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Noncommutative of space-time and the Relativistic Hydrogen Atom

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Abstract. We study the Klein-Gordon equation in a non-commutative space-time as applied to the Hydrogen atom to extract the energy levels, by considering the second-order corrections in the non-commutativity parameter. By comparing to the 2S - 1S transition energies we obtain an upper bound on the non-commutativity parameter. Phenomenologically we show that non-commutativity is the source of lamb shift corrections and spin of the electron.

1. Introduction

The non-commutativity of spatial rotations in three and more dimensions is an idea which is deeply ingrained in our theories. The non-commutativity has received a wide appreciation as an alternative approach to understanding many physical phenomenon such as the ultraviolet and infrared divergences [1], unitarity violation [2], causality [3], and new physics at very short distances of the Planck-length order [4, 5, 6].

The non-commutative field theory is motivated by the natural extension of the usual quantum mechanical commutation relations between position and momentum, by imposing further commutation relations between position coordinates themselves. As in usual quantum mechanics, the non-commutativity of position coordinates immediately implies a set of uncertainty relations between position coordinates analogous to the Heisenberg uncertainty relations between position and momentum; namely:

$$[\hat{x}^{\mu}, \hat{x}^{\nu}]_{*} = i\theta^{\mu\nu}, \tag{1}$$

where $\hat{x}^{\mu}$ are the coordinate operators and $\theta^{\mu\nu}$ are the non-commutativity parameters of dimension of area that signify the smallest area in space that can be probed in principle.We use the symbol $*$ in equation (1) to denote the product of the non-commutative structure. This idea is similar to the physical meaning of the Plank constant in the relation $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$, which as is known is the smallest phase-space in quantum mechanics. The space-time non-commutativity has gained considerable interest in current literature and a search for any possible upper bound on the space-time non-commutativity parameter using recent experimental feedback is very much desirable. The issue of space-time non-commutativity is worth pursuing in its own right because of its deep connection with such fundamental notions as unitarity and causality. It was argued that introduction of space-time non-commutativity spoils unitarity or even causality. Much attention has been devoted in recent times to circumvent these difficulties in formulating

theories with $\theta^{0\nu} \neq 0$ [7, 8, 9, 10]. We do not consider momentum space non-commutative effects as have been done by [11]. Our motivation is to study the effect of non-commutativity on the level of quantum mechanics when time-space non-commutativity is accounted for. One can study the physical consequences of this theory by making detailed analytical estimates for measurable physical quantities and compare the results with experimental data to find an upper bound on the θ parameter. The most obvious natural phenomena to use in hunting for non-commutative effects are simple quantum mechanics systems, such as the hydrogen atom [12, 13, 14]. In the non-commutative space-time one expects the degeneracy of the initial spectral line to be lifted, thus one may say that non-commutativity plays the role of spin.

In this work we present an important contribution to the non-commutative approach to the hydrogen atom. Our goal is to solve the Klein-Gorden equation for the Coulomb potential in a non-commutative space-time up to second-order of the non-commutativity parameter using the Seiberg-Witten maps and the Moyal product. We thus find the non-commutative modification of the energy levels of the hydrogen atom and we show that the non-commutativity is the source of spin as a result of rotation of space-time dimensions.

This paper is organized as follows. In section 2 we derive the corresponding Seiberg-Witten maps up to the second order of θ for the various dynamical fields, and we propose an invariant action of the non-commutative charged scalar field in the presence of an electric field. In section 3, using the generalised Euler-Lagrange field equation, we derive the deformed Klein-Gordon (KG) equation. Applying these results to the hydrogen atom, we solve the deformed KG equation and obtain the non-commutative modification of the energy levels. The last section is devoted to a discussion.

2. Seiberg-Witten maps

Here we look for a mapping $\phi^A \to \hat{\phi}^A$ and $\lambda \to \hat{\lambda}(\lambda, A_\mu)$, where $\phi^A = (A_\mu, \varphi)$ is a generic field, A_μ and φ are the gauge and charged scalar fields respectively (the Greek and Latin indices denote curved and tangent space-time respectively), and λ is the U(1) gauge Lie-valued infinitesimal transformation parameter, such that:

$$\hat{\phi}^A(A) + \hat{\delta}_{\hat{\lambda}} \hat{\phi}^A(A) = \hat{\phi}^A(A + \delta_\lambda A), \tag{2}$$

where δ_λ is the ordinary gauge transformation and $\hat{\delta}_{\hat{\lambda}}$ is a noncommutative gauge transformation which are defined by:

$$\hat{\delta}_{\hat{\lambda}} \hat{\varphi} = i\hat{\lambda} * \hat{\varphi}, \qquad \delta_\lambda \varphi = i\lambda\varphi, \tag{3}$$

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_\mu = \partial_\mu \hat{\lambda} + i\left[\hat{\lambda}, \hat{A}_\mu\right]_*, \qquad \delta_\lambda A_\mu = \partial_\mu \lambda. \tag{4}$$

In accordance with the general method of gauge theories, in the non-commutative space-time, using these transformations one can get at second order in the non-commutative parameter $\theta^{\mu\nu}$ (or equivalently θ) the following Seiberg–Witten maps [15]:

$$\hat{\varphi} = \varphi + \theta\varphi^1 + \theta^2\varphi^2 + \mathcal{O}\left(\theta^3\right), \tag{5}$$

$$\hat{\lambda} = \lambda + \theta\lambda^1(\lambda, A_\mu) + \theta^2\lambda^2(\lambda, A_\mu) + \mathcal{O}\left(\theta^3\right), \tag{6}$$

$$\hat{A}_\xi = A_\xi + \theta A^1_\xi(A_\xi) + \theta^2 A^2_\xi(A_\xi) + \mathcal{O}\left(\theta^3\right), \tag{7}$$

$$\hat{F}_{\mu\xi} = F_{\mu\xi}(A_\xi) + \theta F^1_{\mu\xi}(A_\xi) + \theta^2 F^2_{\mu\xi}(A_\xi) + \mathcal{O}\left(\theta^3\right), \tag{8}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{9}$$

To begin, we consider a non-commutative field theory with a charged scalar particle in the presence of an electrodynamic gauge field in a Minkowski space-time. We can write the action as:

$$\mathcal{S} = \int d^4x \left(\eta^{\mu\nu} \left(\hat{D}_\mu \hat{\varphi} \right)^\dagger * \hat{D}_\nu \hat{\varphi} + m^2 \hat{\varphi}^\dagger * \hat{\varphi} - \frac{1}{4} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} \right), \tag{10}$$

where the gauge covariant derivative is defined as: $\hat{D}_\mu \hat{\varphi} = \left(\partial_\mu + ie\hat{A}_\mu \right) * \hat{\varphi}$.

Next we use the generic-field infinitesimal transformations (3) and (4) and the star-product tensor relations to prove that the action in eq. (10) is invariant. By varying the scalar density under the gauge transformation and from the generalised field equation and the Noether theorem we obtain [16]:

$$\frac{\partial \mathcal{L}}{\partial \hat{\varphi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \hat{\varphi} \right)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \partial_\nu \hat{\varphi} \right)} - \partial_\mu \partial_\nu \partial_\sigma \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \partial_\nu \partial_\sigma \hat{\varphi} \right)} + \mathcal{O}\left(\theta^3 \right) = 0. \tag{11}$$

3. Non-commutative Klein-Gordon equation

In this section we study the Klein-Gordon equation for a Coulomb interaction $(-e/r)$ in the free non-commutative space-time. This means that we will deal with solutions of the U(1) gauge-free non-commutative field equations [17]. For this we use the modified field equations in eq. (11) and the generic field $\hat{A}_\mu$ so that:

$$\delta \hat{A}_\mu = \partial_\mu \hat{\lambda} - ie\hat{A}_\mu * \hat{\lambda} + ie\hat{\lambda} * \hat{A}_\mu, \tag{12}$$

and the free non-commutative field equations:

$$\partial^\mu \hat{F}_{\mu\nu} - ie \left[\hat{A}^\mu, \hat{F}_{\mu\nu} \right]_* = 0. \tag{13}$$

Using the Seiberg-Witten maps (7)–(8) and the choice (13), we can obtain the following deformed Coulomb potential [17]:

$$\hat{a}_0 = -\frac{e}{r} - \frac{e^3}{r^4} \theta^{0j} x_j + \frac{e^5}{2\, r^5} \left[\left(\theta^{0j} \right)^2 - 5 \left(\frac{\theta^{0j} x_j}{r} \right)^2 \right] + \mathcal{O}\left(\theta^3 \right). \tag{14}$$

Using the modified field equation (11) and the generic field $\hat{\varphi}$ so that:

$$\delta_{\hat{\lambda}} \hat{\varphi} = i\hat{\lambda} * \hat{\varphi}, \tag{15}$$

the Klein-Gordon equation in a non-commutative space-time in the presence of the vector potential $\hat{A}_\mu$ can be cast into:

$$\left(\eta^{\mu\nu} \partial_\mu \partial_\nu - m^2 \right) \hat{\varphi} + \left(ie\eta^{\mu\nu} \partial_\mu \hat{A}_\nu - e^2 \eta^{\mu\nu} \hat{A}_\mu * \hat{A}_\nu + 2ie\eta^{\mu\nu} \hat{A}_\mu \partial_\nu \right) \hat{\varphi} = 0. \tag{16}$$

Now using the fact that:

$$\eta^{\mu\nu} \partial_\mu \partial_\nu = -\partial_0^2 + \Delta, \tag{17}$$

and

$$2ie\eta^{\mu\nu} \hat{A}_\mu \partial_\nu = i\frac{2e^2}{r} \partial_0 + 2i\frac{e^4}{r^4} \theta^{0j} x_j \partial_0 - i\frac{e^6}{r^5} \left[\left(\theta^{0j} \right)^2 - 5 \left(\frac{\theta^{0j} x_j}{r} \right)^2 \right] \partial_0, \tag{18}$$

and

$$-e^2 \eta^{\mu\nu} \hat{A}_\mu * \hat{A}_\nu = \frac{e^4}{r^2} + 2\frac{e^6}{r^5} \theta^{0j} x_j + \frac{e^8}{r^6} \left[\left(\theta^{0j} \right)^2 - 4 \left(\frac{\theta^{0j} x_j}{r} \right)^2 \right], \tag{19}$$

then the Klein-Gordon equation (16) up to $\mathcal{O}\left(\theta^3\right)$ takes the form:

$$\left[-\partial_0^2+\Delta-m_e^2+\frac{e^4}{r^2}+i\frac{2e^2}{r}\partial_0+2i\frac{e^4}{r^4}\theta^{0j}x_j\partial_0-i\frac{e^6}{r^5}\left[\left(\theta^{0j}\right)^2-5\left(\frac{\theta^{0j}x_j}{r}\right)^2\right]\partial_0\right.$$
$$\left.+2\frac{e^6}{r^5}\theta^{0j}x_j+\frac{e^8}{r^6}\left[\left(\theta^{0j}\right)^2-4\left(\frac{\theta^{0j}x_j}{r}\right)^2\right]\right]\hat{\varphi}=0. \quad (20)$$

The solution to eq. (20) in spherical polar coordinates (r,θ,ϕ) takes the separable form [18]:

$$\hat{\varphi}(r,\theta,\phi,t)=\frac{1}{r}\hat{R}(r)\hat{Y}\left(\theta,\phi\right)\exp(-iEt). \quad (21)$$

Then eq. (20) reduces to the radial equation:

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)-e^4}{r^2}+\frac{2Ee^2}{r}+E^2-m_e^2+2E\frac{e^4}{r^4}\theta^{0j}x_j+2\frac{e^6}{r^5}\theta^{0j}x_j\right.$$
$$\left.-E\frac{e^6}{r^5}\left[\left(\theta^{0j}\right)^2-5\left(\frac{\theta^{0j}x_j}{r}\right)^2\right]+\frac{e^8}{r^6}\left[\left(\theta^{0j}\right)^2-4\left(\frac{\theta^{0j}x_j}{r}\right)^2\right]\right]\hat{R}(r)=0. \quad (22)$$

In eq. (22) the coulomb potential in non-commutative space-time appears within the perturbation terms:

$$H^{\theta}_{\text{pert}}=2E\frac{e^4}{r^4}\theta^{0j}x_j+2\frac{e^6}{r^5}\theta^{0j}x_j+5E\frac{e^6}{r^5}\left(\frac{\theta^{0j}x_j}{r}\right)^2-E\frac{e^6}{r^5}\left(\theta^{0j}\right)^2-4\frac{e^8}{r^6}\left(\frac{\theta^{0j}x_j}{r}\right)^2+\frac{e^8}{r^6}\left(\theta^{0j}\right)^2, \quad (23)$$

where the first term is the electric dipole–dipole interaction created by the non-commutativity, the second is the electric dipole–quadruple interactions, the third and fourth terms are the electric quadruple–quadruple interaction, and the last two terms are similar to the Van-der–Waals potential energy forces between two atoms. These interactions show us that the effect of space-time non-commutativity on the interaction of the electron and the proton is equivalent to an extension of two nucleus interactions at a considerable distance. This idea effectively confirms the presence of gravity at this level. To investigate the modification of the energy levels by eq. (23), we use the first-order perturbation theory. The spectrum of H_0 and the corresponding wave functions are well known and given by:

$$R_{nl}(r)=\sqrt{\frac{a}{n+\nu+1}}\left(\frac{n!}{\Gamma\left(n+2\nu+2\right)}\right)^{1/2}x^{\nu+1}e^{-x/2}L_n^{2\nu+1}(x)\,, \quad (24)$$

where the relativistic energy levels are given by:

$$E=E_{n,l}=\frac{m_e\left(n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\alpha^2}\right)}{\left[\left(n+\frac{1}{2}\right)^2+\left(l+\frac{1}{2}\right)^2+2\left(n+\frac{1}{2}\right)\sqrt{\left(l+\frac{1}{2}\right)^2-\alpha^2}\right]^{\frac{1}{2}}}\,, \quad (25)$$

and $L_n^{2\nu+1}$ are the associated Laguerre polynomials [19], with the following notations:

$$\nu=-\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\alpha^2},\qquad \alpha=e^2,\qquad a=\sqrt{m_e^2-E^2}. \quad (26)$$

4. Noncommutative corrections of the energy

Now to obtain the modification to the energy levels as a result of the terms (23) due to the non-commutativity of space-time we use perturbation theory. For simplicity, first of all, we take $\theta_i = \delta_{i3}\theta$ $\left(\theta^{0j}x_j = \theta r\cos\vartheta\right)$ and assume that the other components are all zero and also the fact that in the first-order perturbation theory the expectation value of $\cos\vartheta/r^3$, $\cos\vartheta/r^4$, $\cos\vartheta^2/r^5$, $\cos\vartheta^2/r^6$, $1/r^5$ and $1/r^6$ are as follows:

$$\langle\psi^0_{nlm} + \psi^1 \left| H^{\theta(1)}_{\text{pert}} + H^{\theta(2)}_{\text{pert}} \right| \psi^0_{nlm} + \psi^1\rangle, \tag{27}$$

where

$$H^{\theta(1)}_{\text{pert}} = \theta 2e^4\left(E\frac{\cos\vartheta}{r^3} + e^2\frac{\cos\vartheta}{r^4}\right), \tag{28}$$

$$H^{\theta(2)}_{\text{pert}} = \theta^2 e^6\left(5E\frac{\cos\vartheta^2}{r^5} - 4e^2\frac{\cos\vartheta^2}{r^6} - E\frac{1}{r^5} + e^2\frac{1}{r^6}\right), \tag{29}$$

$$\psi^0_{nlm} = R_{nl}(r)Y^m_l(\vartheta,\phi), \tag{30}$$

and

$$\psi^1 = \sum \frac{\langle\psi^0_{nlm}\left|H^{\theta(1)}_{\text{pert}}\right|\psi^0_{nkm}\rangle}{E_{nl} - E_{nk}}\psi^0_{nkm}. \tag{31}$$

By taking into account the fact that

$$\langle\psi^0_{nlm} \mid \cos\vartheta \mid \psi^0_{nl'm'}\rangle = B^{m'}_{l'+1}\delta_{l,l'+1}\delta_{m,m'} + B^{m'}_{l'}\delta_{l,l'-1}\delta_{m,m'}, \tag{32}$$

$$\begin{aligned}\langle\psi^0_{nlm} \mid \cos\vartheta^2 \mid \psi^0_{nl'm'}\rangle &= B^{m'}_{l'+1}B^{m'}_{l'+2}\delta_{l,l'+2}\delta_{m,m'} + B^{m'}_{l'}B^{m'}_{l'-1}\delta_{l'-2}\delta_{m,m'} + \\ &+ \left[B^{m'}_{l'+1}B^{m'}_{l'+1} + B^{m'}_{l'}B^{m'}_{l'}\right]\delta_{l'l}\delta_{mm'},\end{aligned} \tag{33}$$

where

$$B^m_l = \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}, \tag{34}$$

we have

$$\psi^1 = \theta^2\frac{e^4 B^m_l}{E_{nl} - E_{nl-1}}\left[E_{nl}f(3) + e^2 f(4)\right]\psi^0_{nl-1m} + \theta^2\frac{e^4 B^m_{l+1}}{E_{nl} - E_{nl+1}}\left[E_{nl}f(3) + e^2 f(4)\right]\psi^0_{nl+1m}. \tag{35}$$

where $f(k) = \left\langle r^{-k}\right\rangle$ and is completely defined by

$$\begin{aligned}\langle nlm \mid r^{-k} \mid nlm\rangle &= \int_0^\infty R^2_{nl}(r)r^{-k}dr \\ &= \frac{2^{k-1}a^k n!}{(n+\nu+1)\,\Gamma(n+2\nu+2)}\int_0^\infty x^{2\nu+2-k}e^{-x}\left[L^{2\nu+1}_n(x)\right]^2 dx \\ &= f(k) \qquad k = 3,4,5,6.\end{aligned} \tag{36}$$

We use the relation between the confluent hypergeometric function $F(-n;\nu+1;x)$ and the associated Laguerre polynomials $L^\nu_n(x)$, namely:

$$L^\nu_n(x) = \frac{\Gamma(n+\nu+1)}{\Gamma(n+1)\,\Gamma(\nu+1)}F(-n;\nu+1;x), \tag{37}$$

$$\int_0^\infty x^{\nu-1}e^{-x}\left[F(-n;\gamma;x)\right]^2 dx = \frac{n!\Gamma(\nu)}{\gamma(\gamma+1)\cdots(\gamma+n-1)}\left\{1+\frac{n(\gamma-\nu-1)(\gamma-\nu)}{1^2\gamma}+\right.$$
$$+\frac{n(n-1)(\gamma-\nu-2)(\gamma-\nu-1)(\gamma-\nu)(\gamma-\nu+1)}{1^2 2^2\gamma(\gamma+1)}+\cdots$$
$$\left.\cdots+\frac{n(n-1)\cdots 1(\gamma-\nu-n)\cdots(\gamma-\nu+n-1)}{1^2 2^2\cdots n^2\gamma(\gamma+1)\cdots(\gamma+n-1)}\right\}. \quad (38)$$

Equation (36) becomes:

$$\begin{aligned}
\langle nlm \mid r^{-3} \mid nlm'\rangle &= \int_0^\infty R_{nl}^2(r)r^{-3}dr\delta_{mm'} \\
&= \frac{4a^3 n!}{(n+\nu+1)\Gamma(n+2\nu+2)}\int_0^\infty x^{2\nu-1}e^{-x}\left[L_n^{2\nu+1}(x)\right]^2 dx\delta_{mm'} \\
&= \frac{4a^3 n!}{(n+\nu+1)\Gamma(n+2\nu+2)}\left[\frac{\Gamma(n+2\nu+2)}{\Gamma(n+1)\Gamma(2\nu+2)}\right]^2\times \\
&\qquad\times\int_0^\infty x^{2\nu-1}e^{-x}\left[F(-n;2\nu+2;x)\right]^2 dx\delta_{mm'} \\
&= \frac{2a^3}{\nu(2\nu+1)(n+\nu+1)}\left\{1+\frac{n}{(\nu+1)}\right\}\delta_{mm'} = f(3), \quad (39)\\
\langle nlm \mid r^{-4} \mid nlm'\rangle &= \frac{4a^4}{(2\nu-1)\nu(2\nu+1)(n+\nu+1)}\left[1+\frac{3n}{(\nu+1)}+\right. \\
&\quad\left.+\frac{3n(n-1)}{(\nu+1)(2\nu+3)}\right]\delta_{mm'} \\
&= f(4), \quad (40)\\
\langle nlm \mid r^{-5} \mid nlm'\rangle &= \frac{4a^5}{(2\nu-1)(\nu-1)\nu(2\nu+1)(n+\nu+1)}\left[1+\frac{6n}{(\nu+1)}+\right. \\
&\quad\left.+\frac{15n(n-1)}{(\nu+1)(2\nu+3)}+\frac{5n(n-1)(n-2)}{(\nu+1)(2\nu+3)(\nu+2)}\right]\delta_{mm'} \\
&= f(5), \quad (41)\\
\langle nlm \mid r^{-6} \mid nlm'\rangle &= \frac{8a^6}{(2\nu-3)(2\nu-1)(\nu-1)\nu(2\nu+1)(n+\nu+1)}\left[1+\frac{10n}{(\nu+1)}+\right. \\
&\quad +\frac{45n(n-1)}{(\nu+1)(2\nu+3)}+\frac{35n(n-1)(n-2)}{(\nu+1)(2\nu+3)(\nu+2)}+ \\
&\quad\left.+\frac{35n(n-1)(n-2)(n-3)}{2(\nu+1)(2\nu+3)(\nu+2)(2\nu+5)}\right]\delta_{mm'} \\
&= f(6). \quad (42)
\end{aligned}$$

The first-order correction term is:

$$\langle\psi_{nlm}^0\left|H_{\text{pert}}^{\theta(1)}\right|\psi^0\rangle = 0. \quad (43)$$

Therefore, the non-commutativity of space-time to the first order has no effect , so we study the non-commutativity effects at second order.

To second order in θ, equation (27) can be written as:

$$2\langle\psi^1\left|H_{\text{pert}}^{\theta(1)}\right|\psi^0\rangle + \langle\psi_{nlm}^0\left|H_{\text{pert}}^{\theta(2)}\right|\psi^0\rangle. \quad (44)$$

So we have

$$2\langle\psi^1\left|H_{\rm pert}^{\theta(1)}\right|\psi^0\rangle=\theta^2 8e^8\left[\frac{(B_l^m)^2}{E_{nl}-E_{nl-1}}+\frac{\left(B_{l+1}^m\right)^2}{E_{nl}-E_{nl+1}}\right]\left[E_{nl}f(3)+e^2f(4)\right]^2, \tag{45}$$

and

$$\langle\psi_{nlm}^0\left|H_{\rm pert}^{\theta(2)}\right|\psi^0\rangle=\theta^2e^6\left\{\left[(B_l^m)^2+\left(B_{l+1}^m\right)^2\right]\left[5E_{nl}f(5)-4e^2f(6)\right]-E_{nl}f(5)+e^2f(6)\right\}. \tag{46}$$

Putting these results together one gets:

$$\begin{aligned}\Delta E^{\rm nc}(n,l,m) &= \theta^2\alpha^3\left[8\alpha[\frac{(B_l^m)^2}{E_{nl}-E_{nl-1}}+\frac{\left(B_{l+1}^m\right)^2}{E_{nl}-E_{nl+1}}]\left[E_{nl}f(3)+e^2f(4)\right]^2\right.\\ &\left.+([(B_l^m)^2+\left(B_{l+1}^m\right)^2]\left[5E_{nl}f(5)-4e^2f(6)\right]-E_{nl}f(5)+e^2f(6))\right]. \end{aligned}\tag{47}$$

The energy shift is depends on magnetic quantum number, which clearly reflects the existence of spin. Furthermore it is worth noting that the correction terms containing θ^2 are very similar to the spin–spin coupling, thus the non-commutative parameter θ plays the role of spin and thus the degeneracy of levels is completely removed. The energy levels of the hydrogen atom in the framework of the non-commutative Klein- Gordon equation are:

$$\hat{E}=E_{n,l}+\Delta E^{\rm nc}(n,l,m). \tag{48}$$

We showed that the energy-levels shift for $1S$ and $2S$ states are:

$$\Delta E_{\rm 1S}^{\rm nc} = \theta^2\alpha^3\left(-E_{10}f_{\rm 1S}(5)+e^2f_{\rm 1S}(6)\right), \tag{49}$$

$$\Delta E_{\rm 2S}^{\rm nc} = \theta^2\alpha^3\left(-E_{20}f_{\rm 2S}(5)+e^2f_{\rm 2S}(6)\right). \tag{50}$$

One can obtain a limit for θ by comparing the corrections to transition energies obtained using (49) and (50) with the experimental results from hydrogen spectroscopy. We take as test the levels $1S$ and $2S$ because we have the best experimental precision for the transition between them [20]:

$$f_{\rm 1S-2S}=(2446061102474851\pm 34)\,{\rm Hz}. \tag{51}$$

The non-commutative correction to this transition reads:

$$\delta E\,(1{\rm S}-2{\rm S})=636,737\theta^2\,({\rm MeV})^3. \tag{52}$$

Comparing with the result of the experimental value, the bound is found to be given by:

$$\theta\leq(8{\rm GeV})^{-2}. \tag{53}$$

This is in agreement with other results presented for instance in ref [21]. Thus the experimental signature for space-time non-commutativity differs from that for space-space non-commutativity [11].

5. Conclusions

In this work we started from quantum relativistic charged scalar particle in a canonical non-commutative space-time to find the action which is invariant under the infinitesimal gauge transformation. By using the Seiberg-Witten maps and the Moyal product up to second order in the non-commutativity parameter θ, we derived the deformed Klein-Gorden equation for non-commutative Coulomb potential. By solving the deformed Klein-Gordon equation we found that the energy shift up to the second order of θ, is proportional to θ^2, thus we explicitly accounted for spin effects, resulting from the rotation of the time dimension in space. Hence we can say that the Klein-Gordon equation in non-commutative space-time at the second order in θ describes particles with spin. Thus we came to the conclusion that the non-commutative relativistic energy degeneracy is completely removed. This proofs that the noncommutative space-time is responsible for spin effects.

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