $J_1 > 0$ in the case of $k = 3$ and $\gamma = -10$ for second model. Similar to Fig. 4(d), where the paramagnetic phase cover all in third and fourth quadrant ($J_1 < 0$) in the case of $k = 3$ and $\gamma = 10$. The results were different as we see in the case for first model (see Fig. 4(a) and (b)).

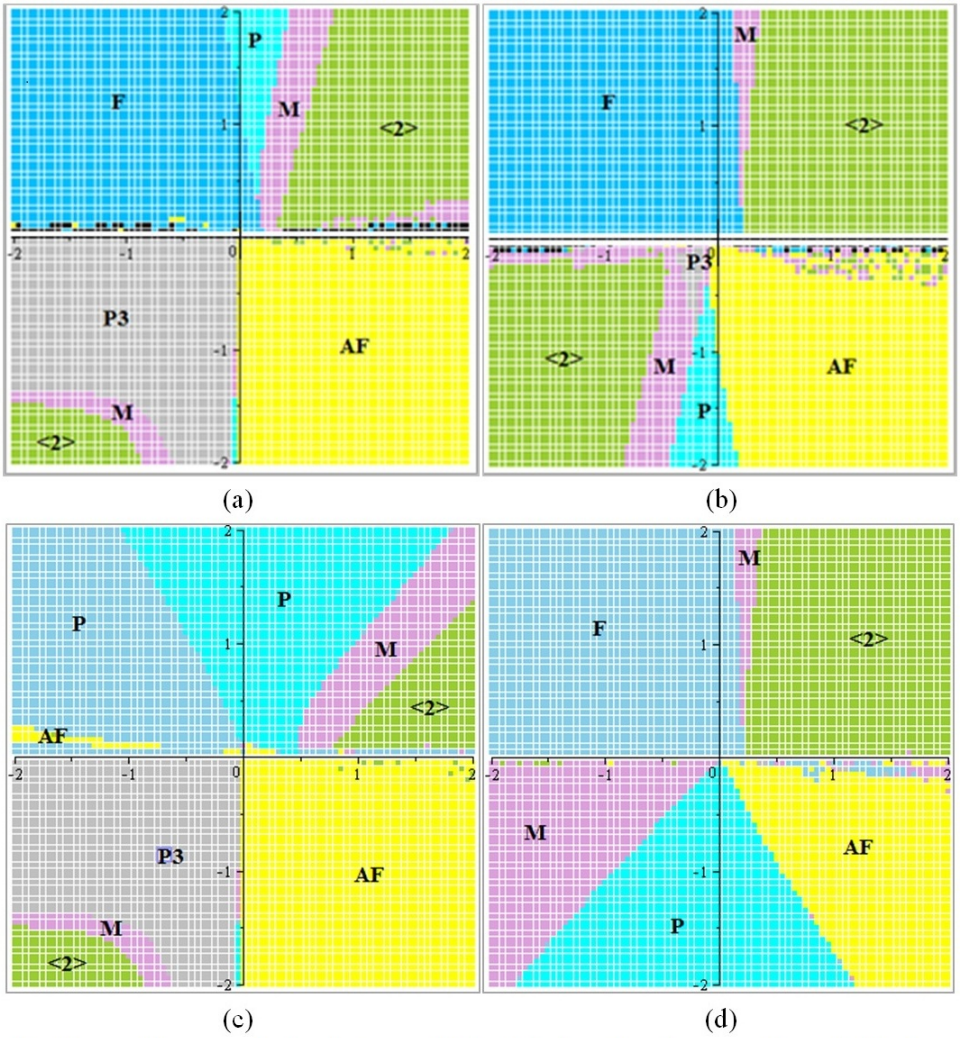


Fig. 5. Phase diagram of the first model for (a) $k = 4$ and $\gamma = -10$ and (b) $k = 4$ and $\gamma = 10$, and for second model with (c) $k = 4$ and $\gamma = -10$ and (d) $k = 4$ and $\gamma = 10$.

Looking on the fig. 5, one can see that the difference between both models occur in the case of $k = 4$ and $\gamma = 10$, as phase of period 3 and antiphase not exist in the third quadrant of our investigation area. (Figs. 5 (c) and (d)). Furthermore, in fig. 6, the phase diagrams were produced with order $k = 10$ and $\gamma = \pm 5$ give an additional in sizes of some phases. In Fig. 6(c), we found that a small phase of antiferromagnetic occur in second quadrant for the second model. Compare to first model, as in first quadrant for second model, we have more region of paramagnetic and modulated phases (see Fig. 6(c)). In addition, one can see in Fig. 6(d), as third quadrant of second model, same region of paramagnetic and modulated phases occur and increase as compare to the first model.

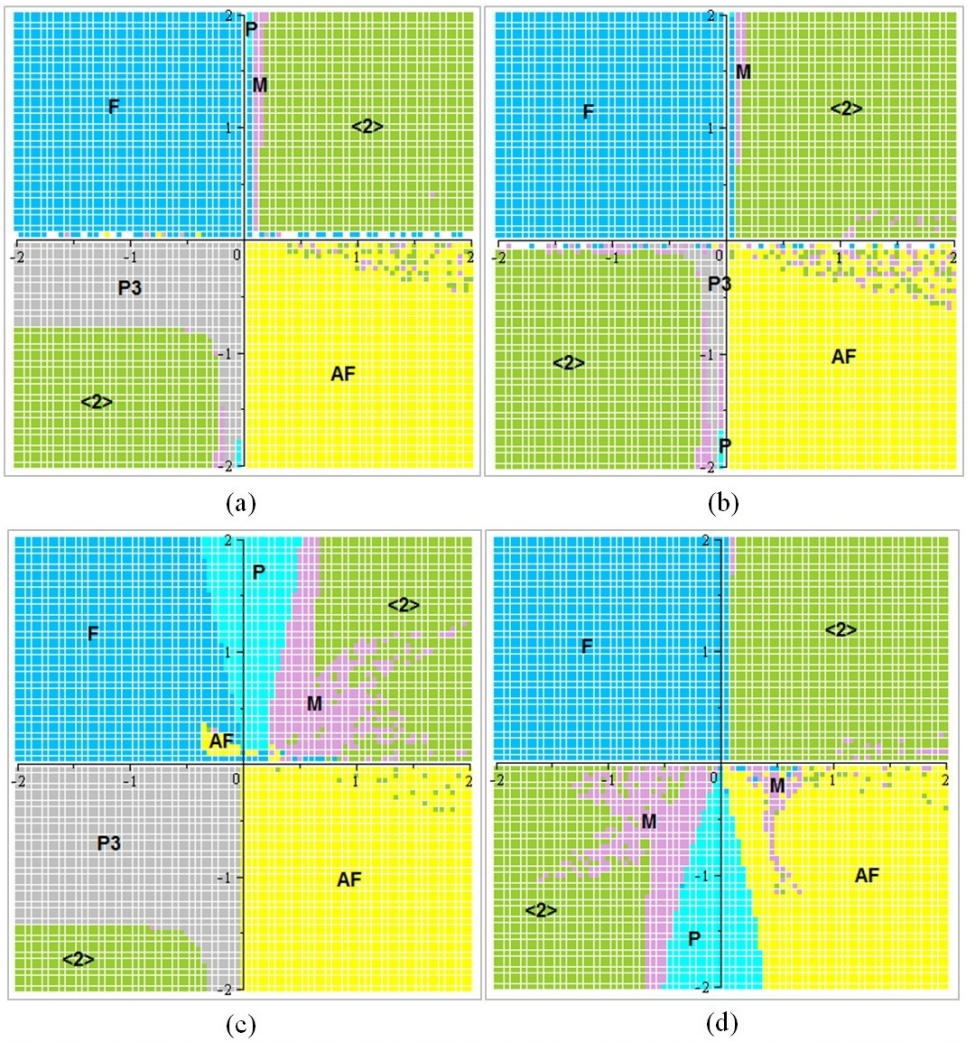


Fig. 6. Phase diagram of the first model for (a) $k = 10$ and $\gamma = -5$ and (b) $k = 10$ and $\gamma = 5$, and for second model with (c) $k = 10$ and $\gamma = -5$ and (d) $k = 10$ and $\gamma = 5$.

From the previously presented resultant phase diagrams, one can see that an investigation on the behaviour of the system (11) and (12) can be performed by applying the numerical approach. The systems were iterated in a large number before the detailed behaviour was studied and visualized on the (α, β) space. As been discussed, differences in the values of k results in differences and changes in the phase. Hence it is very difficult for us to study the behaviour of the systems analytically. Yet, the transition line separating the phases is suggested to be solved analytically. This can be done by linearizing both systems (11) and (12). Later on, the linearized systems and analytical justification of the phase diagrams will be discussed in more detailed.

5. Conclusion

In this paper, we study on relationship between two types of one-level neighbour interactions of Potts model on arbitrary order Cayley tree. One can consider for one-level neighbour interactions, k-tuple interactions and binary interactions. We study on the differences for both types of model on Cayley tree order 3 and order 4 and order 10.

We have found the phase diagram of Potts model (1) and (2) and show that there consist of six phases: ferromagnetic, paramagnetic, antiferromagnetic, period 3, antiphase and modulated

phase. Moreover, we study on the difference of phase diagram for both types of model by fixing γ equal to $\pm 0.1, \pm 1, \pm 5$, and ± 10 . From considerations above, we can see that role of k rather significant because there show the phase diagram quite different for each k in first and second model.

Indeed the first and second model show us the difference, even though we only considering two different way in competing the one level interactions. We can say that this two model are differ to each other. They may give an unique significant to any physical phenomena occur. Further investigation can be consider to understand how this difference attribute appear.

6. Acknowledgments

This work was supported by Fundamental Research Grant Scheme FRGS10-022-0141.

7. References

[1] Bak P 1982 *Rep.Prog.Phys.* **45** 587
[2] Elliott R J 1961 *Phys.Rev* **124** 346
[3] Ostlund S 1981 *Phys.Rev. B* **24** 398
[4] Huse D A 1981 *Phys.Rev. B* **24** 5180
[5] Bak P 1981 *Phys.Rev. Lett.* **46** 791
[6] Jensen M H and Bak P 1983*Phys.Rev.B* **27** 6853
[7] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics*, Academic Press, London/New York
[8] Vannimenus J 1981*Z.Phys. B* **43**, 141
[9] Bak P and von Boehm J 1980 *Phys.rev.B* **21** 5297
[10] Georgii H-O, 1988 *Gibbs Measures and Phase Transitions* de Gruyter, Berlin
[11] Fisher M E and Selke W 1980 *Phys.Rev.Lett.* **44** 1502
[12] Mariz M, Tsalis C, Albuquerque A L 1985 *Jour. Stat. Phys.* **40** 577
[13] Inawashiro S, Thompson C J 1983 *Physics Letters* ,**97A** 245
[14] Inawashiro S, Thompson C J, Honda G 1983 *J. Stat. Phys.* Vol. **33** No. 2 419
[15] Yokoi C S O, Oliveira M J, Salinas S R 1985 *Phys. Rev. Lett.*, **54** 163
[16] Tragtenberg M H R, Yokoi C S O 1995 *Phys. Rev. E*, **52** 2187
[17] Coutinho S, Morgado W A M, Curado E M F, da Silva L 2006 *Physical Review B* **74**(9):094432 1
[18] Ganikhodjaev N N, Mukhamedov F M, Mendes J F F 2006 *Jour. Stat. Mech.* P08012
[19] Ganikhodjaev N N, Mukhamedov F M, Pah C H 2008*Physics Letters A* **373** 33
[20] Katsura S, Takizawa M 1974*Prog. Theor. Phys.* **51** 82
[21] Ganikhodjaev N N, Pah C H, Wahiddin M R B 2004 *J. Math. Phys.* **45** 3645
[22] Ganikhodjaev N N, Rozikov U A 2009 *Mathematical Physics, Analysis and Geometry*,**12**,141
[23] Ganikhodjaev N N, Uuz S 2011 *Physics A* **390**, 4160
[24] Monroe J L, 1992 *J.Stat.Phys.* **67** 1185
[25] Monroe J L 1994 *Phys.Lett. A* **188** 80
[26] Potts R B 1952 *Proc. Cambridge Philos. Soc.* **48** 106
[27] Wu F Y 1982*Rev. Mod. Phys.* **54** 235
[28] Ganikhodjaev N N, Temir S, Akin H, 2009 *J. Stat. Phys.* **137** 701
[29] H. Moraal, *Physica* **92A**:305(1978)
[30] Ganikhodjaev N N and Siti Fatimah Zakaria 2011 *Australian Journal of Basic and Applied Sciences*, **5(8)** 1296
[31] Ganikhodjaev N and Fatimah Abdul Razak 2010 *Mathematical Physics, Analysis and Geometry* **13(1)** 1