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Ising Model with Competing Interactions on Cayley Tree of Order 4: An Analytic Solution.

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Abstract. We investigate an Ising model with two restricted competing interactions (nearest neighbors, and one-level neighbors) on the Cayley tree of order four. We derive a recurrent equation for the Cayley tree of order k. We found an analytic solution for the given interactions in the case of order 4. Our result of the critical curve shows the existence of the phase transition occurs in this model. We also give the calculation of the free energy from the description of Gibbs measure of the given Hamiltonian on Cayley tree of order four.

1. Introduction

There are many mathematical models that can describe the phenomenon of phase transition such as in the Potts model [1], Heisenberg model [2], and Ising model [3, 4]. The Cayley tree Ising model was originally works of Hans Bethe [6]. Unlike the original Ising model, the Cayley tree Ising model takes the basic graph to be a tree rather than set the graph into a lattice point. It is an approximation to Ising model and an alternative model for researchers to mathematically understand about this model especially it's qualitative properties.

The study of the phase transition of the Cayley tree Ising model with competing interactions of ternary and binary on a Cayley tree of order 2, was investigated in [3]. This study is continued by considering a zero external magnetic field and restricted interactions on nearest neighbors and second neighbors to find the exact solution which is also using a recursive method as in [3,4]. A paper on the Ising model with three competing interactions on a Cayley tree [5,10] is investigated with considerations on the external magnetic field and restricted interactions and proved in different cases.

In this paper, a general part is devoted to the Ising model on Cayley tree of order k (see also [8]) and a specific part is provided on finding phase transition curve for the model on Cayley tree of order 4. Note that analytic solution is rarely seen in the study of phase transition, there are only numerical results in [9,11] for studying the Ising model with competing interactions on Cayley tree. Some numerical results for this model are found earlier in [12] that phase transition occurs in Ising model with competing interactions which corresponds to nearest neighbors and one-level neighbors on semi-infinite Cayley tree of order 3 and 4. Our analytic result shall confirm the earlier one.

2. Gibbs Measures & Phase Transition

Let Λ be a finite subset of an infinite lattice, I. Given that $\Omega(\Lambda)$ is the set of all configurations, $\sigma(\Lambda)$ on Λ and $\bar{\sigma}(I/\Lambda)$ be a boundary condition on I/Λ. Assume a Hamiltonian, H is defined on state space, Φ. For any finite $\Lambda \subset I$, a Gibbs measure, P is given by

$$P(\sigma(\Lambda)) = \int_{\Omega} P(d\bar{\sigma}(I/\Lambda)u_{\Lambda}(\sigma(\Lambda)) \qquad (2.1.1)$$

if it satisfies a DLR equation [10]. The conditional Gibbs measure, u_{Λ} in Λ volume under the boundary condition, $\bar{\sigma}(V/\Lambda)$ is given as

$$u_{\Lambda}(\sigma(\Lambda)) = \frac{\exp\left(-\beta H(\sigma(\Lambda)|\bar{\sigma}(I/\Lambda)\right)}{Z_{\Lambda}(\bar{\sigma}(V/\Lambda))}$$

and the normalizing constant, Z or partition function, is defined by

$$Z_{\Lambda}(\bar{\sigma}(V/\Lambda)) = \sum_{\sigma(\Lambda)\epsilon\Omega(\Lambda)} e^{-\frac{1}{kT}H(\sigma(\Lambda)|\bar{\sigma}(V/\Lambda))}$$

where k is the universal constant and T is the temperature.

The non-uniqueness of Gibbs measure of a given Hamiltonian selected from an equilibrium state will produce a phase transition in physical system. We can say that a phase transition occurs with the existence of the non-uniqueness of Gibbs measure. Indeed, phase transition usually occurs at low temperature. In other words, if it is possible to find the exact value of temperature, T^* such that phase transition occurs for all $T < T^*$, then T^* is called a critical value of temperature.

3. The Recurrent Equations for Partition Functions in the Ising Model with Competing Interactions

A semi-infinite Cayley tree of order k (figure 3.1) is a graph with no cycles, each vertex emanates $k+1$ edges. The root, x^0 in the graph is called the 0th level. From the root, x^0 each vertex emanates exactly k edges. From the vertex at these edges, we called the vertices as the vertex of the first level. The vertices from the edges at first level called as the vertex of the second level and so on. A configuration on a tree is assigned as "+" or "−" to each point.

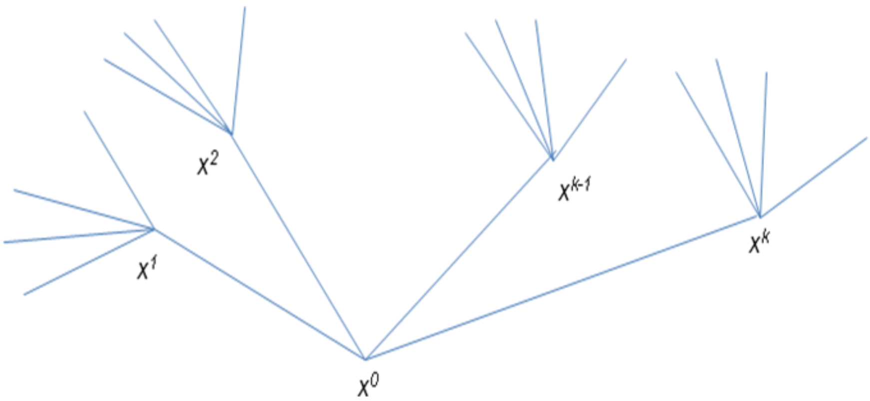


Figure 3.1: Semi-infinite Cayley tree of order k

Let Cayley tree, $\Gamma^k = (V, L), k \geq 1$ where V is given by the set of vertices of the graph, Γ^k and L is the set of edges of Γ^k. Let $x, y \in V$, then the vertices x and y are called *nearest neighbors* $(< x, y >)$ if there exists an edge between them. Note that a collection of pairs $< x, x_1 >, < x_1, x_2 >, \ldots, < x_{d-1}, y >$ are neighboring vertices which called a *path* from x to y. Thus, the distance $d(x, y), x, y \in V$ is the shortest path on the Cayley tree.

For the fixed $x^0 \in V$, we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \; V_n = \bigcup_{k=1}^{n} W_k, \quad L_n = \{l =< x, y >\in L | x, y \in V_n\}.$$

Note that y is the direct successor of x, and x, y are nearest neighboring vertices. In other words, any vertex $x \neq x^0$ has k direct successor and from vertex x^0 has $k+1$ direct successor. Thus, we define the *set of direct successor* of x, $S(x)$ as follows

$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, x \in W_n.$

The vertices $x, y \in V$ are called second neighbor $(> x, y <)$ when there exist vertex $z \in V$ such that $x, y \in S(z)$.

In this model, we consider the spins values in the set $\Phi = \{-1, +1\}$. A configuration, σ on V is defined as a function $x \in V \rightarrow \sigma(x) \in \Phi$ where the set of all configurations coincides with $\Omega = \Phi^V$.

Definition 3.1. *Ising model with competing interactions on Cayley tree is defined by the following Hamiltonian*

$$H(\sigma) = -J \sum_{<x,y>} \sigma(x)\sigma(y) - J_1 \sum_{>x,y<} \sigma(x)\sigma(y) - h \sum_{x \in V} \sigma(x). \qquad (3.1)$$

The first term is the total sum for all nearest neighbors, and the second term is the total sum for all second nearest neighbors. The third term is zero since external magnetic field, h in this case is considered zero. The values of its spin variables, $\sigma(x)$ is considered as ± 1. Both signs refer to the state of the spin "up" or "down". Second nearest neighbor $> x, y <$ is called one-level neighbors $(> \overline{x, y} <)$ if x and y are on the same level, n.

Definition 3.2. *One-level Ising model with competing interactions on Cayley tree is defined by the following Hamiltonian*

$$H(\sigma) = -J \sum_{<x,y>} \sigma(x)\sigma(y) - J_1 \sum_{>\overline{x,y}<} \sigma(x)\sigma(y) - h \sum_{x \in V} \sigma(x) \qquad (3.2)$$

where the sum in the second term is only taken over one-level second neighbors and assume $h = 0$.

In this study, we consider $J > 0$, and $J_1 \in \mathbb{R}$. Let us define the set of all configurations on $V_1 = \{x^0, x^1, \dots, x^k\}$ (see figure 3.1). We consider $\bar{\sigma}(V \backslash V_n)$ as $\bar{\sigma}_n$. The conditional Gibbs measure, $\mu_n(\sigma^n | \bar{\sigma}_n)$ is defined on volume V_n as

$$\mu_n(\sigma^n | \bar{\sigma}_n) = \frac{\exp\{-\beta H(\sigma^n | \bar{\sigma}_n)\}}{Z^n(\bar{\sigma}_n)} \qquad (3.3)$$

where

$$H(\sigma^n | \bar{\sigma}_n) = -J \sum_{<x,y>: x,y \in V_n} \sigma^n(x)\sigma^n(y) - J_1 \sum_{>x,y<: x,y \in V_n} \sigma^n(x)\sigma^n(y)$$
$$-J \sum_{<x,y>: x \in V_n, y \bar{\in} V_n} \sigma^n(x)\bar{\sigma}^n(y) - J \sum_{>x,y<: y \in V_n, x \bar{\in} V_n} \bar{\sigma}^n(x)\sigma^n(y) \qquad (3.4)$$

and partition function is defined by

$$Z^n(\bar{\sigma}_n) = \sum_{\sigma^n \in \Omega_n} \exp\{-\beta H(\sigma^n | \bar{\sigma}_n)\} \qquad (3.5)$$

where $\beta = \frac{1}{kT}$, and T is the inverse temperature.

For Cayley tree, Γ^k of order k, the partition function is divided into two terms

$$Z^n(\bar{\sigma}_n) = Z^n{}_+(\bar{\sigma}_n) + Z^n{}_-(\bar{\sigma}_n) \qquad (3.6)$$

where

$Z^n{}_+(\bar{\sigma}_n) = \sum_{\sigma^0 = +1} \exp\{-\beta H(\sigma^n | \bar{\sigma}_n)\}$ and $Z^n{}_-(\bar{\sigma}_n) = \sum_{\sigma^0 = -1} \exp\{-\beta H(\sigma^n | \bar{\sigma}_n)\}$.

To have general equations relating the partition function of k^{th} order on Cayley tree, all possible configurations of the spins is considered. Let r be the number of spin down $(-)$, $\sigma(y) = -1$ and $k-r$ be the number of spins up (+), $(y) = +1$. The following diagram illustrates spins up and spins down on V_1.

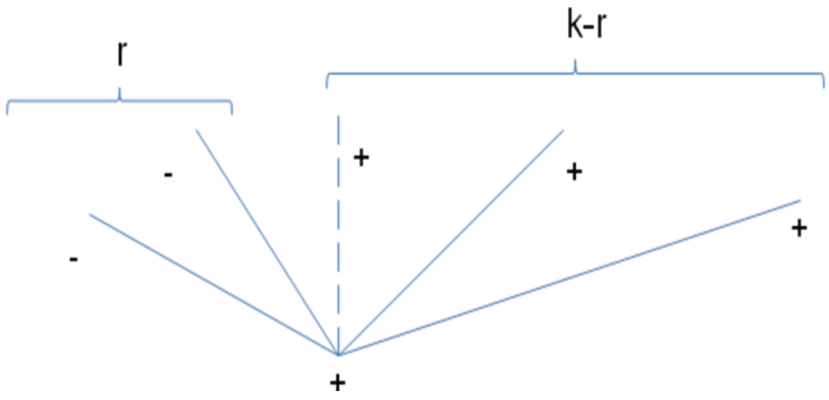
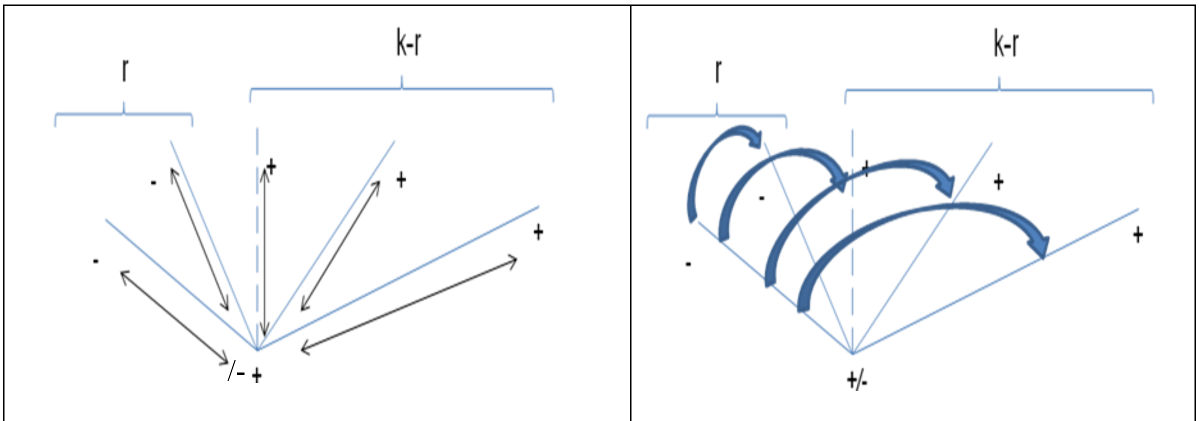


Figure 3.2: Number of spin up (+) and spin down (-) on V_1 with *k* vertices.

Since we are considering ferromagnetic cases where $J, J_1 > 0$, we let $\theta = \exp(\beta J)$ which corresponds to nearest neighbours, and $\theta_1 = \exp(\beta J_1)$ which corresponds to one-level neighbors.



(a) (b)

Figure 3.3: Interactions for positive and negative boundary conditions on (a) nearest neighbors and (b)one-level neighbors

Lemma 3.1. *For a given configuration with r number of spin "$-$" and $(k-r)$ of spin "+" on W_1, one-level interactions give an interaction factor as*

$$\theta_1^v = \theta_1^{\frac{1}{2}k^2-2kr-\frac{1}{2}k+2r^2}. \tag{3.7}$$

Proof. Let $p = C_2^r$ which corresponds to a combination of choosing any two sites with spin down (-), $q = C_2^{k-r}$ corresponds to combination of choosing any two sites with spin up (+), and $s = C_2^k$ is the total combination of spins at one-level neighbors. The total number of interactions, denoted by v is equal to

$$v = p + q - (s - (p+q))v = 2(C_2^r + C_2^{k-r}) - C_2^k = \frac{1}{2}k^2 - 2kr - \frac{1}{2}k + 2r^2.$$

Theorem 3.1. *For any positive integer n and a fixed k^{th} order of Cayley tree, Γ^k the recurrent equation* (3.6) *can be rewritten as the following*

$$Z^n = \sum_{r=0}^{k} C_r^k (Z_+^{n-1})^{k-r} (Z_-^{n-1})^r \, \theta^{k-2r} \theta_1^{\frac{1}{2}k^2-2kr-\frac{1}{2}k+2r^2} + \sum_{r=0}^{k} C_r^k (Z_+^{n-1})^{k-r} (Z_-^{n-1})^r \, \theta^{2r-k} \theta_1^{\frac{1}{2}k^2-2kr-\frac{1}{2}k+2r^2}. \quad (3.8)$$

Proof. Interactions with nearest neighbors and one-level neighbors are illustrated by the figure 3.3.

From Hamiltonian (3.1), exponent of total interactions of nearest neighbors on V_1 can be straightaway calculated as:
$\theta^{-r}\theta^{k-r} = \theta^{-r+k-r} = \theta^{k-2r}$ (with +ve boundary) and
$\theta^{r}\theta^{-(k-r)} = \theta^{-r-k+r} = \theta^{2r-k}$ (with −ve boundary).

Meanwhile, exponent of total interactions of one-level neighbors on V_1 are taking over two competing interactions on the same level (figure 3.2 b). Therefore, from lemma above, we have

$$Z_+^n = \sum_{r=0}^{k} C_r^k (Z_+^{n-1})^{k-r} (Z_-^{n-1})^r \, \theta^{k-2r} \theta_1^{\frac{1}{2}k^2-2kr-\frac{1}{2}k+2r^2}$$

$$Z_-^n = \sum_{r=0}^{k} C_r^k (Z_+^{n-1})^{k-r} (Z_-^{n-1})^r \, \theta^{2r-k} \theta_1^{\frac{1}{2}k^2-2kr-\frac{1}{2}k+2r^2}$$

Substituting into equation (3.4), immediately we have the equation (3.5).

Now, let the ratio be

$$u_n(x^0) = \frac{Z_+^n}{Z_-^n}. \quad (3.9)$$

Theorem 3.2. *For any positive integer n, the ratio* (3.9) *will be as follows*:

$$u_n(x^0) = \frac{\sum_{r=0}^{k} C_r^k u_{n-1}^{k-r} \theta^{k-2r} \theta_1^{\frac{1}{2}k^2-2kr-\frac{1}{2}k+2r^2}}{\sum_{r=0}^{k} C_r^k u_{n-1}^{k-r} \theta^{2r-k} \theta_1^{\frac{1}{2}k^2-2kr-\frac{1}{2}k+2r^2}}. \quad (3.10)$$

Proof. Straightforward from Theorem 3.1 and definition of (3.9).

4. Gibbs Measure and Analysis of Recurrent Equation of Order 4

4.1. Gibbs Measure for Ising model on Cayley Tree of order 4

Let $t: x \to \mathbb{R}$ be a real valued function of $x \in V$. The probability measure, μ_n is defined as follows:

$$\mu_n(\sigma^n) = \frac{\exp\{-\beta H(\sigma^n) + \sum_{x\in W_n} t_x \sigma(x)\}}{Z^n} \qquad (4.1.1)$$

where

$$H(\sigma^n) = -J \sum_{<x,y>:x,y\in V_n} \sigma^n(x)\sigma^n(y) - J_1 \sum_{>x,y<:x,y\in V_n} \sigma^n(x)\sigma^n(y) \qquad (4.1.2)$$

for any $n = 1,2,3,\ldots$ and partition function for V_n is defined as

$$Z^n = Z^n(\beta, f) = \sum_{\tilde{\sigma}^n \in \Box V_n} \exp\{-\beta H(\tilde{\sigma}^n) + \sum_{x\in W_n} t_x \tilde{\sigma}(x)\}. \qquad (4.1.3)$$

where $\beta = \frac{1}{kT}$, and T is inverse temperature.

Let consider consistency condition be as follows: for $\mu_n(\sigma^n), n \geq 1$, the consistency condition is given by

$$\sum_{\sigma_n} \mu_n(\sigma^{n-1}, \sigma_n) = \mu_{n-1}(\sigma^{n-1}), \qquad (4.1.4)$$

where $\sigma_n = \{\sigma(x), x \in W_n\}$.

Let $V_1 \subset V_2 \subset \cdots \bigcup_{n-1}^{\infty} W_n$ and $\mu_1, \mu_2, \ldots$ be a sequence of the probability measures on $\Phi^{V_1}, \Phi^{V_2}, \ldots$ where $\Phi = \{-1, +1\}$ for which it satisfies (4.1.4). Based on Kolmogorov theorem (Kolmogorov, 1956), a unique limiting Gibbs measure, μ_n on Ω existed for positive integer n and $\sigma^n \in \Phi^{V_n}$, the equality is given as

$\mu(\{\sigma | V_n = \sigma^n\}) = \mu_n(\sigma^n)$.

We describe the statement below for the conditions on t_x guaranteeing consistency condition of measures, $\mu_n(\sigma^n)$. In this case we will consider for $k = 4$.

Theorem 4.1. *Probability measure,* $\mu_n(\sigma^n)$, *for* $n = 1,2,\ldots$ *satisfying the consistency condition (4.1.4) if and only if for any* $x \in V$, *it holds the following equation:*

$$t_x = \frac{1}{2}\log \frac{\begin{array}{c}\theta^8\theta_1^8 e^{2(t_p+t_q+t_r+t_s)} + \theta^6\theta_1^2\big(e^{2(t_p+t_q+t_r)} + e^{2(t_p+t_q+t_s)} + e^{2(t_p+t_r+t_s)} \\ + e^{2(t_q+t_r+t_s)}\big) + \theta^2\theta_1^2\big(e^{2t_p} + e^{2t_q} + e^{2t_r} + e^{2t_s}\big) + \theta^4\big(e^{2(t_p+t_q)} \\ + e^{2(t_p+t_r)} + e^{2(t_q+t_r)} + e^{2(t_p+t_s)} + e^{2(t_q+t_s)} + e^{2(t_r+t_s)}\big) + \theta_1^8\end{array}}{\begin{array}{c}\theta_1^8 e^{2(t_p+t_q+t_r+t_s)} + \theta^2\theta_1^2\big(e^{2(t_p+t_q+t_r)} + e^{2(t_p+t_q+t_s)} + e^{2(t_p+t_r+t_s)} \\ + e^{2(t_q+t_r+t_s)}\big) + \theta^6\theta_1^2\big(e^{2t_p} + e^{2t_q} + e^{2t_r} + e^{2t_s}\big) + \theta^4\big(e^{2(t_p+t_q)} \\ + e^{2(t_p+t_r)} + e^{2(t_q+t_r)} + e^{2(t_p+t_s)} + e^{2(t_q+t_s)} + e^{2(t_r+t_s)}\big) + \theta^8\theta_1^8\end{array}}. \qquad (4.1.5)$$

Proof. Necessity. Based on the consistency condition (4.1.4), it gives us the following equation:

$$(Z^n)^{-1} \sum_{\sigma_n} \exp\{-\beta H(\sigma^{n-1}) + \beta J \sum_{x\in W_{n-1}} \sigma(x)\big(\sigma(p) + \sigma(q) + \sigma(r) + \sigma(s)\big) +$$

$$\beta J_1 \sum_{x\in W_{n-1}} \{\sigma(p)\big(\sigma(q) + \sigma(r) + \sigma(s)\big) + \sigma(q)\big(\sigma(r) + \sigma(s)\big) + \sigma(r)\sigma(s)\} +$$

$$\sum_{x\in W_{n-1}}\sum_{p\in S(x)} t_p\sigma(p) + \sum_{x\in W_{n-1}}\sum_{q\in S(x)} t_q\sigma(q) + \sum_{x\in W_{n-1}}\sum_{r\in S(x)} t_r\sigma(r) + \sum_{x\in W_{n-1}}\sum_{s\in S(x)} t_s\sigma(s)\} = (Z^{n-1})^{-1}\exp\{-\beta H(\sigma^{n-1}) + \sum_{x\in W_{n-1}} \sigma(x)t_x\}$$

$$\frac{Z^{n-1}}{Z^n}\exp\{-\beta H(\sigma^{n-1})\}\sum_{\sigma_n}\exp\{\beta J\sum_{x\in W_{n-1}}\sigma(x)\big(\sigma(p)+\sigma(q)+\sigma(r)+\sigma(s)\big) + \beta J_1\sum_{x\in W_{n-1}}\{\sigma(p)\big(\sigma(q)+\sigma(r)+\sigma(s)\big)+\sigma(q)\big(\sigma(r)+\sigma(s)\big)+\sigma(r)\sigma(s)\} + \sum_{x\in W_{n-1}}\sum_{p\in S(x)} t_p\sigma(p) + \sum_{x\in W_{n-1}}\sum_{q\in S(x)} t_q\sigma(q) + \sum_{x\in W_{n-1}}\sum_{r\in S(x)} t_r\sigma(r) + \sum_{x\in W_{n-1}}\sum_{s\in S(x)} t_s\sigma(s)\} = (Z^{n-1})^{-1}\exp\{-\beta H(\sigma^{n-1})\}\exp\sum_{x\in W_{n-1}}\sigma(x)t_x\}. \quad (4.1.6)$$

From (4.1.6), we have

$$\frac{Z^{n-1}}{Z^n}\sum_{\sigma_n}\prod_{x\in W_{n-1}}\exp\{\beta J\ \sigma(x)\big(\sigma(p)+\sigma(q)+\sigma(r)+\sigma(s)\big)+\beta J_1\{\sigma(p)\big(\sigma(q)+\sigma(r)\ +\sigma(s)\big) + \sigma(q)\big(\sigma(r)+\sigma(s)\big)+\sigma(r)\sigma(s)\} + t_p\sigma(p)+t_q\sigma(q)+t_r\sigma(r)+t_s\sigma(s)\} = \prod_{x\in W_{n-1}}\exp\{\sigma(x)t_x\}. \quad (4.1.7)$$

When $\sigma_n = \bigcup_{x\in W_{n-1}}\sigma_n{}^x$, equation (4.1.7) could be derived as follow. Let $x \in W_{n-1}$ and $S(x) = \{\sigma(p),\sigma(q),\sigma(r),\sigma(s)\}$, thus

$$\frac{Z^{n-1}}{Z^n}\prod_{x\in W_{n-1}}\sum_{\sigma_n^x}\exp\{\beta J\,\sigma(x)\big(\sigma(p)+\sigma(q)+\sigma(r)+\sigma(s)\big)+\beta J_1\{\sigma(p)\big(\sigma(q)+\sigma(r)\ +\sigma(s)\big) + \sigma(q)\big(\sigma(r)+\sigma(s)\big)+\sigma(r)\sigma(s)\} + t_p\sigma(p)+t_q\sigma(q)+t_r\sigma(r)+t_s\sigma(s)\} = \prod_{x\in W_{n-1}}\exp\{\sigma(x)t_x\}. \quad (4.1.8)$$

For $\sigma(x) = 1$ and $\sigma(x) = -1$, equation (4.1.8) can be rewritten as

$$\frac{\sum_{\sigma_n^x=\{\sigma(p),\sigma(q),\sigma(r),\sigma(s)\}}\exp\{\beta J\,\sigma(x)\big(\sigma(p)+\sigma(q)+\sigma(r)+\sigma(s)\big)+\beta J_1\{\sigma(p)\big(\sigma(q)+\sigma(r)\ +\sigma(s)\big)+\sigma(q)\big(\sigma(r)+\sigma(s)\big)+\sigma(r)\sigma(s)\} + t_p\sigma(p)+t_q\sigma(q)+t_r\sigma(r)+t_s\sigma(s)\}}{\sum_{\sigma_n^x=\{\sigma(p),\sigma(q),\sigma(r),\sigma(s)\}}\exp\{\beta J\big(\sigma(p)+\sigma(q)+\sigma(r)+\sigma(s)\big)+\beta J_1\{\sigma(p)\big(\sigma(q)+\sigma(r)\ +\sigma(s)\big)+\sigma(q)\big(\sigma(r)+\sigma(s)\big)+\sigma(r)\sigma(s)\} + t_p\sigma(p)+t_q\sigma(q)+t_r\sigma(r)+t_s\sigma(s)\}}$$

$$= \exp\{2t_x\} \quad (4.1.9)$$

where $x \in W_{n-1}$ is fixed.

Below is the basic diagram for V_1 of Cayley tree of order four

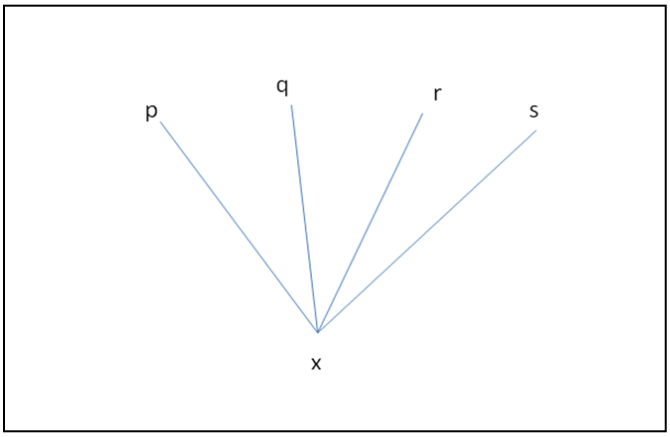


Figure 4.1.1: Basic diagram for Cayley tree of order 4

Based on this diagram, we denote both negative and positive condition as follows:

$$\begin{aligned}
Y_+ = {} & \exp(4\beta J + 6\beta J_1 + t_p + t_q + t_r + t_s) + \exp(2\beta J + t_p + t_q + t_r - t_s) + \\
& \exp(2\beta J + t_p + t_q - t_r + t_s) + \exp(2\beta J + t_p - t_q + t_r + t_s) + \\
& \exp(2\beta J - t_p + t_q + t_r + t_s) + \exp(-2\beta J + t_p - t_q - t_r - t_s) + \\
& \exp(-2\beta J - t_p + t_q - t_r - t_s) + \exp(-2\beta J - t_p - t_q + t_r - t_s) + \\
& \exp(-2\beta J - t_p - t_q - t_r + t_s) + \exp(-2\beta J + t_p + t_q - t_r - t_s) + \\
& \exp(-2\beta J + t_p - t_q + t_r - t_s) + \exp(-2\beta J - t_p + t_q + t_r - t_s) + \\
& \exp(-2\beta J + t_p - t_q - t_r + t_s) + \exp(-2\beta J - t_p + t_q - t_r + t_s) + \\
& \exp(-2\beta J - t_p - t_q + t_r + t_s) + \exp\,(-4\beta J + 6\beta J_1 - t_p - t_q - t_r - t_s) \\
Y_- = {} & \exp(-4\beta J + 6\beta J_1 + t_p + t_q + t_r + t_s) + \exp(-2\beta J + t_p + t_q + t_r - t_s) + \\
& \exp(-2\beta J + t_p + t_q - t_r + t_s) + \exp(-2\beta J + t_p - t_q + t_r + t_s) + \\
& \exp(-2\beta J - t_p + t_q + t_r + t_s) + \exp(2\beta J + t_p - t_q - t_r - t_s) + \\
& \exp(2\beta J - t_p + t_q - t_r - t_s) + \exp(2\beta J - t_p - t_q + t_r - t_s) + \\
& \exp(2\beta J - t_p - t_q - t_r + t_s) + \exp(-2\beta J + t_p + t_q - t_r - t_s) + \\
& \exp(-2\beta J + t_p - t_q + t_r - t_s) + \exp(-2\beta J - t_p + t_q + t_r - t_s) + \\
& \exp(-2\beta J + t_p - t_q - t_r + t_s) + \exp(-2\beta J - t_p + t_q - t_r + t_s) + \\
& \exp(-2\beta J - t_p - t_q + t_r + t_s) + \exp(4\beta J + 6\beta J_1 - t_p - t_q - t_r - t_s).
\end{aligned}$$

From equation (4.1.9), the following equation could be obtained

$$\exp(2t_x) = \frac{Y_+}{Y_-}$$

$$\Rightarrow \qquad t_x = \frac{1}{2}\log\frac{Y_+}{Y_-}. \qquad (4.1.10)$$

Let $\theta = \exp(\beta J)$ and $\theta_1 = \exp(\beta J_1)$, thus

$$\begin{aligned} Y_+ = {} & \theta^4\theta_1{}^6 e^{t_p+t_q+t_r+t_s} + \theta^2 e^{t_p+t_q+t_r-t_s} + \theta^2 e^{t_p+t_q-t_r+t_s} + \theta^2 e^{t_p-t_q+t_r+t_s} \\ & \theta^2 e^{-t_p+t_q+t_r+t_s} + \theta^{-2} e^{t_p-t_q-t_r-t_s} + \theta^{-2} e^{-t_p+t_q-t_r-t_s} + \theta^{-2} e^{-t_p-t_q+t_r-t_s} \\ & \theta^{-2} e^{-t_p-t_q-t_r+t_s} + \theta^{-2} e^{t_p+t_q-t_r-t_s} + \theta^{-2} e^{t_p-t_q+t_r-t_s} + \theta^{-2} e^{-t_p+t_q+t_r-t_s} \\ & \theta^{-2} e^{t_p-t_q-t_r+t_s} + \theta^{-2} e^{-t_p+t_q-t_r+t_s} + \theta^{-2} e^{-t_p-t_q+t_r+t_s} + \theta^{-4}\theta_1{}^6 e^{-t_p-t_q-t_r-t_s} \\ Y_- = {} & \theta^{-4}\theta_1{}^6 e^{t_p+t_q+t_r+t_s} + \theta^{-2} e^{t_p+t_q+t_r-t_s} + \theta^{-2} e^{t_p+t_q-t_r+t_s} + \theta^{-2} e^{t_p-t_q+t_r+t_s} \\ & \theta^{-2} e^{-t_p+t_q+t_r+t_s} + \theta^2 e^{t_p-t_q-t_r-t_s} + \theta^2 e^{-t_p+t_q-t_r-t_s} + \theta^2 e^{-t_p-t_q+t_r-t_s} \\ & \theta^2 e^{-t_p-t_q-t_r+t_s} + \theta^{-2} e^{t_p+t_q-t_r-t_s} + \theta^{-2} e^{t_p-t_q+t_r-t_s} + \theta^{-2} e^{-t_p+t_q+t_r-t_s} \\ & \theta^{-2} e^{t_p-t_q-t_r+t_s} + \theta^{-2} e^{-t_p+t_q-t_r+t_s} + \theta^{-2} e^{-t_p-t_q+t_r+t_s} + \theta^4\theta_1{}^6 e^{-t_p-t_q-t_r-t_s}. \end{aligned}$$

$t_x = \frac{1}{2}\log\frac{Y_+}{Y_-}$ multiply with $e^{t_p+t_q+t_r+t_s}$ for both numerator and denominator. Hence, it implies (4.1.5).

Sufficiency. To show that the consistency condition is sufficient in this case, let assume equation (4.1.5) is accepted, then it gives us as in (4.1.10) and hence (4.1.9). From (4.1.9), equation (4.1.11) could be obtained as follows:

$$\begin{aligned} & \sum_{\sigma_n^x=\{\sigma(p),\sigma(q),\sigma(r),\sigma(s)\}} \begin{matrix} \exp\{\beta J\,\sigma(x)\big(\sigma(p)+\sigma(q)+\sigma(r)+\sigma(s)\big) + \beta J_1\{\sigma(p) \\ \big(\sigma(q)+\sigma(r)\ +\sigma(s)\big) + \sigma(q)\big(\sigma(r)+\sigma(s)\big) + \sigma(r)\sigma(s)\} \\ +t_p\sigma(p) + t_q\sigma(q) + t_r\sigma(r) + t_s\sigma(s)\} \end{matrix} \\ & = b(x)\exp\{\sigma(x)t_x\} \qquad (4.1.11) \end{aligned}$$

where $\sigma(x) = \pm 1$. Equation (4.1.11) implies the following equation

$$\begin{aligned} & \prod_{x\in W_{n-1}} \sum_{\sigma_n^x=\{\sigma(p),\sigma(q),\sigma(r),\sigma(s)\}} \begin{matrix} \exp\{\beta J\,\sigma(x)\big(\sigma(p)+\sigma(q)+\sigma(r)+\sigma(s)\big) + \beta J_1\{\sigma(p) \\ \big(\sigma(q)+\sigma(r)\ +\sigma(s)\big) + \sigma(q)\big(\sigma(r)+\sigma(s)\big) + \sigma(r)\sigma(s)\} \\ +t_p\sigma(p) + t_q\sigma(q) + t_r\sigma(r) + t_s\sigma(s)\} \end{matrix} \\ & = \prod_{x\in W_{n-1}} b(x)\exp\{\sigma(x)t_x\}. \qquad (4.1.12) \end{aligned}$$

Let $B^n = \sum_{x\in W_{n-1}} b(x)$. From equation (4.1.12) we have

$$Z^{n-1}B^{n-1}\mu_{n-1}(\sigma^{n-1}) = Z^n \sum_{\sigma^n} \mu^n(\sigma^{n-1},\sigma_n).$$

It is known that probability measure occurred when each $\mu_n, n \geq 1$. Thus, we have

$$\sum_{\sigma^{n-1}}\sum_{\sigma_n} \mu_n(\sigma^{n-1},\sigma_n) = 1, \ \sum_{\sigma^{n-1}} \mu_{n-1}(\sigma^{n-1}) = 1.$$

Finally, from these equalities, it is obtained that

$$Z^{n-1}B^{n-1} = Z^n \qquad (4.1.13)$$

where it can be concluded that consistency condition (4.1.4) holds.
Based on Theorem 3.1, problem on Gibbs measure is done by describing the solutions of functional equation (4.1.5). According to proposition 3.1, we note that any transformation $\tilde{S}$ of the group G_k induces a shift transformation $\tilde{S}: \Omega \to \Omega$ by

$(\check{S}\sigma)(t) = \sigma(St), t \in G_k, \sigma \in \Omega$

By G_k, we denote the set of all shifts on Ω. A Gibbs measure,μ on Ω is said to be translation-invariant if for any $T \in G_k$ equality $\mu(T(D)) = \mu(D)$ is valid for all $D \in F$.

The solution of the functional equation (4.1.5) is tricky to be analyzed. Hence, it is natural to begin with $t_x = t$ is constant for all $x \in V$. In this case, Gibbs measure corresponding to this solution is clearly to be translation-invariant one [8]. Thus, from (4.1.5), we have

$$u = \frac{\theta^8\theta_1^8 u^4 + 4\theta^6\theta_1^2 u^3 + 6\theta^4 u^2 + 4\theta^2\theta_1^2 u + \theta_1^8}{\theta_1^8 u^4 + 4\theta^2\theta_1^2 u^3 + 6\theta^4 u^2 + 4\theta^6\theta_1^2 u + \theta^8\theta_1^8} \tag{4.1.14}$$

where $u = e^{2t}$.

Corollary 4.1. *Equation 4.1.14 is coincide with the classical result for Ising model when* $\theta_1 = 1$, *and* $J_1 = 0$, *i.e.,*

$$u_n(x^0) = \left(\frac{\theta^2 u_{n-1}(x^0) + 1}{u_{n-1}(x^0) + \theta^2}\right)^4. \tag{4.1.15}$$

4.2. Analysis of the Recurrent Equation

The general recurrent equation (3.5) and its ratio (3.8) are identified in previous section. The recurrent equation obtained for Cayley tree of order 4 is given as below:

$$u_n(x^0) = \frac{{u_{n-1}}^4\theta^4\theta_1^6 + 4{u_{n-1}}^3\theta^2 + 6{u_{n-1}}^2\theta_1^{-2} + 4u_{n-1}\theta^{-2} + \theta^{-4}\theta_1^6}{{u_{n-1}}^4\theta^{-4}\theta_1^6 + 4{u_{n-1}}^3\theta^{-2} + 6{u_{n-1}}^2\theta_1^{-2} + 4u_{n-1}\theta^2 + \theta^4\theta_1^6}. \tag{4.2.1}$$

Equation (4.1.14) describes the fixed points of the equation (4.1.5). If there is more than one solution in equation (4.2.1), then we can say that there is more than one translation-invariant Gibbs measure existed. Thus, based on Theorem 3.1 a phase transition occurs in our case study. Here, we give the analytic solution for the restricted interactions given in our case.

Let the functional equation, $f(u)$ be

$$f(u) = \frac{u^4\theta^4\theta_1^6 + 4u^3\theta^2 + 6u^2\theta_1^{-2} + 4u\theta^{-2} + \theta^{-4}\theta_1^6}{u^4\theta^{-4}\theta_1^6 + 4u^3\theta^{-2} + 6u^2\theta_1^{-2} + 4u\theta^2 + \theta^4\theta_1^6}. \tag{4.2.2}$$

From (4.2.2), we have

$$u = \frac{u\theta^4\theta_1^6 + 4u^3\theta^2 + 6u^2\theta_1^{-2} + 4u\theta^{-2} + \theta^{-4}\theta_1^6}{u^4\theta^{-4}\theta_1^6 + 4u^3\theta^{-2} + 6u^2\theta_1^{-2} + 4u\theta^2 + \theta^4\theta_1^6}. \tag{4.2.3}$$

We simplify equation (4.2.3), thus we have

$u^4\theta^4\theta_1^6 + 4u^3\theta^2 + 6u^2\theta_1^{-2} + 4u\theta^{-2} + \theta^{-4}\theta_1^6 - u^5\theta^{-4}\theta_1^6 - 4u^4\theta^{-2} - 6u^3\theta_1^{-2} - 4u^2\theta^2 - u\theta^4\theta_1^6 = 0$

$$(u-1)(u^4 + \beta u^3 + (\alpha+\beta)u^2 + \beta u + 1) = 0 \tag{4.2.4}$$

where $\alpha = \frac{6\theta^4 - 4\theta^6\theta_1^2}{\theta_1^8}$, $\beta = \frac{4\theta^2 - \theta^8\theta_1^6 + \theta_1^6}{\theta_1^6}$.

From the equation (4.2.4), we can straightaway see that $u_2^* = 1$ is a fixed point. Thus, it is worthwhile to solve the second part of the equation. If $\alpha > 0$, this implies $\gamma \geq 0$, then $u^4 + \beta u^3 + (\alpha + \beta)u^2 + \beta u + 1 = 0$ only has one solution, i.e. $u_2^* = 1$.

Let $t = u + \frac{1}{u}$, $t^2 = u^2 + \frac{1}{u^2}$. Hence, $u^4 + \beta u^3 + (\alpha + \beta)u^2 + \beta u + 1 = 0$ becomes $t^2 + \beta t + (\alpha + \beta) - 2 = 0$

From this quadratic equation, it is found that

$$t = \frac{-\beta \pm \sqrt{\beta^2 - 4(\alpha + \beta - 2)}}{2} \geq 2$$

Given that value of $t = \gamma_1, \gamma_2$. To solve the equation, we let $t = \gamma_1$ or $t = \gamma_2$. It is showed that γ_2 is rejected since

$$\gamma_2 = \frac{-\beta - \sqrt{\beta^2 - 4(\alpha + \beta - 2)}}{2} \leq 2.$$

Now, let γ_1 equals to

$$\gamma_1 = \frac{-\beta + \sqrt{\beta^2 - 4(\alpha + \beta - 2)}}{2} \geq 2$$

If $\alpha \leq 0$, and we have

i) $\beta^2 - 4\alpha - 4\beta + 8 = (\beta - 2)^2 + 4 - 4\alpha > 0$ and

ii) $4 + \beta > -\sqrt{\beta^2 - 4\alpha - 4\beta + 8}$

Therefore,

$$\sqrt{\beta^2 - 4\alpha - 4\beta + 8} \geq 4 + \beta > -\sqrt{\beta^2 - 4\alpha - 4\beta + 8}$$
$$(4 + \beta)^2 < \beta^2 - 4\alpha - 4\beta + 8$$
$$0 \geq 8\beta + 16 + 4(\alpha + \beta - 2)$$
$$0 \geq 3\beta + \alpha + 2.$$

Substituting α, and β, we have

$$0 \geq 12\theta^2\theta_1^2 - 3\theta^8\theta_1^8 + 5\theta_1^8 + 6\theta^4 - 4\theta^6\theta_1^2. \tag{4.2.5}$$

From (4.2.5), we find another two fixed points, u_1^* and u_3^*. Hence, we can say that phase transition occurs when there exist more than one fixed point. In this case, according to equation (4.2.3), we have a non-negative values given by u_1^*, $u_2^* = 1$, and u_3^*. The critical curve we intend to find is

$$0 = 12\theta^2\theta_1^2 - 3\theta^8\theta_1^8 + 5\theta_1^8 + 6\theta^4 - 4\theta^6\theta_1^2. \tag{4.2.6}$$

Let $C(\theta, \theta_1) = 12\theta^2\theta_1^2 - 3\theta^8\theta_1^8 + 5\theta_1^8 + 6\theta^4 - 4\theta^6\theta_1^2 = 0$ be the implicit function of (θ, θ_1). It can be shown that $C(\theta, \theta_1) = 0$ is a one to one function of (θ, θ_1) for the domain of (θ, θ_1) such that,

$$\frac{\partial}{\partial\theta}C(\theta, \theta_1) = -24\theta(\theta\theta_1 - 1)(\theta\theta_1 + 1)(\theta^4\theta_1^6 + \theta^2\theta_1^4 + \theta^2 + \theta_1^2) < 0$$

where $\alpha \leq 0$, i.e., $\theta_1^2 > \frac{3}{2\theta^2}$ which implies $(\theta\theta_1 - 1)$ always positive. Since $C(\theta, \theta_1)$ is always monotone decreasing respect to θ, $C(\theta, \theta_1) = 0$ is a function of θ to θ_1. Assume there is multiple solution of θ_1 for a given θ in (4.2.6), we rewrite (4.2.6) into $(-3\theta^8 + 5)\theta_1^8 + (12\theta^2 - 4\theta^6)\theta_1^2 + 6\theta^4 = 0$

and the necessity condition for this equation to have more than one solution for θ_1 is $-3\theta^8+5>0$ and $12\theta^2-4\theta^6<0$ is impossible.

Hence, we could summarize our result as follows:

Theorem 4.2. *For the Ising model* (3.2) *with h=0, the curve* $C(\theta,\theta_1)=0$ *in the plane* (θ,θ_1) *is a critical curve for phase transitions, namely, for an arbitrary pair of parameters* (θ,θ_1) *above the critical curve the phase transition takes place and for any pair of parameters* (θ,θ_1) *below the critical curve there occurs a single Gibbs state.*

5. The Free Energy

In this section, we derive the free energy for Ising model on Cayley tree of order 4 with restricted competing interactions. Let the partition function $Z_n(\beta,t)$ of the state μ_β^t (which corresponds to the solution of $t=\{t_x, x\epsilon V\}$) of the equation (4.1.5) be as follow:

$$Z_n(\beta,t)=\sum_{\tilde{\sigma}_n\epsilon\Omega_{V_n}}\exp\{-\beta H(\tilde{\sigma}_n)+\sum_{x\epsilon W_n}t_x\tilde{\sigma}(x)\}.$$

Then, the free energy [12, 13] is defined as

$$F(\beta,t)=-\lim_{n\to\infty}\frac{3}{\beta(4^{n+1}-1)}\ln Z_n(\beta,t). \qquad (5.2.1)$$

The aim of this section is to prove the following Theorem.

Theorem 5.2.1.(i) *The free energy exists for all t, and is given by the formula*

$$F(\beta,t)=-\lim_{\beta\to\infty}\lim_{n\to\infty}\frac{3}{(4^{n+1}-1)}\sum_{k=0}^{n}\sum_{x\epsilon W_{n-k}}A_\beta(J,J_1,t_p,t_q,t_r,t_s), \qquad (5.2.2)$$

where $p=p(x),q=q(x).r=r(x),s=s(x)$ *are direct successors of* x;

$$A_\beta(J,J_1,t_p,t_q,t_r,t_s)=a_{\beta J}(2\beta J_1+p-s)+a_{\beta J}(-2\beta J_1+p-s)+a_{\beta J}(2\beta J_1+q-s)$$
$$+a_{\beta J}(-2\beta J_1+q-s)+a_{\beta J}(2\beta J_1+r-s)+a_{\beta J}(-2\beta J_1+r-s)+\tilde{a}(\beta J,\beta J_1);$$
$$a_\beta(x)=\frac{1}{4}\ln[4\cosh(x-\beta)\cosh(x+\beta); \qquad (5.2.3)$$
$$\tilde{a}_{\beta J,\beta J_1}(D)=\frac{1}{2}\ln D.$$

(ii) *For any solution* $t=\{t_x, x\epsilon V\}$ *of (4.1.5)*

$$F(\beta,t)=F(\beta,-t),$$

where $-t=\{-t_x, x\epsilon V\}$.

6. Conclusion

We have calculated a recurrent equation of partition function for a ferromagnetic Ising model with competing interactions on Cayley tree of arbitrary order *k*. For the case of order *k*=4, we show that phase transition occurs, i.e., Theorem 4.2, in this model and the critical curve is given by equation (4.2.6). Free energy for the same model is also calculated.

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