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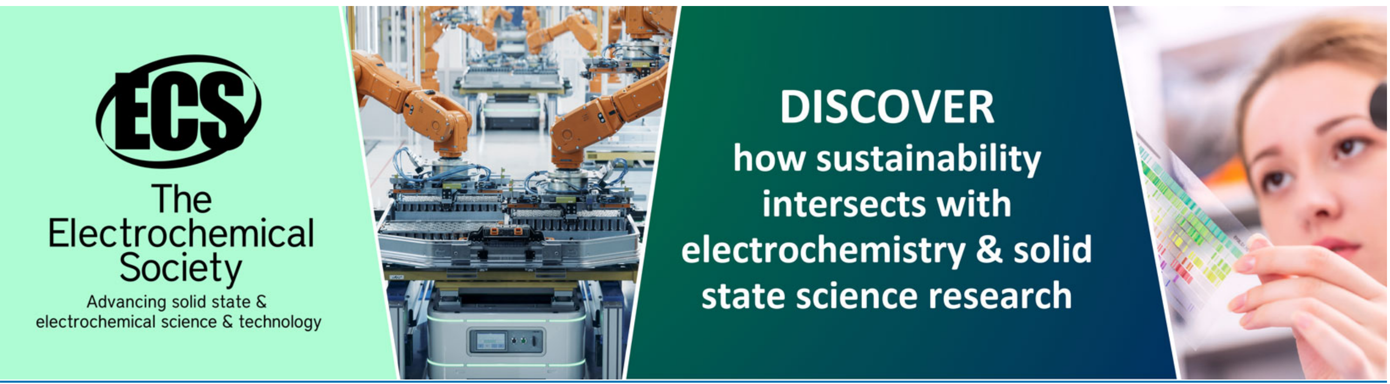

On Invariant Measure of the Circle Maps

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Abstract. Let f be piecewise smooth circle homeomorphisms with break points and the rotation number ρ is irrational. We provide a necessary condition for the absolute continuity of f-invariant measure with respect to Lebesgue measure.

1. Introduction and Statement of Results

Let f be an orientation preserving homeomorphism of the circle $S^1 \equiv \mathbb{R}/\mathbb{Z}$ with lift $F : \mathbb{R} \to \mathbb{R}$ continuous, strictly increasing and $F(x + 1) = F(x) + 1, x \in \mathbb{R}$. The circle homeomorphism f is then defined by $f(x) = F(x) \mod 1$. The most important arithmetic characteristic of the homeomorphism f of the circle S^1 is the *rotation number*

$$\rho(f) = \lim_{i\to\infty} \frac{F^i(x)}{i} \mod 1.$$

Here and below, for a given map F, F^i denotes its i-th iteration. Denjoy in [2] proved that, if f is a circle diffeomorphism with irrational rotation number $\rho = \rho(f)$ and $\log Df$ is of bounded variation, then f is conjugate to the pure rotation $f_\rho : x \to x + \rho \mod 1$, that is, there exists an essentially unique homeomorphism φ of the circle with $\varphi \circ f = f_\rho \circ \varphi$. This classical result of Denjoy can be extended to circle homeomorphisms with break points. It is well known, that circle homeomorphisms f with irrational rotation number ρ admit a unique f- invariant probability measure μ. Since the conjugating map φ and the f-invariant measure μ are related by $\varphi(x) = \mu([0, x])$ (see [1]), regularity properties of the conjugating map φ imply corresponding properties of the density of the absolutely continuous invariant measure μ. This problem of smoothness of density function of μ for the smooth diffeomorphisms now very well understood by several authors [8], [9] and [10]. An important class of circle homeomorphisms are the homeomorphisms with break points so-called class **P-homeomorphisms**.

The class of **P-homeomorphisms** consists of orientation preserving circle homeomorphisms f whose lifts F are differentiable except in finite number points, so called *break points* of f, at which the left and right derivatives, denoted respectively by Df_- and Df_+ exist, and such that,

(i) there exist constants $0 < c < C < \infty$ with $c < Df(x) < C$ for all $S^1 \backslash BP(f), c < Df_-(x_b) < C$ and $c < Df_-(x_b) < C$ for all $x_b \in BP(f)$, with $BP(f)$ the set of break points of f on S^1;

(ii) $\log Df$ has bounded variation on S^1.

The ratio $\sigma(x_b) := Df(x_{b-})/Df(x_{b+})$ is called the *jump ratio* of f in x_b. The total variation of $\log Df$ we denote by v i.e. $v = Var_{S^1} \log Df$. f-invariant measures of P-homeomorphisms with one break point was first studied by A. Dzhalilov and K. Khanin in [3]. The main result of them is as follows:

Theorem 1.1. *Let $f \in C^{2+\alpha}(S^1 \setminus \{b\})$, $\alpha > 0$ be circle homeomorphism with one break point $b = b(f)$. Assume that the rotation number $\rho = \rho(f)$ of f is irrational. Then the f-invariant measure μ is mutually singular with Lebesgue measure ℓ.*

Piecewise-linear (PL) circle homeomorphisms with two break points are the simplest examples of homeomorphisms with two break points and feature in many areas of mathematics, for example, group theory and homotopy theory (see [12]). The f-invariant measures of such homeomorphisms were studied for the first time in [7]. There was shown that, the f-invariant measure of the (PL) circle homeomorphism f with two break points and irrational rotation number is absolutely continuous with respect to Lebesgue measure if and only if both break points lie on the same trajectory. General (non (PL)) class of P-homeomorphisms with two break points have been studied by A. Dzhalilov and I. Liousse in [4]. The main result of [4] is the following theorem.

Theorem 1.2. *Let $f \in C^2(S^1 \setminus \{a, b\})$, be circle homeomorphism with two break points. Assume that the rotation number $\rho = \rho(f)$ of f is irrational of bounded type i.e. the coefficients in the continued fraction expansion of ρ are bounded. Then the f-invariant measure is absolutely continuous with respect to Lebesgue measure if and only if both break points lie on the same trajectory and the product of their jump ratios is 1.*

The f-invariant measures of P-homeomorphisms with several break points for the case product of jump ratios not equal to one were studied by A. Dzhalilov D. Mayer and U. Safarov in [5]. The main result of [5] is the following statement.

Theorem 1.3. *Let $f \in C^2(S^1 \setminus \{b_i\})$, be circle homeomorphism with several break points $b_i = b_i(f)$. Assume that the rotation number $\rho = \rho(f)$ of f is irrational and product of jump ratios not equal to one. Then the f-invariant measure μ is singular with respect to Lebesgue measure ℓ.*

Note that the question of the regularity of the invariant measure of circle homeomorphisms with several break points that do not lie on the same trajectory and such that the product of the jumps is trivial, that is, product of jump ratios equal to one remains open. Let rotation number ρ be an irrational. We use the continued fraction $\rho = 1/(k_1 + 1/(k_2 + ...)) := [k_1, k_2, ..., k_n, ...)$ of the rotation number, which is understood as the limit of the sequence of convergents $p_n/q_n = [k_1, k_2, ..., k_n]$. The sequence of positive integer k_n with $n \geq 1$, which are called *incomplete multiples*, is uniquely determined for irrational ρ. The coprimes p_n and q_n satisfy the recursive relations $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$ for $n \geq 1$, where we set for convenience, $p_{-1} = 0$, $q_{-1} = 1$, and $p_0 = 1$, $q_0 = k_1$. In this paper we will study P-homeomorphism f with irrational rotation number having several number break points. Our purpose in this paper is to give a necessary condition for the absolute continuity of the f-invariant measure μ w.r.t. Lebesgue measure ℓ. Our main result is the following:

Theorem 1.4. *Let f be a P-homeomorphism with irrational rotation number $\rho = \rho(f)$. If the f-invariant measure μ is absolutely continuous w.r.t. Lebesgue measure ℓ, then for all $\delta > 0$ and for all $0 \leq k \leq k_{n+1}$*

$$\lim_{n\to\infty} \ell(x : |\log Df^{kq_n + q_{n-1}}(x)| \geq \delta) = 0.$$

Notice that, Theorem 1.4 was proved in [4] for the case $k = 0$ and this fact played key role to prove main result (Theorem 1.2) of the paper [4]. To study the f-invariant measures, Theorem 1.4 provides an opportunity to analyze high iteration of $f^{q_{n-1}}$ i.e. we can analyze $f^{kq_n+q_{n-1}}$, for all $0 \le k \le k_{n+1}$.

2. Notations, terminology, background

The assertions listed below, which are valid for any orientation-preserving homeomorphism $f \in P$ with irrational rotation number ρ, constitute classical Denjoy's theory. Their elementary proofs can be found in [4] and [5].

I. Dynamical partition. We define the n-th *fundamental segment* $\Delta_0^n = \Delta^n(\xi)$ as the circle arc $[\xi, f^{q_n}(\xi)]$ if n is even and $[f^{q_n}(\xi), \xi]$ if n is odd. Let $\xi \in S^1$ and $\Delta_0^n(\xi)$ be n-th fundamental segment. We denote two sets of closed intervals of order $n : q_n$ "lengthy" intervals: $\{\Delta_i^{n-1} = f^i(\Delta_0^{n-1}),\ 0 \le i < q_n\}$ and q_{n-1} "short" intervals: $\left\{\Delta_j^n = f^j(\Delta_0^n),\ 0 \le j < q_{n-1}\right\}$. The "lengthy" and "short" intervals are mutually disjoint except for the endpoints and cover the whole circle. The partition obtained by the above construction will be denoted by $\mathbf{P}_n$ and called the n-th *dynamical partition* of the point ξ. Obviously the partition $\mathbf{P}_{n+1}$ is a refinement of the partition $\mathbf{P}_n$. Indeed the "short" intervals are members of $\mathbf{P}_{n+1}$ and each "lengthy" interval $\Delta_i^{n-1} \in \mathbf{P}_n,\ 0 \le i < q_n$, is partitioned into $k_{n+1} + 1$ intervals belonging to $\mathbf{P}_{n+1}$ such that

$$\Delta_i^{n-1} = \Delta_i^{n+1} \cup \bigcup_{s=0}^{k_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^n. \tag{1}$$

II. Generalized Denjoy estimate. Let $\xi_0 \in S^1$ be a continuity point of Df^{q_n}, then the following inequality holds: $e^{-v} \le Df^{q_n}(\xi_0) \le e^v$, where $v = Var_{S^1} \log Df$.

III. Generalized Finzi estimate. Suppose $\xi \in S^1$, $\eta \in \Delta^{n-1}(\xi)$ and ξ, η are continuity points of Df^{q_n}. Then for any $0 \le k < q_n$ the following inequality holds: $|\log Df^k(\xi) - \log Df^k(\eta)| \le v$.

Let $(S^1, \mathfrak{G}, \mu)$ be a measure space and $T : S^1 \to S^1$ be a measurable map.

Definition 2.1. *The set $A \in \mathfrak{G}$ is said to be **invariant** with respect to the measurable T, if $A = T^{-1}A$.*

Definition 2.2. *A measurable map $T : S^1 \to S^1$ is said to be **ergodic** with respect Lebesgue measure ℓ if the Lebesgue measure $\ell(A)$ of any invariant set A equals 0 or 1.*

Let $\xi_0 \in S^1$, denote by $V_n = \Delta^n(\xi_0) \cup \Delta^{n-1}(\xi_0)$. Now we equip S^1 with the usual metric $|x - y| = \inf\{|\widetilde{x} - \widetilde{y}|$, where $\widetilde{x}, \widetilde{y}$ are ranges over the lifts of $x, y \in S^1$ respectively$\}$.

Lemma 2.3. *Let f be a P-homeomorphism with irrational rotation number ρ. Suppose $\xi \in V_n$ and ξ be a continuity point of Df^{q_n}. Then for any $0 \le k < q_n$ the following inequality holds:*

$$e^{-v}\frac{|f^k(V_n)|}{|V_n|} \le Df^k(\xi) \le e^v \frac{|f^k(V_n)|}{|V_n|}. \tag{2}$$

Proof. Let the system of intervals $\mathcal{I} = \{I : I \subset V_n, \text{and the map } Df^{q_n} \text{ is continuous on } I\}$ be continuity intervals of Df^{q_n}. Let $\xi \in \Delta^{n-1}(\xi_0)$. Then, by the mean value theorem, for any $0 \le k < q_n$, we have

$$\frac{|f^k(\Delta^{n-1}(\xi_0))|}{Df^k(\xi)|\Delta^{n-1}(\xi_0)|} = \frac{Df^k(z_1)|I_1| + Df^k(z_2)|I_2| + ... + Df^k(z_d)|I_d|}{Df^k(\xi)|\Delta^{n-1}(\xi_0)|}, \tag{3}$$

where $z_i \in I_i \subset \Delta^{n-1}(\xi_0)$ and $I_i \in \mathcal{I}$, $1 \le i \le d$. If $\xi \in \Delta^n(\xi_0)$ then we have

$$\frac{|f^k(\Delta^n(\xi_0))|}{Df^k(\xi)|\Delta^n(\xi_0)|} = \frac{Df^k(y_1)|J_1| + Df^k(y_2)|J_2| + ... + Df^k(y_t)|J_t|}{Df^k(\xi)|\Delta^n(\xi_0)|}, \tag{4}$$

where $y_i \in J_i \subset \Delta^n(\xi_0)$ and $J_i \in \mathcal{I}$, $1 \le i \le t$. Apply generalized Finzi estimate to the right-hand side of relations (12) and (13) we get

$$e^{-v} \le \frac{|f^k(\Delta^{n-1}(\xi_0))|}{Df^k(\xi)|\Delta^{n-1}(\xi_0)|} \le e^v \text{ and } e^{-v} \le \frac{|f^k(\Delta^n(\xi_0))|}{Df^k(\xi)|\Delta^n(\xi_0)|} \le e^v.$$

Finally, we get

$$e^{-v} \le \frac{|f^k(V_n)|}{Df^k(\xi)|V_n|} \le e^v.$$

□

Lemma 2.4. *Let f be a P-homeomorphism of the circle S^1 with irrational rotation number ρ, then f is ergodic with respect to Lebesgue measure ℓ.*

Proof. Suppose that there exists an invariant set A such that $0 < \ell(A) < 1$. Then by the Lebesgue Density Theorem, A has a density point z. We fix an arbitrary $\epsilon > 0$. By definition of density points, we can find a $\delta > 0$ such that for any interval $[a, b]$ satisfying the conditions $z \in [a, b]$, $[a, b] \subset (z - \delta, z + \delta)$, we have $\ell(A \cap [a, b]) \ge (1 - \epsilon)\ell([a, b])$, or, in other words, $\ell(A^c \cap [a, b]) < \epsilon\ell([a, b])$, where A^c denotes the complement of A. Now, we choose such n that $V_n = \Delta^n(z) \cup \Delta^{n-1}(z) \subset (z - \delta, z + \delta)$. We can check that $\bigcup_{k=0}^{q_n-1} T^k(V_n) = S^1$ and each point of the circle belongs to at most two intervals of this cover. Hence, the set A^c is invariant with respect to f, using above lemma we get:

$$\ell(A^c) = \sum_{k=0}^{q_n-1} \ell(A^c \cap f^k(V_n)) = \sum_{k=0}^{q_n-1} \int_{A^c \cap V_n} Df^k(x) d\ell \le \frac{e^v \ell(A^c \cap V_n)}{\ell(V_n)} \sum_{k=0}^{q_n-1} \ell(f^k(V_n)) \le 2e^v \epsilon.$$

Since ϵ was arbitrary, $\ell(A^c) = 0$. The theorem is proved. □

Lemma 2.5. *Let f be a P-homeomorphisms with irrational rotation number. Then conjugation map φ between f and f_ρ is either absolutely continuous or singular.*

Proof. Consider a P-homeomorphism f of the circle S^1 with irrational rotation number ρ. Let φ be a conjugating map between f and f_ρ, i.e.

$$\varphi \circ f = f_\rho \circ \varphi. \tag{5}$$

We know that conjugation function φ is strictly increasing function on S^1. Then $D\varphi$ exists almost everywhere on S^1. Denote by $A = \{x : x \in S^1, D\varphi(x) > 0\}$. It is clear that the set A is mod 0 invariant with respect to f. Since the class P-homeomorphism is ergodic with respect to the Lebesgue measure. Hence, the Lebesgue measure of A is either null or full. If Lebesque measure of A is null then φ is a singular function, if it is full then φ is an absolutely continuous function. □

Lemma 2.6. *Let f be a P-homeomorphism with irrational rotation number and φ be a conjugating map between f and f_ρ. Then the following properties are equivalent:*

(i) the map φ is singular;

(ii) the map φ^{-1} is singular;

(iii) the f-invariant measure μ and the Lebesgue measure ℓ is mutually singular.

Proof. Since the conjugating map φ and the invariant measure μ are related by $\varphi(x) = \mu([0, x])$ i.e. φ is a distribution function of measure μ. Considering this we get equivalence of relations (i) and (iii). We will prove that the properties (i) and (ii) are equivalent. Let the map φ be a singular function. Therefore, there exists a measurable subset $C \subset S^1$ such that $\ell(C) = 0$ and $\ell(\varphi(C)) > 0$. Denote by $B = \bigcup_{n\in\mathbb{Z}} f^n(C)$. It is clear that the subset $B \subset S^1$ is a invariant subset w.r.t. f and the Lebesgue measure of B is null, since f is P-homeomorphism. Using by equation (5) it is easy to see that the set $\varphi(B)$ is invariant w.r.t. f_ρ. Due to ergodicity of f_ρ w.r.t. Lebesgue measure, follows that $\ell(\varphi(B)) = 0$ or 1. But, since $\ell(\varphi(C)) > 0$, implies that $\ell(\varphi(B)) = 1$. The map φ^{-1} displays $S^1 \setminus \varphi(B)$ into $S^1 \setminus B$. Using above assertions we get $\ell(S^1 \setminus \varphi(B)) = 0$ and $\ell(S^1 \setminus B) = 1$. Therefore, φ^{-1} is singular. Similarly we can show that if φ^{-1} is singular, then φ is also singular. □

Using lemmas 2.5 and 2.6 we get the following remark.

Remark 2.7. *Let f be a P-homeomorphism with irrational rotation number and φ be a conjugating map between f and f_ρ. Then the following properties are equivalent:*

(i) the map φ is absolutely continuous;

(ii) the map φ^{-1} is absolutely continuous;

(iii) the f-invariant measure μ and the Lebesgue measure ℓ is mutually absolutely continuous i.e. they are equivalent.

3. Prove of Main Theorem

Proof. Since φ is conjugation map between f and f_ρ i.e. $\varphi \circ f = f_\rho \circ \varphi$. Then for all $0 \le k \le k_{n+1}$ we have

$$f^{kq_n+q_{n-1}} = \varphi^{-1} \circ f_\rho^{kq_n+q_{n-1}} \circ \varphi. \tag{6}$$

By assumption of Theorem 1.4 and Remark 2.7 the conjugation map φ is absolutely continuous. From remark 2.7 implies that the map φ^{-1} is absolutely continuous homeomorphism. Below we will use the following well known fact (see [6]). Let $[a, b], [c, d] \subset \mathbb{R}$ be two intervals, let $v \in AC([c, d])$ (i.e. absolutely continuous on $[c, d]$) and $u : [a, b] \to [c, d]$ be monotone and $u \in AC([a, b])$, then $v \circ u \in AC([a, b])$ and the chain rule equality holds i.e.

$$D(v(u(x))) = Dv(u(x))Du(x) \quad \text{a.e.} \quad (\text{w.r.t. } \ell) \tag{7}$$

Since $\varphi,\ \varphi^{-1} \in AC(S^1)$ and $f_\rho^{kq_n+q_{n-1}}$ is linear, using above fact and (6) we get the following relation.

$$Df^{kq_n+q_{n-1}}(x) = D\varphi^{-1}(f_\rho^{kq_n+q_{n-1}}(\varphi(x)))D\varphi(x) \text{ a.e.} \quad (\text{w.r.t. } \ell). \tag{8}$$

Hence, $\varphi^{-1}(\varphi(x)) = x$ and $\varphi,\ \varphi^{-1} \in AC(S^1)$ we have

$$D\varphi^{-1}(\varphi(x))D\varphi(x) = 1 \quad \text{a.e.} \quad (\text{w.r.t. } \ell). \tag{9}$$

Using equalities (8) and (9), we get

$$\log Df^{kq_n+q_{n-1}}(x) = \log D\varphi^{-1}(f_\rho^{kq_n+q_{n-1}}(\varphi(x))) - \log D\varphi^{-1}(\varphi(x)) \text{ a.e.} \quad (\text{w.r.t. } \ell). \tag{10}$$

By assumption of Theorem 1.4 and Remark 2.7 the f-invariant measure μ and the Lebesgue measure ℓ are equivalent measures and therefore the relations (8), (9) and (10) hold with respect to f-invariant measure μ. Integrating last equality by measure $d\mu$ we have

$$\int_{S^1} |\log Df^{kq_n+q_{n-1}}(x)|d\mu = \int_{S^1} |\log D\varphi^{-1}(f_\rho^{kq_n+q_{n-1}}(\varphi(x))) - \log D\varphi^{-1}(\varphi(x))|d\mu. \tag{11}$$

Now we will show that the right site of (11) tending to zero when $n \to \infty$. Since, $\varphi_*\mu = \ell$ (i.e. $\mu(B) = \ell(\varphi(B))$ for any Borel set B) the equality $d\mu = D\varphi d\ell$ holds. Using this equality and changing of variable $x \to \varphi(x)$ we can write the right site of equality (11) as follows.

$$\int_{S^1} |\log D\varphi^{-1}(f_\rho^{kq_n+q_{n-1}}(\varphi(x))) - \log D\varphi^{-1}(\varphi(x))|d\mu =$$

$$= \int_{S^1} |\log D\varphi^{-1}(f_\rho^{kq_n+q_{n-1}}(x)) - \log D\varphi^{-1}(x)|d\ell. \tag{12}$$

Due to Remark 2.7 $\log D\varphi^{-1} \in L_1(S^1, d\ell)$. Well known fact that the class $C([a,b])$ of continuous functions on $[a,b]$ is dense (in $\|\cdot\|_{L_1}$) in $L_1([a,b], d\ell)$ (see [11]). From this fact implies that for any $\epsilon > 0$ there exists a continuous function ψ_ϵ and ℓ-integrable function ϕ_ϵ such that $\log D\varphi^{-1} = \psi_\epsilon + \phi_\epsilon$ and $\|\phi_\epsilon\|_{L_1} \le \epsilon$. Using this fact, we have

$$\|\log D\varphi^{-1}(f_\rho^{kq_n+q_{n-1}}) - \log D\varphi^{-1}\|_{L_1} \le$$

$$\le \|\psi_\epsilon \circ f_\rho^{kq_n+q_{n-1}} - \psi_\epsilon\|_{L_1} + \|\phi_\epsilon \circ f_\rho^{kq_n+q_{n-1}} - \phi_\epsilon\|_{L_1}.$$

Using property of dynamical partition it is easy to see that $(x, f_\rho^{kq_n+q_{n-1}}(x)) \subset (x, f_\rho^{q_{n-1}}(x))$ for all $0 \le k \le k_{n+1}$ and $|f_\rho^{q_{n-1}}(x) - x| \le q_n^{-1}$, for any $x \in S^1$. As ψ_ϵ is uniformly continuous on S^1 and by exponential refinement $f_\rho^{kq_n+q_{n-1}}(x)$ uniformly tends to x, there exists a positive integer $n_0 = n_0(\epsilon)$ such that for all $n \ge n_0$, $\|\psi_\epsilon \circ f_\rho^{kq_n+q_{n-1}} - \psi_\epsilon\|_{L_1} \le \epsilon$. Since the function $f_\rho^{kq_n+q_{n-1}}(x)$ is linear function, by changing of variable $x \to f_\rho^{kq_n+q_{n-1}}(x)$ we have $\|\phi_\epsilon \circ f_\rho^{kq_n+q_{n-1}}\|_{L_1} = \|\phi_\epsilon\|_{L_1}$. Therefore,

$$\|\log D\varphi^{-1}(f_\rho^{kq_n+q_{n-1}}) - \log D\varphi^{-1}\|_{L_1} \le 3\epsilon. \tag{13}$$

Since $\epsilon > 0$ was arbitrary and sufficiently small and using relations (11)-(13) we get

$$\lim_{n\to\infty} \int_{S^1} |\log Df^{kq_n+q_{n-1}}(x)|d\mu = 0. \tag{14}$$

Denote by $B(\delta, k, n) = \{x : |\log Df^{kq_n+q_{n-1}}(x)| \ge \delta\}$. It is clear that

$$\int_{S^1} |\log Df^{kq_n+q_{n-1}}(x)|d\mu \ge \delta\mu(B(\delta, k, n)).$$

Using this inequality together with relation (14) we get $\lim\limits_{n\to\infty} \mu(B(\delta, k, n)) = 0$. Therefore, the f-invariant measure μ and the Lebesgue measure ℓ are equivalent measures we get

$$\lim_{n\to\infty} \ell(B(\delta, k, n)) = 0.$$

□

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